Clay Mathematics Institute
Summer School 2014

CMI

Periods and Motives
Feynman amplitudes in the 21st century

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Contents

1 Summary

The theory of motives was introduced by Grothendieck as a "universal cohomology theory" to explain the common properties of different cohomology theories. One may describe motives as an intermediate step between algebraic varieties and their linear invariants (cohomology). Motives therefore provide a conceptual framework for the study of periods. For instance the periods that arise from mixed Tate motives are combinations of multiple zeta values. The theory of motives is an vibrant area of current research in modern mathematics. As an example we mention a recent letter of Deligne to Drinfeld, in which he proved the finiteness of the number of irreducible lisse $\mathbb{Q}_\ell$-sheaves with bounded ramification, up to isomorphism and up to twist, on a smooth variety defined over a finite field. This result is inspired by motivic considerations, and its proof relies on Lafforgue's Langlands correspondence over curves.

Multiple zeta values (MZVs) are real numbers defined by iterated sums. Kontsevich discovered a representation of MZVs in terms of iterated integrals. MZVs are special values of multiple polylogarithms (MPLs). MZVs and MPLs stand at the intersection of several mathematical areas, including algebraic geometry, Lie group theory, algebra and combinatorics.

In perturbative quantum field theory (QFT) Feynman diagrams appear as coefficients of power expansions in coupling parameters. These diagrams encode Feynman amplitude, i.e., highly intricate integrals over a large number of variables. The correspondence between diagrams and integrals is established via the Feynman rules of a QFT, which associate to any Feynman diagram a Feynman amplitude. Their efficient calculation is of foremost importance for theoretical predictions. Thanks to the pathbreaking work of Bloch, Connes, Esnault, Goncharov, Kreimer, Marcolli and others, beautiful relations between Feynman amplitudes and periods has emerged.

1.1 Scientific Program

The school will consist of three weeks of lecture courses supplemented by exercise and problem sessions. It aims at introducing in pedagogical lectures foundational as well as advanced aspects of this fascinating subject.

The courses of the fourth week of this school aim at a higher level. In five mini-courses experts will present topics of high interest in current research directions in the field including some of the main open problems.

– Week I-III: Main Lecture Courses

Spencer BLOCH (University of Chicago, USA)
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José I. BURGOS GIL (ICMAT-CSIC, Madrid, Spain)
`http://www.icmat.es/miembros/burgos/`

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`http://www.mi.fu-berlin.de/users/esnault/`
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Matilde MARCOLLI (California Institute of Technology, USA)
`www.its.caltech.edu/~matilde/`

– Week IV: Mini-Courses

Each of the five mini-courses (each 3hrs) during the last week focus on topics in the field that are at forefront of current research in the field.

For more information go to: `www.icmat.es/summerschool2014/`

2 Main Lecture Courses

2.1 Lectures on the Mathematics of Feynman Amplitudes

Spencer Bloch (Chicago)

Outline: The purpose of this course is to develop a rigorous background in the Mathematics of Feynman amplitudes. We will be guided by physics, but the lectures will be unabashedly mathematical in character. The level will of course be dictated by the level of the attendees, but I would hope that people had encountered the basic elements of classical topology like singular homology and cohomology, as well as the rudiments of Hodge theory; in particular the decomposition of complex cohomology into pieces of type (p, q).

I. Introduction
   Passage from the path integral to Feynman integrals

II. Three settings for Feynman amplitudes
   A. Symmetric matrices (First Symanzik polynomial)
   B. Quaternionic Hermitian matrices (Second Symanzik polynomial)
   C. Skew matrices (Twister integrals)
   D. Universal determinants and transversality

III. Examples
   A. 1 loop, dilogarithms
   B. 2 loops, "??"
   C. Feynman amplitudes and arithmetic-examples of Broadhurst

IV. Thresholds
   A. Thresholds and discriminants
   B. Thresholds and cuts
   C. Thresholds at 1 loop
      - degenerations of the triangle graph
   D. Cutkosky rules

V. Motives associated to Feynman graphs
   Work of Bloch, Esnault, Kreimer; and Brown, Schnetz

References

2.2 Motivic multiple zeta values

José I. Burgos Gil (ICMAT-CSIC)

Outline: The famous Riemann zeta function

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s}, \quad \Re(s) > 1$$

is one of the most studied objects in mathematics. But it still hides many mysteries and enigmas such as the Riemann hypothesis. The values of this function at positive integers is another example of these mysteries. Euler already knew how to compute the values at even positive integers

$$\zeta(2k) = (-1)^{k+1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!},$$

where B_{2k} are rational numbers called the Bernoulli numbers. That is, values of the zeta function at even integers can be expressed in terms of powers of π and rational numbers.

However, nobody has been able to express the values of the zeta function at odd integers in terms of known numbers like π. In fact, it is expected, that this should not be possible. More precisely we *expect that the numbers* π, $\zeta(3)$, $\zeta(5)$, $\zeta(7)$, *etc., are algebraically independent over* $\mathbb{Q}$. Although it is fair so say that we are very far from being able to prove this. In fact, only little is known, such as Apéry's celebrated proof of the irrationality of $\zeta(3)$, and Ball's and Rivoal's result showing that in the above sequence there are infinitely many irrational numbers.

As it is often done in mathematics, in order to simplify a problem, we turn to a generalization of it. In this case we introduce multiple zeta values

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > n_2 > \dots > n_k > 0} \frac{1}{n_1^{s_1} \dots n_k^{s_k}}.$$

with s_k positive integers and $s_1 > 1$. Multiple zeta values (MZVs for short) have the advantage that the product of two MZVs is a linear combination of MZVs, as can be easily seen from the series expansion above. Therefore we have reduced the problem of algebraic dependence between zeta values to a problem of $\mathbb{Q}$-linear dependence between MZVs. The MZV $\zeta(s_1, \dots, s_k)$ has weight $\sum s_i$ and length, or depth, k. Euler was the first to study MZVs of length 2 and proved many relations among them, like

$$\zeta(2,1) = \zeta(3).$$

MZVs were mainly popularized by Zagier in the 1990s, who discovered that they satisfy vast numbers of relations. For example, there are a priori $2^{13} = 8192$ possible such numbers in weight 15, but in reality they form a vector space over $\mathbb{Q}$ of dimension at most 28.

MZVs have deep properties, and have appeared in recent years in connection with many of topics of surprising diversity, including knot invariants, Galois representations, periods of mixed Tate motives, and calculations of integrals associated to Feynman diagrams in perturbative quantum field theory (pQFT). The significance of MZVs in the context of pQFT results from progress made in computational high energy physics, which led to an enormous amount of data, thereby uncovering beautiful connections between particle physics and problems in number theory and algebraic geometry (see lectures by S. Bloch and M. Marcolli).

Denote by V_n the $\mathbb{Q}$-vector space generated by the MZVs of weight n, with the convention that 1 is a multiple zeta value of weight zero. We write V for the subspace of $\mathbb{R}$ generated by all MZVs. Zagier has conjectured that

$$V = \bigoplus_n V_n,$$

that is, there are no relations among MZVs of different weights. Moreover, the dimension of V_n is the number d_n defined by the generating function

$$\sum_n d_n X^n = \frac{1}{1 - X^2 - X^3}.$$

These numbers satisfy the recurrence relation $d_0 = 1$, $d_1 = 0$, $d_2 = 1$ and $d_n = d_{n-2} + d_{n-3}$. Hoffman [Hof97] further conjectured that a basis of the space V_n, $n \geq 2$, can be obtained by the multiple zeta values that have only 2 and 3 as entries. Again, we are far from being able to prove these conjectures. However, thanks to the works of Goncharov [DG05] [Gon05], Terasoma [Ter02] and Brown [Bro12] (see also [Bro10] and [Del12]), we know that parts of the conjectures hold. Namely, that $\dim V_n \leq d_n$ and that the basis predicted by Hoffman is a set of generators of the whole space of MZV's. The only known approach to prove these results is through the theory of motives.

The theory of motives aims at associating to each "piece" of an algebraic variety a "motive", that contains all the cohomological information of the variety. It originated from Grothendieck's work in the 1960s. Though, a completely satisfactory theory of motives has not yet been developed. Motives should live in a suitable abelian category, but we do not know whether it exists. Thanks to the work of Voevodski, and many others, we dispose of a good candidate for the derived category of the category of motives.

Among motives, there exisits a particularly simple class, called mixed Tate motives. Roughly speaking, mixed Tate motives are the those that can be obtained from affine spaces by relatively simple algebraic manipulations. The theory of mixed Tate motives over a number field is well established, and the structure of the category of mixed Tate motives is well understood.

To each motive one can associate certain numbers called periods. Goncharov and Terasoma independently identified MZVs as periods of mixed Tate motives over $\mathbb{Z}$. This suffice to prove that

$$\dim V_n \leq d_n.$$

Recently Brown [Bro12] proved that every period of a mixed Tate motive, defined over $\mathbb{Z}$, can be expressed as a combination of MZVs and powers of $2\pi i$. This allowed him to define the algebra of motivic MZVs, which maps surjectively to the algebra of "numerical" MZVs. The algebra of motivic MZVs satisfies the conjectures of Zagier and Hoffman and has a rich structure. In fact, it is a Hopf algebra.

The aim of this series of lectures is to explain the results of Brown, Goncharov and Terasoma, as well as some of the required prerequisites.

Plan of the course.

1. Multiple zeta values and their combinatorial properties.
2. Mixed Hodge structures.
3. Mixed Hodge structures of the fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
4. Polylogarithms and MZVs as periods.
5. Mixed Tate Motives
6. The Tannakian formalism.
7. The motivic fundamental groupoid of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.
8. Motivic multiple zeta values and proof of the main theorems.
9. Examples and applications.

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2.3 An introductory course on ℓ-adic Galois representations of function fields

Hélène Esnault, Lars Kindler, Kay Rülling (FU Berlin)

Note: *This is a combined lecture course + advanced course. Dr. Kindler and Dr. Rülling will give an introductory lecture course during the first three weeks of the school. It aims at preparing for Prof. Esnault's advanced course in last week of the school.*

Outline: In the introductory course (15 hours) on ℓ-adic Galois representations at the Clay Mathematics Institute Summer School 2014 at ICMAT Madrid, we plan to cover the following subjects:

Infinite Galois theory

We briefly recall the Galois theory of infinite algebraic field extensions. A basic reference is [Neu99, Ch. IV, §1].

Ramification Groups

If L/K is a finite Galois extension of local fields with Galois group G, we define the filtration of G by its ramification subgroups, first in the "lower numbering", then in the "upper numbering". Next, we recall the main properties of this filtration, e.g. the theorems of Herbrand, and of Hasse–Arf. The main reference is [Ser79, Ch. IV], but see also [Neu99]. This will allow us to define a ramification filtration (with upper numbering) on the absolute Galois group of K.

The Swan conductor

Given a local field with absolute Galois group G, and $\rho : G \to \mathrm{GL}_r(\mathbb{Z}_\ell)$ a continuous representation of the profinite group G (ℓ a prime different from the residue characteristic of K), we use the ramification filtration of G to define the Swan conductor of ρ. A nice reference is [Kat88, Ch. 1].

The Swan representation

After recalling some background from the representation theory of finite groups, we construct the Swan representation Sw_G attached to a finite Galois extension L/K of local fields with Galois group G; this is a projective $\mathbb{Z}_\ell[G]$-module closely related to the Swan conductor. The details of the construction are somewhat scattered in the literature. Nice overviews can be found in [Ser77, Ch. 19], [Lau81], [Lau87, §2.1], [Ser60]. We will also use material from [Ser79, Ch. VI].

The étale fundamental group

We will recall the definition and some of the main properties of the étale fundamental group of an algebraic variety. The relation to infinite Galois theory will be made clear. The standard reference is [GR71]. [Mur67], [Len08] and [Sza09] are also helpful and of a more introductory nature.

ℓ-adic sheaves

If X is a variety, we define a $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ as a projective system of constructible $\mathbb{Z}/\ell^n$-sheaves $\mathcal{F}_n$, satisfying certain conditions. If the $\mathcal{F}_n$ are locally constant, $\mathcal{F}$ is called lisse $\mathbb{Z}_\ell$-sheaf. We explain how lisse $\mathbb{Z}_\ell$-sheaves are related to representations of the étale fundamental group of X, and we also cover variants like $\mathbb{Q}_\ell$-sheaves and $\overline{\mathbb{Q}}_\ell$-sheaves.

The literature can seem daunting; the main reference is [GIJB77, Exp. V and VI], but see also [Del77, Rapport] and [Del80, §1] for overviews. Further expositions are [Sta13, Tag03N1] and in [FK88, §12]. For an absolutely state-of-the-art point of view see [BS13].

The formula of Grothendieck–Ogg–Shafarevich

Let $\overline{C}$ be a smooth proper curve over an algebraically closed field of characteristic $\neq \ell$ and $C \subset \overline{C}$ a dense open subset. If $\mathcal{F}$ is a lisse $\mathbb{Q}_\ell$-sheaf on C, then we define the Swan conductor $\mathrm{Sw}_c(\mathcal{F})$ for every closed point in $\overline{C} \setminus C$. The lisse $\mathbb{Q}_\ell$-sheaf $\mathcal{F}$ is called tame at c if $\mathrm{Sw}_c(\mathcal{F}) = 0$. The formula of Grothendieck–Ogg–Shafarevich computes the Euler characteristic $\chi_c(\mathcal{F})$ of $\mathcal{F}$ in terms of the numbers $\mathrm{Sw}_c(\mathcal{F})$, $\operatorname{rank} \mathcal{F}$ and $\chi_c(\mathbb{Q}_\ell)$. The main reference is [Kat88, Ch. 2], but see also [GIJB77, Exp. X] and [Ray66].

Higher dimensions *

Thanks to the Swan conductor and the formula of Grothendieck–Ogg–Shafarevich, the ramification theory of ℓ-adic sheaves on smooth curves is fairly well understood. In higher dimensions, one has to leave the world of local fields, and things become more complicated. One approach utilizes "higher local fields", and was developed by by Bloch, Kato, Saito and may others. This approach will not be discussed in our lectures; if you are interested, see [Kat91, §4] and [Sai10] for an overview and many more references.

Instead, if time permits, we will introduce an alternative approach, going back to ideas of G. Wiesend. If X is a variety of dimension > 1, one studies the ramification of a lisse sheaf $\mathcal{F}$ by "probing" it with smooth curves. More precisely, one studies the ramification of $\phi^*\mathcal{F}$ for *every* map $\phi : C \to X$ with C a regular curve, and tries to draw conclusions about $\mathcal{F}$ on X. This is a current area of research, and if there is time, we will introduce some related concepts. Advanced references are [EK11], [Dri12], [EK12], [KS13].

* If time permits.

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2.4 Feynman integrals, periods and motives

Matilde Marcolli (Caltech)

Outline: The lectures will cover some aspects of Feynman integrals (and some closely related mathematical generalizations) in momentum and configuration spaces.

Topics covered:

(1) Algebraic renormalization: the Connes–Kreimer Hopf algebra of Feynman graphs; algebraic Feynman rules; Rota–Baxter algebras and Birkhoff factorization

(2) Algebro-geometric Feynman rules arising from classes in the Grothendieck ring and Chern classes of singular varieties; graph hypersurfaces

(3) Feynman amplitudes in momentum and configuration spaces: algebraic and analytic versions; Fourier transform and Green functions; Bochner–Martinelli kernels for Feynman graphs.

(4) Gegenbauer polynomials and harmonic functions: the x-space method.

(5) Mathematical generalizations (complexifications) of Feynman amplitudes in configuration spaces: Configuration spaces of graphs and their wonderful compactifications and their motives; Rota–Baxter algebras of configuration spaces and renormalized period integrals

(6) Algebraic varieties arising from quantum field theory and geometry over the "field with one element"

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3 Mini-Courses

3.1 Joseph Ayoub (Universität Zürich, Switzerland)

Motives, motivic Galois groups and periods

Outline: The notion of "motive" was invented by A. Grothendieck to serve as a universal cohomology theory for algebraic varieties. Roughly speaking, given an algebraic variety X over a field k there should exist cohomological invariants $H^i_{\mathcal{M}}(X,\mathbb{Q})$, non zero only for $0 \le i \le 2\dim(X)$, called *motives*, that determines the classical cohomological invariants such as the mixed Hodge structures on the singular cohomology $H^i_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q})$ (for every choice of an embedding $k \hookrightarrow \mathbb{C}$) and the Galois representations on the ℓ-adic cohomology $H^i_{\text{ét}}(X \otimes_k \overline{k}, \mathbb{Q}_\ell)$ (for every choice of an algebraic closure $\overline{k}/k$). Moreover, these motives $H^i_{\mathcal{M}}(X,\mathbb{Q})$ should be algebraically defined and completely determined by the geometry of algebraic cycles (or, equivalently, by K-theory). Finally, the motives associated to X should govern the transcendence properties of the periods attached to X.

The above picture is of course largely conjectural. However, there have been some partial advances in the past years and the goal of the mini-course is to report on some of these advances. We hope to cover the following topics.

Motives à la Voevodsky

Thanks to the work of Voevodsky, Morel–Voevodsky and others, we now have a very satisfactory construction of the *triangulated* category of motives. Unfortunately, this is much weaker than having an *Abelian* category of motives as demanded by Grothendieck,[†] but at least the relation with algebraic cycles and K-theory is very strong and as expected. (We recommend the introduction of [8] for a summary of the relations between motives, algebraic cycles and K-theory.)

Let k be a base field. (Our main interest is when the field k has characteristic zero, but the constructions make sense over general base-schemes.) Voevodsky's triangulated category of motives $\mathbf{DM}(k,\mathbb{Q})$ is constructed in a four-steps "enlargement" of the category Sm/k of smooth k-varieties:

1. One considers the category of finite correspondences $\mathbf{SmCor}(k)$ from [8, Chapter 2]. The objects of this category are smooth k-varieties while the group of morphisms between X and Y is the group $\mathbf{Cor}(X,Y)$ of finite correspondences. (When k has characteristic zero, finite correspondences are exactly the elements of the group completion of the monoid of multivalued morphisms; see the introduction of the beautiful paper [17].)
2. One considers the category $\mathbf{Str}(\mathrm{Sm}/k,\mathbb{Q})$ of étale sheaves with transfers from [15, Lecture 6]. These are contravariant functors from $\mathbf{SmCor}(k)$ to the category of $\mathbb{Q}$-vector spaces which are étale sheaves when restricted to Sm/k. A typical object of the category is $\mathbb{Q}_{\mathrm{tr}}(X)$, for $X \in \mathrm{Sm}/k$, whose values on U is $\mathbf{Cor}(U,X) \otimes \mathbb{Q}$.

[†] For instance, one drawback of this is that we don't know how to define the individual $H^i_{\mathcal{M}}(X,\mathbb{Q})$'s but we have a "motivic complex" that "contains" all the $H^i_{\mathcal{M}}(X,\mathbb{Q})$'s built together in a way that we don't yet understand.

3. The category $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ is the Verdier localization of the derived category $\mathbf{Str}(\text{Sm}/k, \mathbb{Q})$ with respect to the smallest triangulated subcategory closed under infinite sums and containing the complexes of the forms

$$[\mathbb{Q}_{\text{tr}}(X) \xrightarrow{s_0} \mathbb{Q}_{\text{tr}}(\mathbb{A}^1 \times X)].$$

This construction appears essentially in [15, Lecture 9].‡

4. The category $\mathbf{DM}(k, \mathbb{Q})$ is obtained from $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ by inverting the Tate motive $T = \mathbb{Q}_{\text{tr}}(\mathbb{P}^1, \infty)[-2]$ with respect to the tensor product. The "naive" way of doing this is explained in [8, page 192].§ To do this correctedly, one appeals to the theory of symmetric spectra (see [11]) borrowed from stable homotopy theory, but we will probably pass over the technical details as they are irrelevant to our purposes. In any case, there is a natural functor

$$\Sigma_T^\infty : \mathbf{DM}^{\text{eff}}(k, \mathbb{Q}) \to \mathbf{DM}(k, \mathbb{Q})$$

which is fully faithful. (Full faithfulness is a deep theorem of Voevodsky whose proof can be found in [8] !) The image of $\mathbb{Q}_{\text{tr}}(X)$ by this functor is denoted by $\text{M}(X)$; this is the *motive* attached to X.

Remark. — It is quite remarkable that the above construction can be *simplified* without changing the outcome ! Indeed, one can forget about transfers and consider simply the category $\mathbf{Shv}(\text{Sm}/k, \mathbb{Q})$ of étale sheaves. Then, a Verdier localization of the derived category $\mathbf{D}(\mathbf{Shv}(\text{Sm}/k, \mathbb{Q}))$ as in (3) gives the category $\mathbf{DA}^{\text{eff,ét}}(k, \mathbb{Q})$ which is equivalent to $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ by a theorem of Morel and Cisinksi–Déglise. (See [3, Appendice B] for a simplified proof.) Also, inverting the Tate motive gives the category $\mathbf{DA}^{\text{ét}}(k, \mathbb{Q})$ which is equivalent to $\mathbf{DM}(k, \mathbb{Q})$. Although for the purpose of the mini-course there will be no good reasons for not using the simpler construction, we will stick to the categories $\mathbf{DM}^{\text{eff}}(k, \mathbb{Q})$ and $\mathbf{DM}(k, \mathbb{Q})$.¶

Motivic Galois groups

The construction of $\mathbf{DM}(k, \mathbb{Q})$ explained above can be repeated in other contexts. For instance, one can replace the category Sm/k by the category CpVar of complex analytic smooth varieties. We denote by $\mathbf{AnDM}(\mathbb{Q})$ the triangulated category obtained in this way.

‡In fact, it also appear in [8] but there the emphasis is on the Nisnevich topology. However, it is an easy exercise to check that with rational coefficients, the étale and Nisnevich topologies yield the same notion of sheaves with transfers.

§Note that the "naive" way yield the correct category when restricted to *geometric motives*.

¶In fact, depending on the situation, one can prefer the $\mathbf{DM}$-construction or the $\mathbf{DA}$-construction:

- Over a perfect field, the $\mathbf{DM}$-construction is very useful thanks to the Voevodsky's theory of "homotopy invariant presheaves with transfers" yielding an explicit model for the $\mathbb{A}^1$-localization functor. Such a model is not available for the $\mathbf{DA}$-construction.
- The categories $\mathbf{DA}^{\text{ét}}(S, \Lambda)$ over general basis and for general coefficients rings have been studied extensively in [1, 2, 6] and their general functoriality is completely understood. This is not the case for the categories $\mathbf{DM}(S, \Lambda)$ for which the *localization axiom* is not known to hold in enough generality.

It is an easy exercise to show that the category $\mathbf{AnDM}(\mathbb{Q})$ is canonically equivalent to the derived category of $\mathbb{Q}$-vector spaces $\mathbf{D}(\mathbb{Q})$. (See [3, §2.1] for a proof for the $\mathbf{DA}$-version.) On the other hand, given an embedding $k \hookrightarrow \mathbb{C}$, the functor $\mathrm{Sm}/k \to \mathrm{CpVar}$, $X \mapsto X(\mathbb{C})$, induces an "obvious" functor $\mathrm{An}^* : \mathbf{DM}(k,\mathbb{Q}) \to \mathbf{AnDM}(\mathbb{Q})$. The Betti realization can be defined as the composition of

$$\mathrm{B}^* : \mathbf{DM}(k,\mathbb{Q}) \xrightarrow{\mathrm{An}^*} \mathbf{AnDM}(\mathbb{Q}) \simeq \mathbf{D}(\mathbb{Q}).$$

The next ingredient for the construction of motivic Galois groups is the so-called *weak Tannakian formalism* developed in [3, §1]. Namely, given a monoidal functor $f : \mathcal{M} \to \mathcal{E}$ (e.g., $f = \mathrm{B}^*$) satisfying the following properties (see [3, Hypothèse 1.40])

(a) f admits a monoidal section e;

(b) f and e admit right adjoints g and c respectively;

(c) the natural morphism $g(A') \otimes B \to g(A' \otimes f(B))$ is an isomorphism for $A' \in \mathcal{E}$ and $B \in \mathcal{M}$,

the objet $\mathrm{H} = fg(\mathbf{1}) \in \mathcal{E}$ (where $\mathbf{1}$ is the unit for the tensor product) is naturally a Hopf algebra in $\mathcal{E}$. (This is [3, Théorème 1.45] whose proof is given in [3, §1].) Moreover, there is a universal factorization of f as

$$\mathcal{M} \xrightarrow{\tilde{f}} \mathbf{coMod}(\mathrm{H}) \xrightarrow{\mathrm{oub}} \mathcal{E}.$$

It is not difficult to show that the Betti realization B^* satisfies the conditions (a)–(c) above. We denote $\mathcal{H}_{\mathrm{mot}}(k) = \mathrm{B}^*\mathrm{B}_*\mathbb{Q}$ (where B_* is the right adjoint to B^*). This is the motivic Hopf algebra of k.

Note that, by construction, $\mathcal{H}_{\mathrm{mot}}(k)$ is an objet of $\mathbf{D}(\mathbb{Q})$. However, one has:

Conjecture. — *The homology of $\mathcal{H}_{\mathrm{mot}}(k)$ vanish except in degree zero.*

The right half of this conjecture is known by [3, Corollaire 2.105]. More precisely, one has $\mathrm{H}_i(\mathcal{H}_{\mathrm{mot}}(k)) = 0$ for $i < 0$. This is sufficient to insure that $\mathbf{H}_{\mathrm{mot}}(k) := \mathrm{H}_0(\mathcal{H}_{\mathrm{mot}}(k))$ inherits the Hopf algebra structure of $\mathcal{H}_{\mathrm{mot}}(k)$. It is then possible to make the following:

Definition. — *The motivic Galois group of k (associated to the embedding $k \hookrightarrow \mathbb{C}$) is* $\mathbf{G}_{\mathrm{mot}}(k) := \mathrm{Spec}(\mathbf{H}_{\mathrm{mot}}(k))$.

Remark. — There is a different construction of the motivic Galois group of a field due to M. Nori [16] (see [14] for a published accout of the construction). Nori's approach is based on an ingenious construction of an Abelian category of mixed motives. Unfortunately, this Abelian category of motives has some serious disadvantages:

- its construction is not purely algebraic and one needs to fix a Weil cohomology theory (e.g., singular cohomology) as an input;
- there is no interpretation of ext-groups between Nori's motives in terms of algebraic cycles or K-theory.

Nevertheless, it can shown [9] that the motivic Galois group constructed above is isomorphic to Nori's motivic Galois group. However, we feel it is important and more satisfactory to have a construction of the motivic Galois group built-in in the framework of Voevodsky's motives.

Periods

In $\mathbf{DM}^{\mathrm{eff}}(k;\mathbb{Q})$ there is a natural well-known objet, namely the complex of sheaves $\Omega^{\bullet}_{/k}$ whose values on a smooth k-variety X is the global sections of the de Rham complex $\Omega^{\bullet}_{X/k}(X)$. (In fact, it is not obvious that $\Omega^{\bullet}_{/k}$ is a complex of sheaves *with transfers* but this is indeed true by [13]; alternatively, one can view $\Omega^{\bullet}_{/k}$ as an objet of $\mathbf{DA}^{\mathrm{eff},\text{ét}}(k,\mathbb{Q})$ which we know to be the same as $\mathbf{DM}^{\mathrm{eff}}(k;\mathbb{Q})$.) We also note that $\Omega^{\bullet}_{/k}$ can be enhanced to an objet of $\mathbf{DM}(k;\mathbb{Q})$ that we denote by $\boldsymbol{\Omega}_{/k}$. In any case, it is an easy exercise to show that

$$\hom_{\mathbf{DM}(k,\mathbb{Q})}(\mathrm{M}(X),\boldsymbol{\Omega}_{/k}[i]) \simeq H^i_{\mathrm{dR}}(X).$$

A similar formula holds for $\mathrm{B}_*\mathbb{Q}$:

$$\hom_{\mathbf{DM}(k,\mathbb{Q})}(\mathrm{M}(X),\mathrm{B}_*\mathbb{Q}[i]) \simeq H^i_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{Q}).$$

(Again, an easy exercise!) The Grothendieck comparison theorem between algebraic de Rham cohomology and singular cohomology [10] can now be restated as an isomorphism in $\mathbf{DM}(k,\mathbb{Q})$:

$$\boldsymbol{\Omega}_{/k}\otimes_k\mathbb{C} \simeq (\mathrm{B}_*\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}.$$

Going back to the definition of the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k)$, one deduces the following facts:

(i) $\mathcal{P}(k) := \mathrm{B}^*\boldsymbol{\Omega}_{/k}$ is a comodule over the motivic Hopf algebra $\mathcal{H}_{\mathrm{mot}}(k)$;

(ii) $\mathcal{P}(k)\otimes_k\mathbb{C}$ is isomorphic (as a comodule) to $\mathcal{H}_{\mathrm{mot}}(k)\otimes_{\mathbb{Q}}\mathbb{C}$. Said differently, $\mathcal{P}(k)$ is a torsor over $\mathcal{H}_{\mathrm{mot}}(k)$.

Also, note that the vanishing of $\mathrm{H}_i(\mathcal{H}_{\mathrm{mot}}(k))$, for $i<0$, implies the vanishing of $\mathrm{H}_i(\mathcal{P}(k))$ for $i<0$. (In fact, this is proved in the other way round !) As before, we set $\mathbf{P}(k) := \mathrm{H}_0(\mathcal{P}(k))$ and we let $\mathbf{T}(k) = \mathrm{Spec}(\mathbf{P}(k))$.

Definition. — $\mathbf{T}(k)$ *is a torsor over the motivicGalois group* $\mathbf{G}_{\mathrm{mot}}(k)$*; it is called the* torsor of periods.

The torsor $\mathbf{T}(k)$ is defined over k (i.e., is a k-scheme). The Grothendieck comparison theorem yields a $\mathbb{C}$-point $\mathrm{comp}\in\mathbf{T}(k)(\mathbb{C})$. The following is a famous conjecture of Grothendieck and Kontsevich–Zagier:

Conjecture: *If k is a number field, the evaluation map*

$$\mathrm{comp}^*:\mathbf{P}(k)=\mathcal{O}(\mathbf{T}(k))\to\mathbb{C}$$

is injective.

If time permits, I will also give a very concrete formulation of the previous conjecture using algebraic functions on polydiscs and their partial derivatives (see the introduction of [7]).

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3.2 Francis Brown (CNRS, IHÉS, Bures-sur-Yvette, France)

Multiple modular values

Outline: It has been observed since the 1990's that there is an intricate relationship between amplitudes in perturbative quantum field theory, and arithmetically interesting periods in number theory. This is especially striking for the case of multiple zeta values, which are periods of curves of genus zero, which are ubiquitous in high energy physics. Nonetheless, multiple zeta values are insufficient for many purposes, and there is an increasingly large family of examples of amplitudes in quantum field theory which cannot be expressed as multiple zeta values. Often the obstruction comes from a modular form. This motivates searching for larger classes of numbers (and functions) with which to express the basic quantities of particle physics.

This course will be a leisurely introduction to a new class of numbers, that I wish to call multiple modular values since they simultaneously generalise multiple zeta values and periods of modular forms. These quantities are iterated integrals which build on the iterated modular symbols defined by Manin, and closely related to the periods of the relative Malcev completion of fundamental groups studied by Hain.

I will review basic aspects of the theory of modular forms, their periods and L-functions, group cohomology and iterated integrals before defining multiple modular values and some of their basic properties. If time permits, I will describe the motivic coaction on the corresponding Hodge-motivic objects.

3.3 James Drummond (CNRS)

Bootstraps for scattering amplitudes in N=4 super Yang–Mills theory

Outline: Over recent years great progress has been made in understanding the scattering amplitudes of gauge theories. Pedagogical introductions to the subject can be found in e.g. [1, 2]. The maximally supersymmetric N=4 theory has been a focus of many of the recent developments; in the planar limit this theory exhibits a relation between the scattering amplitudes and light-like Wilson loops [3, 4, 5, 6, 7] and consequently an additional symmetry called dual conformal symmetry.

Dual conformal symmetry fixes the form of the amplitudes for four or five external particles [8, 9]. The simplest non-trivial amplitudes therefore arise at six points. Perturbative calculations of these amplitudes reveal that they are always described in terms of a certain class of multiple polylogarithms [10, 11]. These functions are studied systematically in [12].

Based on this observation, a kind of S-matrix bootstrap technique can be developed to calculate the perturbation expansion avoiding the calculation of any loop integrals [13, 14, 15, 16, 17]. This technique has allowed the calculation of explicit results for the leading orders of perturbation theory and has an interesting interplay with non-perturbative techniques based on the operator product expansion for light-like Wilson loops [18, 19, 20, 21] as well as with the BFKL expansion of scattering amplitudes in multi-Regge kinematics [22, 23, 24, 25].

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3.4 Claude Duhr (Durham University, UK)

Feynman integrals, scattering amplitudes and the Hopf algebra of multiple polylogarithms

Outline: The computation of scattering amplitudes in quantum field theory to higher loop orders generically requires the computation of complicated multi-loop Feynman integrals depending on many external variables describing the kinematics of the scattering process. It is known that large classes of phenomenologically interesting Feynman integrals evaluate to polylogarithmic functions, which, due to the dependence on the external variables, generically fall into the class of multiple polylogarithms [1, 2, 3, 4]. It is then clear that a better understanding of the computation of scattering amplitudes from the physics side goes hand-in-hand with the understanding of the mathematical properties of multiple polylogarithms.

In particular, multiple polylogarithms form a Hopf algebra [5, 6], and this Hopf algebra structure can be turned into a practical tool to organize and simplify computations of Feynman integrals in the cases where they evaluate to this restricted class of functions. However, the Hopf algebra has the unwanted feature that it is defined modulo $\zeta(2)$, which implies that a lot of physically relevant information about the scattering amplitude is lost. Based on ideas recently introduced by Francis Brown in the context of motivic multiple zeta values [7], we conjecture that it is possible to remedy this problem by extending the Hopf algebra $\mathcal{H}$ of multiple polylogarithms to a trivial comodule $\mathbb{Q}[i\pi]\otimes\mathcal{H}$ [8] where all the relevant information is kept. In a first part of the lecture we define this comodule and we illustrate how our conjecture allows us to derive complicated functional identities among multiple polylogarithms (with rational functions as arguments) that are exact in the sense that both sides of the identity yield the same numerical answer.

Armed with this machinery, we discuss in a second part of the lecture how Feynman integrals fit into the picture. Indeed, Feynman integrals are characterized by a very specific branch cut structure, and we show how combining this branch cut structure with the aforementioned comodule puts severe constraints on the polylogarithmic functions that can appear in the final answer [9, 10, 11].

In a third part of the lecture, we review several applications of the Hopf algebraic ideas in the context of scattering amplitudes and Feynman integrals. The first application is the simplification of analytic expressions for multi-loop Feynman integrals. Indeed, it is well-known that explicit results for loop integrals generically lead to very long an complicated expressions. Concentrating on the example of two-loop integrals and building upon the ideas of ref. [12], we show how Hopf algebraic tools can be used to find, if they exist, simpler analytic expressions for Feynman integrals. In particular, we review a (conjecturally) necessary and sufficient condition for a two-loop integral to be expressible in terms of classical polylogarithms only [12], and we present an approach to find the set of classical polylogarithms through which the answer can be expressed [13]. As a second application, we discuss the analytic continuation of Feynman integrals to the physical scattering region. Feynman integrals are usually computed in Euclidean kinematics where the result is real. The results must then be carefully analytically continued to the physical region, where the scattering amplitude develops imaginary parts. We show how the analytic continuation can be performed recursively in the weight, providing a prescription for how to perform the analytic continuation of large classes of Feynman integrals.

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3.5 Hélène Esnault (FU Berlin, Germany)

ℓ-adic Galois representations of function fields over finite fields

Outline: As an offspring of his work on the Weil conjectures, Deligne conjectured that if X is a normal connected scheme of finite type over a finite field of characteristic p, and V is an irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaf of rank r, with finite determinant, then:

1. V has weight 0,
2. there is a number field $E(V) \subset \overline{\mathbb{Q}}_\ell$ containing all the coefficients of the characteristic polynomials $\det(1 - tF_x|V_x)$, where x runs through the closed points of X and F_x is the geometric Frobenius at the point x,
3. V admits ℓ'-companions for all prime numbers $\ell' \neq p$.

As an application of his Langlands correspondence for GL_r, Lafforgue proved (a), (b) and (c) for X a smooth curve, out of which one deduces (a) in general. Using Lafforgue's results, Deligne showed (b). Using (b) and ideas of Wiesend, Drinfeld showed (c) assuming in addition X to be smooth.

Those conjectures were formulated with the hope that a more motivic statement could be true, which would say that those lisse sheaves come from geometry. Thus it is natural to expect:

Theorem 1 (Deligne)**.** *There are only finitely many irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves up to twist on X with suitably bounded ramification at infinity.*

Deligne proved this result in a letter to Drinfeld. His proof uses Lafforgue's Langlands correspondence, the theory of ramification and a beautiful study of the coarse moduli space of generalized sheaves with bounded ramification.

The aim of this course is to explain Deligne's proof of Theorem 1 following a recent paper by Esnault and Kerz‖ and discuss some of its applications.

To introduce the needed language and to put the result in context, we will start by explaining the Weil conjectures and the ingredients of Deligne's proof of them. In particular we will review étale cohomology, ℓ-adic sheaves and motivic ℓ-adic sheaves, Frobenius action, weight theory and the Weil group. With this language we will state precisely the main result and derive some consequences.

Then we will review the main tools used in the proof of this result, like Lafforgue's result on the Langlands correspondence on curves, ramification theory and the theory of moduli spaces. Finally we will give an account of the proof of Theorem 1.

‖http://arxiv.org/abs/1208.0128

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