The conjecture of Birch and Swinnerton-Dyer and anticyclotomic Iwasawa theory

– Study group –

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1 Introduction

The conjecture of Birch and Swinnerton-Dyer (BSD) is widely regarded to be one of the most important open problems in modern number theory.

Let E be an elliptic curve defined over $\mathbb{Q}$ and let $r \coloneqq \operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q})$ be the rank (as an abelian group) of its Mordell–Weil group. We can associate to E its 'Hasse–Weil L-series' $L(E,s)$, which is a holomorphic function due to the known modularity of E. The 'weak form' of BSD concerns the order of vanishing of $L(E,s)$ at $s=1$ and states that

$$\operatorname{ord}_{s=1} L(E,s) = r. \tag{1}$$

The 'full' BSD conjecture predicts a formula for the first non-vanishing coefficient in the Taylor series expansion of $L(E,s)$ at $s=1$:

$$\tfrac{1}{r!} \cdot L^{(r)}(E,1) = \frac{|Ш| \cdot \Omega \cdot \mathrm{Reg} \cdot \mathrm{Tam}}{|E(\mathbb{Q})_{\mathrm{tor}}|^2}, \tag{2}$$

where the right hand side is composed of important invariants of E. In particular, Ш is the *Tate–Shafarevich group* parameterising E-torsors that become trivial over every completion of $\mathbb{Q}$, a group whose finiteness is an important open conjecture in its own right, Ω is the Néron period, Reg is the regulator of, Tam is the product of Tamagawa numbers, and $|E(\mathbb{Q})_{\mathrm{tor}}|$ is the number of points in $E(\mathbb{Q})$ that have finite order.

For a more detailed introduction to the conjecture of Birch–Swinnerton-Dyer, and further reading, we refer to the survey articles [3, 9, 17, 18, 19, 20].

The aim of this study group is to discuss how anticyclotomic Iwasawa theory can be used to study the conjecture of Birch–Swinnerton-Dyer. For example, we will prove the following result towards (1) that is still among the best currently available.

Theorem (Gross-Zagier, Kolyvagin)**.** *Let E be an elliptic curve over $\mathbb{Q}$ with Tate–Shafarevich group* Ш *and Hasse–Weil L-series $L(E,s)$. If $\operatorname{ord}_{s=1} L(E,s) \leq 1$, then* Ш *is finite and (1) holds.*

2 Schedule

Time: Thursdays, 3.30 – 5pm
Place: Aula Naranja, ICMAT

Below we briefly outline the talks in this study group.

(1) 5 March: **Introduction and statement of the BSD conjecture**
Dominik Bullach
State the conjecture in its original form [1, Conj. A], and explain how this is related to the current formulation (by work of Goldfeld [6]). You should also define all quantities in (2). State the generalisation to elliptic curves over number fields. End with an overview of the study group.

(2) 12 March: **Selmer groups** *Javier Aguilera Villegas*
This talk will cover [11, § 1.1] with additional details (from [15, Ch. I] and [12, § 1 and § 2]). Introduce p-adic representations [15, Ch. I,§ 1], with examples [15, Ch. I, Ex. 1.4, 1.5, 2.1]. Define Selmer structures [12, Def. 2.1.1] (this uses local conditions [12, Def. 1.1.1]) and give examples [12, Def. 1.1.6, Ex. 2.1.2, Ex. 2.1.3]. We will also need propagated Selmer structures [12, Ex. 1.1.2] and cartesian Selmer structures [12, Def. 1.1.4]. Finally, review local duality [12, § 1.3] [11, § 1.1] and global duality [12, Thm. 2.3.4].

(3) 19 March: **The Cassels–Tate pairing** *Daniel Macías Castillo*
This talk covers [11, §§ 1.3 – 1.4].

- w/c 23 March: "**Iwasawa theory over function fields**" (workshop)
- w/c 30 March: **Easter break**

(4) 9 April: **Modules over principal Artinian rings, and Kolyvagin systems**
TBA
This talk first covers [11, § 1.5] and then [11, § 1.2]. For the second part, define the finite-singular comparison isomorphisms [12, § 1.2] and then define Kolyvagin systems [11, Def. 1.2.3].

- w/c 13 April: "**Drinfeld modular forms and related topics**" (workshop)

(5) 23 April: **Heegner points** *Maxim Mornev*
Summarise the necessary background from the theory of complex multiplication [5, § 3.1], introduce Heegner points [5, § 3.2], and explain important properties of these [5, § 3.4]. State the Gross–Zagier formula [5, Thm. 3.20] and Kolyvagin's theorem [5, Thm. 3.21]. Explain how the Gross–Zagier–Kolyvagin Theorem [5, Thm. 3.22] is deduced from Kolyvagin's Theorem (this is the 'Sketch of proof' following [5, Thm. 3.22]).
Another useful reference is [7], and some historical background can be found in [2].

(6) 30 April: **Bounding Selmer groups** *TBA*
Complete the proof of Kolyvagin's theorem [11, Thm. A] by covering [11, §§ 1.6 – 1.7].
(See also [8], [5, Ch. 10], and [14, Thm.] for different accounts of Kolyvagin's argument).

(7) 7 May: **The Gross–Zagier formula** *TBA*
Explain how the Gross–Zagier formula stated in talk (5) (= [4, Thm. 5]) is deduced from the technical main result [10, Thm. I.6.1] [4, Thm. 2] of Gross and Zagier (this is the content of [4, § 2]). Sketch the proof of [4, Thm. 2] (this is [4, § 3]). Another useful source might be [16].

(8) 14 May: **Introduction to Iwasawa theory** *TBA*
Define the Iwasawa algebra [13, Def. 5.2.1]. State its basic algebraic properties, review the structure theorem for Iwasawa modules, and define the characteristic ideal and Iwasawa invariants of a torsion Iwasawa module [13, § 3 up to Def. 5.3.9]. Explain the statement of the Heegner point main conjecture [11, Thm. B].

(9) 21 May **Kolyvagin systems over Λ** *TBA*
This talk covers [11, §§ 2.1 – 2.2].

(10) 28 May: **The Heegner point main conjecture** *TBA*
Complete the proof of [11, Thm. B] by covering [11, § 2.3].

- w/c 1 June: "**Arithmetic and applications of Anderson motives**" (workshop)
- w/c 8 June: "**Zeta, multizeta and L-values in positive characteristic**" (workshop)

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