Lectures on the geometric Langlands conjecture and non-abelian Hodge theory

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Chapter 1

Introduction

In these lectures which are a slightly expanded version of the survey [DP09a] we will focus on the aspects of the geometric Langlands Conjecture and that relate it to non abelian Hodge theory. This relation was unravelled in works by many mathematicians and physicists, but we will emphasize the point of view that evolved in a series of works by the authors, starting from the outline in [Don89], through the recent proof of the classical limit conjecture in [DP06], and leading to the works in progress [DP09b], [DPS09b], and [DPS09a].

The Langlands program is the *non-abelian* extension of class field theory. The abelian case is well understood. Its geometric version, or geometric class field theory, is essentially the theory of a curve C and its Jacobian $J = J(C)$. This abelian case of the Geometric Langlands Conjecture amounts to the well known result that any rank one local system (or: line bundle with flat connection) on the curve C extends uniquely to J, and this extension is natural with respect to the Abel-Jacobi map. The structure group of a rank one local system is of course just the abelian group $\mathbb{C}^\times = GL_1(\mathbb{C})$. The geometric Langlands conjecture is the attempt to extend this classical result from $\mathbb{C}^\times$ to all complex reductive groups G. This goes as follows.

The Jacobian is replaced by the moduli $\boldsymbol{\mathcal{B}un}$ of principal bundles V on C whose structure group is the Langlands dual group ${}^LG$ of the original G. The analogues of the Abel-Jacobi maps are the *Hecke correspondences* $\boldsymbol{\mathcal{H}}\mathbf{ecke} \subset \boldsymbol{\mathcal{B}un} \times \boldsymbol{\mathcal{B}un} \times C$. These parametrize quadruples (V, V', x, β) where x is a point of C, while V, V' are bundles on C, with an isomorphism $\beta : V_{|C-x} \to V'_{|C-x}$ away from the point x having prescribed order of blowing up at x. (In case $G = \mathbb{C}^\times$ these become triples (L, L', x) where the line bundle L' is obtained from L by tensoring with some fixed power of the line bundle $\mathcal{O}_C(x)$. By fixing L and varying x we see that this is indeed essentially the Abel-Jacobi map.) For $GL(n)$ and more complicated groups, there are many ways to specify the allowed order of growth of β, so there is a collection of Hecke correspondences, each inducing a Hecke operator on various categories of objects on $\boldsymbol{\mathcal{B}un}$. The resulting Hecke operators form a commutative algebra. The Geometric Langlands Conjecture says that an irreducible G-local system on C determines a $\mathcal{D}$-module (or a perverse sheaf) on $\boldsymbol{\mathcal{B}un}$ which is a simultaneous eigensheaf for the action of the Hecke operators - this turns out to be the right generalization of naturality with respect to the Abel-Jacobi map. (A perverse sheaf is, roughly, a local system on a Zariski open subset of $\boldsymbol{\mathcal{B}un}$, extended in a natural way across the complement.) Fancier versions of the conjecture recast this as an equivalence of derived categories: of $\mathcal{D}$-modules on $\boldsymbol{\mathcal{B}un}$ versus coherent sheaves on the moduli $\boldsymbol{\mathcal{L}oc}$ of local systems. Our discussion of the geometric Langlands conjecture occupies section 2 of these lecture notes. There are many related conjectures and extensions, notably to *punctured* curves via parabolic bundles and local systems. Some of these make an appearance in section 6.

Great progress has been made towards understanding these conjectures [Dri80, Dri83, Dri87], [Lau87], [BD03], [Laf02], [FGKV98], [FGV01], [Gai01], [Lau03], including proofs of some versions of the conjecture for GL_2 [Dri83] and later, using Lafforgue's spectacular work [Laf02], also for GL_n [FGV01, Gai01]. The conjecture is unknown for other groups, nor in the parabolic case. Even for $GL(n)$, the proof is indirect: no construction of non-abelian Hecke eigensheaves is known, except perhaps for the work of Bezrukavnikov-Braverman [BB07] over finite fields, which is very much in the spirit of the approach discussed in these lectures.

The work surveyed here is based on an abelianization of the geometric Langlands conjecture in terms of Higgs bundles. A Higgs bundle is a pair (E, θ) consisting of a vector bundle E on C with a ω_C-valued endomorphism $\theta : E \to E \otimes \omega_C$, where ω_C is the canonical bundle of C. More generally, a G-Higgs bundle is a pair (E, θ) consisting of a principal G-bundle E with a section θ of $ad(E) \otimes \omega_C$, where $ad(E)$ is the adjoint vector bundle of E. Hitchin [Hit87b] studied the moduli $\boldsymbol{\mathcal{H}iggs}$ of such Higgs bundles (subject to an appropriate stability condition) and showed that it is an algebraically integrable system: it is algebraically symplectic, and it admits a natural map $h : \boldsymbol{\mathcal{H}iggs} \to B$ to a vector space B such that the fibers are Lagrangian subvarieties. In fact the fiber over a general point $b \in B$ (in the complement of the *discriminant hypersurface*) is an abelian variety, obtained as Jacobian or Prym of an appropriate spectral cover $\overline{C}_b$. The description in terms of spectral covers is somewhat ad hoc, in that it depends on the choice of a representation of the group G. A uniform description is given in terms of generalized Pryms of *cameral covers*, cf. [Don93, Fal93, Don95, DG02]. The results we need about Higgs bundles and the Hitchin system are reviewed in section 3.1.

In old work [Don89], we defined abelianized Hecke correspondences on $\boldsymbol{\mathcal{H}iggs}$ and used the Hitchin system to construct eigensheaves for them. That construction is described in section 3.2. After some encouragement from Witten and concurrent with the appearance of [KW06], complete statements and proofs of these results finally appeared in [DP06]. This paper also built on results obtained previously, in the somewhat different context of large N duality, geometric transitions and integrable systems, in [DDP07a, DDP07b, DDD$^+$06]. The case of the groups GL_n, SL_n and $\mathbb{P}GL_n$ had appeared earlier in [HT03], in the context of hyperkahler mirror symmetry. The main result of [DP06] is formulated as a duality of the Hitchin system: There is a canonical isomorphism between the bases B, ${}^LB$ of the Hitchin system for the group G and its Langlands dual ${}^LG$, taking the discriminant in one to the discriminant in the other. Away from the discriminants, the corresponding fibers are abelian varieties, and we exhibit a canonical duality between them. The old results about abelianized Hecke correspondences and their eigenseaves then follow immediately. These results are explained in section 4 of the present lectures.

It is very tempting to try to understand the relationship of this abelianized result to the full geometric Langlands conjecture. The view of the geometric Langlands correspondece pursued in [BD03] is that it is "quantum" theory. The emphasis in [BD03] is therefore on quantizing Hitchin's system, which leads to the investigation of *opers*. One possibility, discussed in [DP06] and [Ari02, Ari08], is to view the full geometric Langlands conjecture as a quantum statement whose "classical limit" is the result in [DP06]. The idea then would be to try to prove the geometric Langlands conjecture by deforming both sides of the result of [DP06] to higher and higher orders. Arinkin has carried out some deep work in this direction [Ari02, Ari05, Ari08]. But there is another path.

In these lectures we explore the tantalizing possibility that the abelianized version of the geometric Langlands conjecture is in fact equivalent, via recent breakthroughs in non-abelian Hodge theory, to the full original (non-abelian) geometric Langlands conjecture, not only to its 0-th order or "classical" approximation. Instead of viewing the solution constructed in [DP06] as a classical limit of the full solution, it is interpreted as the $z = 0$ incarnation of a twistor-type object that also has a $z = 1$ interpretation which is identified with the full solution.

Non abelian Hodge theory, as developed by Donaldson, Hitchin, Corlette, Simpson [Don87, Hit87a, Cor88, Sim92, Cor93, Sim97], and many others, establishes under appropriate assumptions the equivalence of local systems and Higgs bundles. A richer object (harmonic bundle or twistor structure) is introduced, which specializes to both local systems and Higgs bundles. This is closely related to Deligne's notion of a z-connection: at $z = 1$ we have ordinary connections (or local systems), while at $z = 0$ we have Higgs bundles. Depending on the exact context, these specialization maps are shown to be diffeomorphisms or categorical equivalences. The projective (or compact Kähler) case and the one dimensional open case were settled by Simpson twenty years ago - but the open case in higher dimension had to await the recent breakthroughs by Mochizuki [Moc06, Moc09, Moc07a, Moc07b], Sabbah [Sab05], Jost-Yang-Zuo [JYZ07], Biquard [Biq97], and others. This higher dimensional theory produces an equivalence of parabolic local systems and parabolic Higgs bundles. This is quite analogous to what is obtained in the compact case, except that the objects involved are required to satisfy three key conditions discovered by Mochizuki. In section 5.1 we review these exciting developments, and outline our proposal for using non-abelian Hodge theory to construct the automorphic sheaves required by the geometric Langlands conjecture. This approach is purely mathematical of course, but it is parallel to physical ideas that have emerged from the recent collaborations of Witten with Kapustin, Gukov and Frenkel [KW06, GW06, FW08], where the geometric Langlands conjecture was placed firmly in the context of quantum field theory.

Completion of these ideas depends on verification that Mochizuki's conditions are satisfied in situations arising from the geometric Langlands conjecture. This requires a detailed analysis of instability loci in moduli spaces. Particularly important are the *wobbly* locus of non-very-stable bundles, and the *shaky* locus, roughly the Hitchin image of stable Higgs bundles with an unstable underlying bundle. In section 6.1 we announce some results about these loci for rank 2 bundles. These lead in some cases to an explicit construction (modulo solving the differential equations inherent in the non-abelian Hodge theory) of the Hecke eigensheaf demanded by the geometric Langlands correspondence.

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Chapter 2

Review of the geometric Langlands conjecture

2.1 The geometric Langlands conjecture in general

In a nutshell the Geometric Langlands Conjecture predicts (see e.g. [BD03]) the existence of a canonical equivalence of categories

$$\boxed{\mathfrak{c} : D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O}) \xrightarrow{\cong} D_{\mathrm{coh}}({}^{L}\mathbf{Bun}, \mathcal{D}),} \tag{GLC}$$

which is uniquely characterized by the property that $\mathfrak{c}$ sends the structure sheaves of points $\mathbb{V}$ in $\boldsymbol{\mathcal{L}oc}$ to Hecke eigen $\mathcal{D}$-modules on ${}^{L}\mathbf{Bun}$:

$${}^{L}H^{\mu}\left(\mathfrak{c}(\mathcal{O}_{\mathbb{V}})\right) = \mathfrak{c}(\mathcal{O}_{\mathbb{V}}) \boxtimes \rho^{\mu}(\mathbb{V}).$$

To introduce properly the objects appearing in the geometric Langlands correspondence we will need to introduce certain background geometric data first. We start with fixing

- a smooth compact Riemann surface C;
- a pair of Langlands dual complex reductive groups G, ${}^{L}G$.

Recall that if we write $\mathfrak{g}$ and ${}^{L}\mathfrak{g}$ for the Lie algebras of G and ${}^{L}G$ and we fix maximal tori $T \subset G$ and ${}^{L}T \subset {}^{L}G$ with Cartan subalgebras by $\mathfrak{t} \subset \mathfrak{g}$ and ${}^{L}\mathfrak{t} \subset {}^{L}\mathfrak{g}$, then group theoretic Langlands duality can be summarized in the relation between character lattices

$$\begin{array}{ccccccc}
\mathsf{root}_{\mathfrak{g}} & \subset & \mathsf{char}_{G} & \subset & \mathsf{weight}_{\mathfrak{g}} & \subset & \mathfrak{t}^{\vee} \\
 & & & & & & \\
\mathsf{coroot}_{\mathfrak{g}} & \subset & \mathsf{cochar}_{G} & \subset & \mathsf{coweight}_{\mathfrak{g}} & \subset & \mathfrak{t} \\
\| & & \| & & \| & & \| \\
\mathsf{root}_{[{}^{L}\mathfrak{g}]} & \subset & \mathsf{char}_{[{}^{L}G]} & \subset & \mathsf{weight}_{[{}^{L}\mathfrak{g}]} & \subset & {}^{L}\mathfrak{t}^{\vee}
\end{array}$$

Here $\mathsf{root}_{\mathfrak{g}} \subset \mathsf{weight}_{\mathfrak{g}} \subset \mathfrak{t}^{\vee}$ are the root and weight lattice corresponding to the root system on $\mathfrak{g}$ and $\mathsf{char}_G = \mathrm{Hom}(T, \mathbb{C}^{\times})$ denotes the character lattice of G.

With this data we can associate various moduli stacks:

$\boldsymbol{\mathcal{B}un}$, $^L\boldsymbol{\mathcal{B}un}$: the moduli stacks of principal G, ^LG bundles V on C,

$\boldsymbol{\mathcal{L}oc}$, $^L\boldsymbol{\mathcal{L}oc}$: the moduli stacks of G, ^LG local systems $\mathbb{V} = (V, \nabla)$ on C.

For reductive structure groups we also need the rigidified versions $\mathbf{Bun}$, $^L\mathbf{Bun}$, $\mathbf{Loc}$, $^L\mathbf{Loc}$, in which the connected component of the generic stabilizer is "removed". For semi-simple groups this step is unnecessary (see 2.1).

Rigidification is a general construction for algebraic stacks which is described in detail in [AOV08, Appendix A]. For convenience we recall the main properties of the rigification process next.

Suppose $\mathscr{X}$ is an algebraic stack over $\mathbb{C}$. Let $I_{\mathscr{X}} = \mathscr{X} \times_{\mathscr{X}\times\mathscr{X}} \mathscr{X}$ be the stabilizer stack of $\mathscr{X}$, and let $\mathsf{H} \subset I_{\mathscr{X}}$ be a flat group subscheme. Explicitly this means that for every scheme S and every section $\xi \in \mathscr{X}(S)$ we are given a group scheme $\mathsf{H}_{\xi} \subset \underline{\mathrm{Aut}}_S(\xi)$ so that the various H_{ξ}'s are compatible under pullbacks. Note that this in particular implies that for every ξ we have $\mathsf{H}_{\xi} \triangleleft \underline{\mathrm{Aut}}_S(\xi)$. So heuristically H is a normal subgroup of the generic stabilizers for $\mathscr{X}$. In [AOV08, Appendix A] Abramovich-Olsson-Vistoli show that there exists a unque up to equivalence algebraic stack $\mathscr{X} /\!\!/ \mathsf{H}$ equipped with a morphism $\varrho : \mathscr{X} \to \mathscr{X} /\!\!/ \mathsf{H}$ so that

- $\mathscr{X}$ is a flat gerbe over $\mathscr{X} /\!\!/ \mathsf{H}$;
- for each (S, ξ) the natural morphism $\underline{\mathrm{Aut}}_S(\xi) \to \underline{\mathrm{Aut}}_S(\varrho(\xi))$ is surjective with kernel H.

Going back to the stacks of bundles it is useful to introduce the notions of a regularly stable bundle and a regularly simple local system. By definition these are objects whose automorphism group coincides with the generic group of automorphisms, namely the center $Z(G)$, $Z(^LG)$ of the structure group G, ^LG. A subgroup of $Z(G)$ and $Z(^LG)$ will give a normal flat subgroup in the inertia of the moduli stacks and as explained above we can pass to a quotient by such subgroups to obtain rigidified stacks. In particular we have

$$\mathbf{Bun} = \boldsymbol{\mathcal{B}un} /\!\!/ Z_0(G), \qquad \mathbf{Loc} = \boldsymbol{\mathcal{L}oc} /\!\!/ Z_0(G),$$

$$^L\mathbf{Bun} = {}^L\boldsymbol{\mathcal{B}un} /\!\!/ Z_0(^LG), \qquad ^L\mathbf{Loc} = {}^L\boldsymbol{\mathcal{L}oc} /\!\!/ Z_0(^LG),$$

where $Z_0(G)$, $Z_0(^LG)$ denote the connected components of $Z(G)$, $Z(^LG)$.

b

Remark 2.1. *It is instructive to note that the rigidified stacks often specialize to familiar geometric objects. For instance if G, ^LG are semi-simple groups, then we are rigidifying by the trivial subgroup and so* $\mathbf{Bun} = \boldsymbol{\mathcal{B}un}$, $\boldsymbol{\mathcal{L}oc} = \mathbf{Loc}$, *etc.. Note also that if the center of G is connected, then both* $\mathbf{Bun}$ *and* $\mathbf{Loc}$ *are generically varieties: in this case the open substacks* $\mathbf{Bun}^{rs} \subset \mathbf{Bun}$, $\mathbf{Loc}^{rs} \subset \mathbf{Loc}$ *parameterizing regularly stable bundles or regularly simple local systems coincide with the GIT-moduli spaces of regularly stable bundles and regularly simple local systems respectively.*

The geometric Langlands conjecture relates categories of sheaves on the stack $\boldsymbol{\mathcal{L}oc}$ of G-local systems and the rigidified stack ${}^L\mathbf{Bun}$ of ${}^LG$-bundles. The appearance of the rigidfied moduli ${}^L\mathbf{Bun}$ in the geometric Langlands conjecture is necessary (see Remark 2.4) to ensure the matching of components of the two categories involved in (GLC).

The relevant categories of sheaves are both of the same type. For a sheaf of algebras $\mathcal{A}$ on an algebraic stack $\mathscr{X}$ we will write $D_{\mathrm{coh}}(\mathscr{X}, \mathcal{A})$ for the derived category of complexes of $\mathcal{A}$-modules whose cohomology sheaves are coherent $\mathcal{A}$-modules. The sheaves of algebras $\mathcal{A}$ that we will be primarily interested in will be $\mathcal{A} = \mathcal{O}_{\mathscr{X}}$ - the structure sheaf of $\mathscr{X}$, or $\mathcal{A} = \mathcal{D}_{\mathscr{X}}$ - the sheaf of algebraic differential operators on $\mathscr{X}$, or $\mathcal{A} = \mathrm{Sym}^\bullet T_{\mathscr{X}}$ - the symmetric algebra on the tangent sheaf of $\mathscr{X}$. For instance in the left hand side of the (GLC) we have $\mathcal{A} = \mathcal{O}$ and on the right hand side we have $\mathcal{A} = \mathcal{D}$.

To formulate the characteristic property of $\mathfrak{c}$ we also need the Hecke correspondences ${}^L\mathbf{Hecke}^\mu \subset {}^L\mathbf{Hecke}$ defined for all dominant cocharacter $\mu \in \mathsf{cochar}^+_{[{}^LG]}$ as follows:

^{L}Hecke: the moduli stack of quadruples (V, V', x, β), where

- V, V' are principal ${}^LG$-bundles on C,
- $x \in C$,
- $\beta : V_{|C-\{x\}} \widetilde{\to} V'_{|C-\{x\}}$.

^{L}Hecke$^\mu$: the closed substack of ${}^L\mathbf{Hecke}$ of quadruples (V, V', x, β) such that if $\lambda \in \mathsf{char}^+_{[{}^LG]}$ is a dominant cocharacter and if ρ^λ is the irreducible representation of ${}^LG$ with highest weight λ, then β induces an inclusion of locally free sheaves $\rho^\lambda(\beta) : \rho^\lambda(V) \hookrightarrow \rho^\lambda(V') \otimes \mathcal{O}_C(\langle \mu, \lambda \rangle x)$.

These stacks are equipped with natural projections

$$\begin{tikzcd} & {}^L\mathbf{Hecke} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ {}^L\mathbf{Bun} & & {}^L\mathbf{Bun} \times C \end{tikzcd} \qquad \begin{tikzcd} & {}^L\mathbf{Hecke}^\mu \arrow[dl, "p^\mu"'] \arrow[dr, "q^\mu"] & \\ {}^L\mathbf{Bun} & & {}^L\mathbf{Bun} \times C \end{tikzcd}$$

where $p(V, V', x, \beta) := V$, $q(V, V', x, \beta) := V'$, and p^μ and q^μ are the restrictions of p and q to ${}^L\mathbf{Hecke}^\mu$. Moreover

- p^μ, q^μ are proper representable morphisms which are locally trivial fibrations in the etale topology;
- ${}^L\mathbf{Hecke}^\mu$ is smooth if and only if μ is a minuscule weight of G (see e.g. [EM99]);
- ${}^L\mathbf{Hecke}$ is an ind-stack and is the inductive limit of all ${}^L\mathbf{Hecke}^\mu$'s;
- p and q are formally smooth[1] locally trivial fibrations whose fibers are ind-schemes, the fibers of q are all isomorphic to the affine grassmanian for ${}^LG$.

[1]An unpleasant feature of this general setup is that even though p and q are formally smooth, they are not smooth. This follows from the fact that the affine Grassmanian can not be approximmated even locally by smooth finite dimensional varieties (see Appendix A). This requires extra care in handling homological questions on the Hecke cycles.

The Hecke functor ${}^L H^\mu$ is defined as the integral transform

$$\begin{aligned} {}^L H^\mu : \; D_{\mathrm{coh}}({}^L\boldsymbol{\mathcal{B}un}, \mathcal{D}) &\longrightarrow D_{\mathrm{coh}}({}^L\boldsymbol{\mathcal{B}un}, \mathcal{D}) \\ \mathscr{M} &\longmapsto q^\mu_! \left((p^\mu)^* \mathscr{M} \otimes {}^L I^\mu \right) \end{aligned}$$

where ${}^L I^\mu$ is the Goresky-MacPherson middle perversity extension $j_{!*}\left(\mathbb{C}\left[\dim {}^L\mathbf{Hecke}^\mu\right]\right)$ of the trivial rank one local system on the smooth part $j : \left({}^L\mathbf{Hecke}^\mu\right)^{\mathrm{smooth}} \hookrightarrow {}^L\mathbf{Hecke}^\mu$ of the Hecke stack.

Remark 2.2. *Similarly we can define Hecke operators ${}^L H^{\mu,x}$ labeled by a cocharacter $\mu \in \mathsf{cochar}^+({}^L G)$ and a point $x \in C$. To construct these operators we can repeat the definition of the ${}^L H^\mu$'s but instead of ${}^L I^\mu$ we need to use the intersection cohomology sheaf on the restricted Hecke correspondence*

$${}^L\mathbf{Hecke}^{\mu,x} := {}^L\mathbf{Hecke}^\mu \underset{{}^L\mathbf{Bun}\times{}^L\mathbf{Bun}\times C}{\times} \left({}^L\mathbf{Bun} \times {}^L\mathbf{Bun} \times \{x\}\right).$$

The operators ${}^L H^{\mu,x}$ are known to generate a commutative algebra of endofucntors of $D_{\mathrm{coh}}({}^L\boldsymbol{\mathcal{B}un}, \mathcal{D})$ [BD03], [Gai01]. In particular it is natural to look for $\mathcal{D}$-modules on ${}^L\mathbf{Bun}$ that are common eigen-modules of all the ${}^L H^{\mu,x}$.

A $\mathcal{D}$-module $\mathscr{M}$ on ${}^L\mathbf{Bun}$ is a *Hecke eigen module* with eigenvalue $\mathbb{V} \in \boldsymbol{\mathcal{L}oc}$ if for every $\mu \in \mathsf{char}^+(G)$ we have

$${}^L H^\mu(\mathscr{M}) = \mathscr{M} \boxtimes \rho^\mu(\mathbb{V}).$$

This setup explains all the ingredients in (GLC). According to the conjecture (GLC) the derived category of coherent $\mathcal{O}$-modules on $\boldsymbol{\mathcal{L}oc}$ is equaivalent to the derived category of coherent $\mathcal{D}$-modules on ${}^L\mathbf{Bun}$. Moreover this equivalence transforms the skyscraper sheaves of points on $\boldsymbol{\mathcal{L}oc}$ into Hecke eigen $\mathcal{D}$-modules on ${}^L\mathbf{Bun}$

Remark 2.3. *Alternatively we can characterize uniquely the geometric Langlands equivalence $\mathfrak{c}$ by the property that $\mathfrak{c}$ intertwines the action of the tensorization functors on $D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O})$ with the action of the Hecke functors on $D_{\mathrm{coh}}({}^L\boldsymbol{\mathcal{B}un}, \mathcal{D})$. The tensorization functors $W^{\mu,x} : D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O}) \to D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O})$ are endofunctors labelled by the same data as the Hecke functors: pairs (x, μ), where $x \in C$ is a closed point and $\mu \in \mathsf{cochar}^+({}^L G) = \mathsf{char}^+(G)$ is a dominant cocharacter for ${}^L G$, or equivalently a dominant character for G. Given such a pair (x, μ) one defines the tensorization functor $W^{\mu,x}$ as*

$$\begin{aligned} W^{\mu,x} : \; D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O}) &\longrightarrow D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O}) \\ \mathscr{F} &\longmapsto \mathscr{F} \otimes \rho^\mu\left(\mathscr{V}_{|\boldsymbol{\mathcal{L}oc}\times\{x\}}\right). \end{aligned}$$

The equivalent characteristic property of $\mathfrak{c}$ now can be formulated as the requirement that $\mathfrak{c} \circ W^{\mu,x} = {}^L H^{\mu,x}$.

Remark 2.4. *The categories related by the conjectural geometric Langlands correspondence admit natural orthogonal decompositions. For instance note that the center of G is contained in the stabilizer of any point $\mathbb{V}$ of $\boldsymbol{\mathcal{L}oc}$ and so $\boldsymbol{\mathcal{L}oc}$ is a $Z(G)$-gerbe over the full rigidification*

$$\underline{\boldsymbol{\mathcal{L}oc}} := \boldsymbol{\mathcal{L}oc} /\!\!/ Z(G) = \mathbf{Loc} /\!\!/ \pi_0(Z(G))$$

of $\boldsymbol{\mathcal{L}oc}$. (In fact by the same token as in Remark 2.1, the stack $\underline{\boldsymbol{\mathcal{L}oc}}$ is generically a variety.) Furthermore the stack $\boldsymbol{\mathcal{L}oc}$ is in general disconnected and

$$\pi_0(\boldsymbol{\mathcal{L}oc}) = \pi_0(\underline{\boldsymbol{\mathcal{L}oc}}) = H^2(C, \pi_1(G)_{\mathrm{tor}}) = \pi_1(G)_{\mathrm{tor}}$$

where $\pi_1(G)_{\mathrm{tor}} \subset \pi_1(G)$ is the torsion part of the finitely generated abelian group $\pi_1(G)$. Thus we get an orthogonal decomposition

$$D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O}) = \coprod_{(\gamma,\alpha)\in\pi_1(G)_{\mathrm{tor}}\times Z(G)^\wedge} D_{\mathrm{coh}}(\underline{\boldsymbol{\mathcal{L}oc}}_\gamma, \mathcal{O}; \alpha), \tag{2.1}$$

where $Z(G)^\wedge = \mathrm{Hom}(Z(G), \mathbb{C}^\times)$ is the character group of the center and $D_{\mathrm{coh}}(\underline{\boldsymbol{\mathcal{L}oc}}_\gamma, \mathcal{O}; \alpha)$ is the derived category of α-twisted coherent $\mathcal{O}$-modules on the connected component $\underline{\boldsymbol{\mathcal{L}oc}}_\gamma$.

Similarly the group of connected components $\pi_0(Z({}^LG))$ is contained in the stabilizer of any point of ${}^L\mathbf{Bun}$ and so is a $\pi_0(Z({}^LG))$-gerbe over ${}^L\underline{\mathbf{Bun}} := {}^L\mathbf{Bun} /\!\!/ \pi_0(Z({}^LG))$. Also the stack ${}^L\mathbf{Bun}$ can be disconnected and

$$\pi_0({}^L\mathbf{Bun}) = \pi_0({}^L\underline{\mathbf{Bun}}) = H^2(C, \pi_1({}^LG)) = \pi_1({}^LG).$$

Hence we have an orthogonal decomposition

$$D_{\mathrm{coh}}({}^L\mathbf{Bun}, \mathcal{D}) = \coprod_{(\alpha,\gamma)\in\pi_1({}^LG)\times\pi_0(Z({}^LG))^\wedge} D_{\mathrm{coh}}({}^L\underline{\mathbf{Bun}}_\alpha, \mathcal{D}; \gamma), \tag{2.2}$$

where $D_{\mathrm{coh}}({}^L\underline{\mathbf{Bun}}_\alpha, \mathcal{D}; \gamma)$ is the derived category of γ-twisted coherent $\mathcal{D}$-modules on the connected component ${}^L\underline{\mathbf{Bun}}_\alpha$.

Finally, observe that the group theoretic Langlands duality gives natural identifications

$$\begin{aligned} \pi_1({}^LG) &= Z(G)^\wedge \\ Z_0({}^LG) &= (\pi_1(G)_{\mathrm{free}})^\wedge \\ \pi_0(Z({}^LG)) &= (\pi_1(G)_{\mathrm{tor}})^\wedge, \end{aligned}$$

where again $\pi_1(G)_{\mathrm{tor}} \subset \pi_1(G)$ is the torsion subgroup, $\pi_1(G)_{\mathrm{free}} = \pi_1(G)/\pi_1(G)_{\mathrm{tor}}$ is the maximal free quotient, and $Z({}^LG)$ is the center of ${}^LG$, and $Z_0({}^LG)$ is its connected component.

In particular the two orthogonal decompositions (2.1) and (2.2) are labeled by the same set and one expects that the conjectural equivalence $\mathfrak{c}$ from (GLC) idenitifies $D_{\mathrm{coh}}(\underline{\boldsymbol{\mathcal{L}oc}}_\gamma, \mathcal{O}; -\alpha)$ with $D_{\mathrm{coh}}({}^L\underline{\mathbf{Bun}}_\alpha, \mathcal{D}; \gamma)$. The minus sign on α here is essential and necessary in order to get a duality transformation that belongs to $SL_2(\mathbb{Z})$. This behavior of twistings was analyzed and discussed in detail in [DP08].

2.2 The geometric Langlands conjecture for $GL_n(\mathbb{C})$

Suppose $G = GL_n(\mathbb{C})$. Then ${}^LG = GL_n(\mathbb{C})$ and **Bun** is the rigidified stack of rank n vector bundles on C. In this case the stack **Bun** can be described easily as a solution to a moduli problem. Namely for a complex scheme S we can identify the sections of **Bun** over S with the groupoid $\mathbf{Bun}(S)$ whose objects are the rank n algebraic vector bundles on $S \to C$, and in which an isomorphism between two bundles $V \to S \times C$ and $W \to S \times C$ is given by a pair (A, ϕ), where $A \in \mathrm{Pic}(S)$ is a line bundle on S, and ϕ is an isomorphism $\phi : V \xrightarrow{\cong} W \otimes p_S^* A$.

Similarly $\boldsymbol{\mathcal{L}oc}$ can be identified with the stack of rank n vector bundles C equipped with an integrable connection. In this case the algebra of Hecke operators is generated by the operators H^i given by the special Hecke correspondences

$$\mathbf{Hecke}^i := \left\{ (V, V', x) \;\middle|\; \begin{array}{l} V \text{ and } V' \text{ are locally free sheaves of rank} \\ n \text{ such that } V \subset V' \subset V(x) \text{ and} \\ \mathrm{length}(V'/V) = i. \end{array} \right\}$$

The operators H^i correspond to the fundamental weights of $GL_n(\mathbb{C})$ which are all minuscule. In particular all $\mathbf{Hecke}^i$'s are smooth. The fibers of the projection $q^i : \mathbf{Hecke}^i \to \mathbf{Bun} \times C$ are all isomorphic to the Grassmanian $Gr(i, n)$ of i-dimensional subspaces in an n-dimensional space.

In this case the geometric Langlands correspondence $\mathfrak{c}$ is characterized uniquely by the Hecke eigen-property for the operators H^i. Explicitly for every irreducible rank n local system $\mathbb{V} = (V, \nabla)$ the conjecture (GLC) predicts the existence of a unique irreducible coherent $\mathcal{D}$-module $\mathfrak{c}(\mathcal{O}_{\mathbb{V}})$ on **Bun** so that

$$H^i\left(\mathfrak{c}(\mathcal{O}_{\mathbb{V}})\right) = \mathfrak{c}(\mathcal{O}_{\mathbb{V}}) \boxtimes \wedge^i \mathbb{V}$$

for $i = 1, 2, \ldots, n$. This is the version of (GLC) that will be of primary interest in these lectures.

Example 2.5. *Suppose $G \cong GL_1(\mathbb{C}) \cong {}^LG$. Then $\mathbf{Bun} = \mathbf{Pic}(C)$ is the Picard variety of C. Here there is only one interesting Hecke operator*

$$H^1 : D_{\mathrm{coh}}(\mathbf{Pic}(C), \mathcal{D}) \to D_{\mathrm{coh}}(C \times \mathbf{Pic}(C), \mathcal{D})$$

which is simply the pull-back $H^1 := \mathsf{aj}^*$ via the classical Abel-Jacobi map

$$\begin{array}{rrcl} \mathsf{aj} : & C \times \mathbf{Pic}^d(C) & \longrightarrow & \mathbf{Pic}^{d+1}(C) \\ & (x, L) & \longmapsto & L(x). \end{array}$$

In this case the geometric Langlands correspondence $\mathfrak{c}$ can be described explicitly. Let $\mathbb{L} = (L, \nabla)$ be a rank one local system on C. Since $\pi_1(\mathbf{Pic}^d(C))$ is the abelianization of $\pi_1(C)$ and the monodromy representation of $\mathbb{L}$ is abelian, it follows that we can view $\mathbb{L}$ as a local *system on each component $\mathbf{Pic}^d(C)$ of $\mathbf{Pic}(C)$. With this setup we have*

$$\mathfrak{c}(\mathbb{L}) := \begin{pmatrix} \text{the unique translation invariant} \\ \text{rank one local system on } \mathbf{Pic}(C) \\ \text{whose restriction on each compo-} \\ \text{nent } \mathbf{Pic}^d(C) \text{ has the same mon-} \\ \text{odromy as } \mathbb{L} \end{pmatrix}.$$

The local system $\mathfrak{c}(\mathbb{L})$ can be constructed effectively from $\mathbb{L}$ (see e.g. [Lau90]):

- *Pullback the local system $\mathbb{L}$ to the various factors of the d-th Cartesian power $C^{\times d}$ of C and tensor these pullbacks to get rank one local system $\mathbb{L}^{\boxtimes d}$ on $C^{\times d}$;*
- *By construction $\mathbb{L}^{\boxtimes d}$ is equipped with a canonical $\mathsf{S_d}$-equivariant structure compatible with the standard action of the symmetric group $\mathsf{S_d}$ on $C^{\times d}$. Pushing forward $\mathbb{L}^{\boxtimes d}$ via $g_d : C^{\times d} \to C^{(d)} = C^{\times d}/\mathsf{S_d}$ and passing to $\mathsf{S_d}$ invariants we get a rank one local system $\left(g_{d*}\mathbb{L}^{\boxtimes d}\right)^{\mathsf{S_d}}$ on $C^{(d)}$;*
- *For $d > 2g-2$ the Abel-Jacobi map $\mathsf{aj}^d : C^{(d)}\, \mathbf{Pic}^d(C)$ is a projective bundle over $\mathbf{Pic}^d(C)$ and so by pushing forward by aj^d we get a rank one local system which we denote by $\mathfrak{c}(\mathbb{L})_{|\,\mathbf{Pic}^d(C)}$. In other words*

  $$\mathfrak{c}(\mathbb{L})_{|\,\mathbf{Pic}^d(C)} := \mathsf{aj}^d_* \left[\left(g_{d*}\mathbb{L}^{\boxtimes d}\right)^{\mathsf{S_d}} \right].$$

- *Translation $(\bullet)\otimes\omega_C$ by the canonical line bundle transports the local systems $\mathfrak{c}(\mathbb{L})_{|\,\mathbf{Pic}^d(C)}$ to components $\mathbf{Pic}^d(C)$ of $\mathbf{Pic}(C)$ with $d \leq 2g-2$.*

Roughly, our goal in these lectures is to argue that one should be able to reduce the case of a general group to the previous example by using Hitchin's abelianization. The idea is that the correspondence $\mathfrak{c}$ should (in the case $G = GL_n(\mathbb{C})$ decompose as

$$\mathfrak{c} = \mathsf{quant}_{\mathbf{Bun}} \circ \mathsf{FM} \circ \mathsf{quant}_C^{-1} \tag{2.3}$$

where quant_C is a suitable quantization procedure for coherent sheaves on $T^\vee C$, $\mathsf{quant}_{\mathbf{Bun}}$ is a suitable quantization procedure for coherent sheaves on $T^\vee\,\mathbf{Bun}$, and FM is a suitable algebro-geometric Fourier-Mukai transform for coherent sheaves on $T^\vee\,\mathbf{Bun}$. Furthermore the two quantization procedures quant_C and $\mathsf{quant}_{\mathbf{Bun}}$ are apropriately understood non-abelian Hodge corespondences.

We will try to make this idea more precise in the remainder of these lectures. To that end we need to introduce the Hitchin integrable system which allows us to abelianize the moduli stack of Higgs bundles. We will also need some details about the non-abelian Hodge correspondence on schemes or stacks.

Chapter 3

Higgs bundles, the Hitchin system, and abelianization

3.1 Higgs bundles and the Hitchin map

As in the previous chapter fixing the curve C and the groups G, ${}^LG$ allows us to define moduli stacks of Higgs bundles:

$\boldsymbol{\mathcal{H}iggs}$**,** ${}^L\boldsymbol{\mathcal{H}iggs}$**:** the moduli stacks of ω_C-valued G, ${}^LG$ Higgs bundles (E, φ) on C. Here E is a principal G, or ${}^LG$ bundle on C and $\varphi \in H^0(C, \mathrm{ad}(E) \otimes \omega_C)$.

and their rigidified versions **Higgs**, ^{L}**Higgs** in which the connected component of the center of the generic stabilizer ($= Z_0(G)$, $Z_0({}^LG)$) is "removed" [AOV08, Appendix A].

Hitchin discovered [Hit87b] that the moduli stack $\boldsymbol{\mathcal{H}iggs}$ has a natural symplectic structure and comes equipped with a complete system of commuting Hamiltonians. These are most conveniently organized in a remarkable map $h : \boldsymbol{\mathcal{H}iggs} \to B$ to a vector space B, known as the *Hichin map*.

The target of this map, also known as the *Hitchin base*, is the cone

$$B := H^0(C, (\mathfrak{t} \otimes \omega_C)/W),$$

where as before $\mathfrak{t}$ is our fixed Cartan algebra in $\mathfrak{g} = \mathrm{Lie}(G)$, and W is the Weyl group of G.

To construct the Hitchin map h one considers the adjoint action of G on $\mathfrak{g}$. For every principal G-bundle E the quotient map $\mathfrak{g} \to \mathfrak{g}//G$ induces a map between the total spaces of the associated fiber bundles

$$\begin{array}{ccc} E \times_{\mathrm{Ad}} \mathfrak{g} & \twoheadrightarrow & E \times_{\mathrm{Ad}} (\mathfrak{g}//G) \\ \| & & \| \\ \mathrm{ad}(E) & & C \times (\mathfrak{g}//G) \end{array}$$

which induces a (polynomial) map between the fiber bundles

$$\mathrm{ad}(E) \otimes \omega_C \to (\mathfrak{g} \otimes \omega_C)//G. \tag{3.1}$$

The map (3.1) combines with the canonical identification $\mathfrak{g}//G = \mathfrak{t}/W$ given by Chevalley's restriction theorem [Hum72, Section 23.1] to yield a natural map of fiber bundles

$$\nu_E : \mathrm{ad}(E) \otimes \omega_C \to (\mathfrak{t} \otimes \omega_C)/W.$$

This construction gives rise to the Hitchin map:

$$\begin{aligned} h: \quad \boldsymbol{\mathcal{H}iggs} &\longrightarrow B := H^0(C, (\mathfrak{t} \otimes \omega_C/W) \\ (E, \varphi) &\longrightarrow \text{“}\varphi \bmod W\text{”} := \nu_E(\theta). \end{aligned}$$

Slightly less canonically if $r = \dim \mathfrak{t} = \operatorname{rank} \mathfrak{g}$ we can choose homogeneous G-invariant polynomials $I_1, I_2, \ldots, I_r \in \mathbb{C}[\mathfrak{g}]$ such that $\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[I_1, \ldots, I_r]$. With this choice we get an identification

$$B = H^0(C, (\mathfrak{t} \otimes \omega_C)/W) \cong \oplus_{s=1}^r H^0(C, \omega_C^{\otimes d_s}),$$

with $d_s = \deg I_s$, and we can rewrite the Hitchin map as

$$\begin{aligned} h: \quad \boldsymbol{\mathcal{H}iggs} &\longrightarrow B = \oplus_{s=1}^r H^0(C, \omega_C^{\otimes d_s}) \\ (E, \varphi) &\longrightarrow (I_1(\varphi), \ldots, I_r(\varphi)). \end{aligned}$$

The points of the Hitchin base admit a natural geometric interpretation as certain Galois covers of C with Galois group W called *cameral covers*. By definition the cameral cover associated with a point $b \in B$ is the cover $p_b : \widetilde{C}_b \to C$ obtained as the fiber product

$$\begin{tikzcd} \widetilde{C}_b \arrow[r] \arrow[d, "p_b"'] & \mathrm{tot}(\mathfrak{t} \otimes \omega_C) \arrow[d] \\ C \arrow[r, "b"'] & \mathrm{tot}(\mathfrak{t} \otimes \omega_C)/W \end{tikzcd}$$

Repeating the same construction for the tautological section $C \times B \to \mathrm{tot}(\mathfrak{t} \otimes \omega_C)/W$ we also get the universal cameral cover

$$\begin{tikzcd} \widetilde{\mathcal{C}} \arrow[r] \arrow[d] & \mathrm{tot}(\mathfrak{t} \otimes \omega_C) \arrow[d] \\ C \times B \arrow[r] & \mathrm{tot}(\mathfrak{t} \otimes \omega_C)/W \end{tikzcd}$$

which by construction restricts to $\widetilde{C}_b$ on the slice $C \times \{b\} \subset C \times B$.

Deformation theory for principal bundles on C together with Serre duality gives (see e.g. [Fal93, BL94]) a natural identification

$$\boldsymbol{\mathcal{H}iggs} \cong T^\vee \boldsymbol{\mathcal{B}un}$$

of the stack of Higgs bundles with the cotangent stack $T^\vee \boldsymbol{\mathcal{B}un}$ to the stack of bundles. This gives rise to the symplectic structure on $\boldsymbol{\mathcal{H}iggs}$. The Hitchin map $h : \boldsymbol{\mathcal{H}iggs} \to B$

is a completely integrable system structure on $\boldsymbol{\mathcal{H}iggs}$. Its generic fibers are abelian group stacks which are also Lagrangian for the natural symplectic structure. Concretely the fiber $h^{-1}(b)$ is identified with an appropriately defined Prym stack for the cameral cover $p_b : \widetilde{C}_b \to C$, i.e. $h^{-1}(b)$ is a special W-isotypic piece for the W-action on the stack of (decorated) line bundles on $\widetilde{C}_b$. The details of this picture were worked out in various situations in [Hit87b, Fal93, Don93, Don95, Sco98, DG02]. The most general result in this direction is [DG02, Theorem 4.4] according to which:

- the covering map $\tilde{p} : \widetilde{\mathcal{C}} \to C \times B$ determines an abelian group scheme $\mathcal{T}$ over $C \times B$;
- if $\Delta \subset B$ is the discriminant divisor parametrizing $b \in B$ for which $\tilde{p}_b : \widetilde{C}_b \to C$ does not have simple Galois ramification, then the restriction

  $$h : \boldsymbol{\mathcal{H}iggs}_{|B-\Delta} \to B - \Delta$$

  is a principal homogeneous stack over the commutative group stack $\boldsymbol{\mathcal{T}ors}_{\mathcal{T}}$ on $B - \Delta$ parametrizing $\mathcal{T}$-torsors along C.

Remark 3.1. *For every $\mu \in \mathsf{char}(G)$ we can also consider the associated spectral cover $\overline{\mathcal{C}}^{\mu} \to C \times B$. It is the quotient of $\widetilde{\mathcal{C}}$ by the stabilizer of μ in W. Very often, e.g. for classical groups and the fundamental weight [Hit87b, Don93] the fiber of the Hitchin map can also be described as a stack of (decorated) line bundles on the spectral cover. For instance if $G = GL_n(\mathbb{C})$ and we use the highest weight of the n-dimensional fundamental representation of G, then the associated spectral cover $\overline{C}_b \to C$ is of degree n, and the fiber of the Hitchin map $h^{-1}(b)$ can be identified with the stack $\mathcal{P}ic(\overline{C}_b)$ of all line bundles on $\overline{C}_b$.*

3.2 Using abelianization

From the point of view of the Geometric Langlands Conjecture the main utility of the Hitchin map is that it allows us to relate the highly non-linear moduli $\boldsymbol{\mathcal{B}un}$ to an object that is essentially "abelian".

The basic idea is to combine the Hitchin map with the projection ${}^L\boldsymbol{\mathcal{H}iggs} \to {}^L\boldsymbol{\mathcal{B}un}$, $(E, \varphi) \to E$. More precisely we have a diagram

$$\begin{tikzcd} & {}^L\boldsymbol{\mathcal{H}iggs}_{|{}^LB-{}^L\Delta} \arrow[dl] \arrow[dr, "h"] & \\ {}^L\boldsymbol{\mathcal{B}un} & & {}^LB - {}^L\Delta \end{tikzcd}$$

in which the fibers of $h : \boldsymbol{\mathcal{H}iggs}_{|B-\Delta} \to B - \Delta$ are commutative group stacks and each fiber of h dominates $\boldsymbol{\mathcal{B}un}$.

We can use this diagram to reformulate questions about $\mathcal{O}$-modules or $\mathcal{D}$-modules on ${}^L\boldsymbol{\mathcal{B}un}$ to questions about $\mathcal{O}$-modules or $\mathcal{D}$-modules on fibers of h. This process is known as *abelianization* and has been applied successfully to answer many gometric questions about the moduli of bundles. The fact that each fiber of $h : \boldsymbol{\mathcal{H}iggs}_{|B-\Delta} \to B - \Delta$ is an isotypic component of the moduli of line bundles on the corresponding cameral cover, and the fact (see Example 2.5) the Geometric Langlands Correspondence can be constructed explicitly for rank one local systems, suggests that abelianization can be used to give a construction of the functor $\mathfrak{c}$ (GLC) in general.

There are various ways in which one can employ abelianization to produce a candidate for the functor $\mathfrak{c}$:

- One possibility is to apply a version of the generalizations of the Fourier transform due to Laumon, Rothstein, and Polishchuk [Lau96, Rot96, PR01, Pol08] along the fibers of h. The most successful implementation of this approach to date is the recent work of Frenkel-Teleman [FT09] who used the generalized Fourier transform to give a construction of the correspondence (GLC) for coherent sheaves on a formal neighborhood of the substack of opers (see [BD05, BD03]) inside $\boldsymbol{\mathcal{L}oc}$.

- Another approach is to study the deformation quantization of a Fourier-Mukai transform along the fibers of h. This is the main component of Arinkin's approach [Ari02, Ari08] to the quasi-classical geometric Langlands correspondence. This approach was recently utilized by Bezrukavnikov and Braverman [BB07] who proved the geometric Langlands correspondence for curves over finite fields for $G = {}^L G = \mathrm{GL}_n$.

- Last but not least, in the recent work of Kapustin-Witten [KW06] the geometric Langlands correspondence $\mathfrak{c}$ is interpreted physically in two different ways. On one hand it is argued that the existence of the conjectural map (GLC) is a mirror symmetry statement relating the A and B-type branes on the hyper-Kähler moduli spaces of Higgs bundles. On the other hand Kapustin and Witten use a gauge theory/string duality to show that the functor $\mathfrak{c}$ can be thought of as an electric-magnetic duality between supersymmetric four-dimensional Gauge theories with structure groups G and ${}^L G$ respectively. This suggests that $\mathfrak{c}$ can be understood as a conjugation of the Fourier-Mukai transform along the Hitchin fibers with two non-abelian Hodge correspondences. Some non-trivial tests of this proposal were performed in [KW06] and in the work of Frenkel-Witten [FW08] who elaborated further on this conjectural construction.

Before we explain the non-abelian Hodge theory approach in more detail we will briefly discuss yet another way of extracting a functor $\mathfrak{c}$ from abelianization that was proposed in [Don89]. This proposal shares many of the same ingredients as the other approaches and highlights the important issues that one has to overcome. It also has the advantage of being manifestly algebraic. In this approach one starts with a local system $\mathbb{V}$ on C and uses it together with some geometry to construct a pair (M, δ) where M is a bundle on **Higgs**, and $\delta : M \to M \otimes \Omega^1_{\mathbf{Higgs}/B}$ is a meromorphic relative flat connection acting along the fibers of the Hitchin map $\mathsf{h} : \mathbf{Higgs} \to B$. Furthermore by construction the bundle (M, δ) is a Hecke eigen $\mathcal{D}$-module with eigenvalue (V, ∇) but with respect to an abelianized version

$$ {}_{\mathrm{ab}}H^i : D_{\mathrm{coh}}(\mathbf{Higgs}, \mathcal{O}) \to D_{\mathrm{coh}}(\mathbf{Higgs} \times C, \mathcal{O}) $$

of the Hecke functors. These are defined again for $i = 1, \ldots, n-1$ as integral transforms with respect to the trivial local system on the abelianized Hecke correspondences

$$\begin{tikzcd} & {}_{\mathrm{ab}}\,\mathbf{Hecke}^i \arrow[dl, "{}_{\mathrm{ab}}p^i"'] \arrow[dr, "{}_{\mathrm{ab}}q^i"] & \\ \mathbf{Higgs} & & \mathbf{Higgs}\times C \end{tikzcd}$$

The correspondences ${}_{\mathrm{ab}}\,\mathbf{Hecke}^i$ are the "Higgs lifts" of the correspondences $\mathbf{Hecke}^i$ from section 2.2, that is

$${}_{\mathrm{ab}}\,\mathbf{Hecke}^i = \left\{ ((V,\varphi),(V',\varphi'),\beta,x) \;\middle|\; \begin{array}{l} (V,V',\beta,x)\in\mathbf{Hecke}^i \text{ and } \beta \text{ fits in a commu-} \\ \text{tative diagram} \quad \begin{tikzcd} V \arrow[r, "\beta"] \arrow[d, "\varphi"'] & V' \arrow[d, "\varphi'"] \\ V\otimes\omega_C \arrow[r, "\beta\otimes\mathrm{id}"'] & V'\otimes\omega_C \end{tikzcd} \end{array} \right\}$$

Here the maps ${}_{\mathrm{ab}}p^i$, ${}_{\mathrm{ab}}q^i$ are induced from the maps p^i, q^i and so the fiber of ${}_{\mathrm{ab}}q^i$ over $((V',\varphi'),x)$ is contained in the fiber of q^i over (V',x), which as we saw before is isomorphic to $Gr(i,n)$. In fact from the definition we see that the fiber of ${}_{\mathrm{ab}}q^i$ over $((V',\varphi'),x)$ consists of the i-dimensionals subspaces in V'_x which are φ'-invariant. Thus if the point x is not a ramification point of the spectral cover of (V',φ') it follows that the fiber of ${}_{\mathrm{ab}}q^i$ consists of finitely many points in $Gr(i,n)$.

The construction of (M,δ) occupies the remainder of this section. The approach depends on one global choice: we will fix a theta characteristic $\zeta\in\mathbf{Pic}^{g-1}(C)$, $\zeta^{\otimes 2}=\omega_C$. To simplify the discussion we will assume that $G={}^LG=GL_n(\mathbb{C})$. As we saw in the previous section the choice of the fundamental n-dimensional representation of $GL_n(\mathbb{C})$ gives rise to a universal n-sheeted spectral cover

$$\begin{tikzcd} \overline{\mathcal{C}} \arrow[d, "\pi"'] \\ C\times B \end{tikzcd}$$

Suppose now we have $\mathbb{V}=(V,\nabla)$ - a rank n vector bundle with an integrable connection on C. The fiber over $V\in\boldsymbol{\mathcal{B}un}$ of the projection $\boldsymbol{\mathcal{H}iggs}\cong T^\vee\boldsymbol{\mathcal{B}un}\to\boldsymbol{\mathcal{B}un}$ is just the fiber $T^\vee_V\boldsymbol{\mathcal{B}un}=H^0(C,\mathrm{ad}(V)\otimes\omega_C)$ of the cotangent bundle to $\boldsymbol{\mathcal{B}un}$. Restricting the Hitchin map to $T^\vee_V\boldsymbol{\mathcal{B}un}$ and pullng back the universal spectral cover we get a cover

$$\begin{tikzcd} \overline{\mathcal{C}}_V \arrow[r] \arrow[d, "\pi_V"'] & \overline{\mathcal{C}} \arrow[d, "\pi"] \\ C\times T^\vee_V\boldsymbol{\mathcal{B}un} \arrow[r, "h_V"'] & C\times B \end{tikzcd}$$

Using the spectral correspondence [Hit87b, Don95, DG02] we can find a holomorphic line bundle L on $\overline{\mathcal{C}}_V$ such that

- $\pi_{V*}L\cong p_C^*\left(V\otimes\zeta^{\otimes-(n-1)}\right)$,
- ∇ induces a (relative over $T^\vee_V\boldsymbol{\mathcal{B}un}$) holomorphic connection D on L.

Indeed, by definition the vector bundle $p_C^* V$ on $C \times T_V^\vee \boldsymbol{\mathcal{B}un}$ comes equipped with a tautological Higgs field $\varphi \in H^0(C \times T_V^\vee \boldsymbol{\mathcal{B}un}, p_C^*(\mathrm{ad}(V) \otimes \omega_C))$, characterized uniquely by the property that for every $\theta \in T_V^\vee \boldsymbol{\mathcal{B}un}$ we have $\varphi_{|C\times\{\theta\}} = \theta$. The cover $\overline{\mathcal{C}}_V \to C \times T_V^\vee \boldsymbol{\mathcal{B}un}$ is simply the spectral cover of $(p_C^* V, \varphi)$ and hence comes equipped with a natural line bundle L', such that $\pi_{V*} L' = p_C^* V$. Notice that for every $\theta \in T_V^\vee \boldsymbol{\mathcal{B}un}$ the restriction of the line bundle L' to the spectral curve $\overline{C}_{h_V(\theta)} = \overline{\mathcal{C}}_{V|C\times\{\theta\}}$ has degree $n(n-1)(g-1)$ and so does not admit a holomorphic connection. To correct this problem we can look instead at $L'_{|\overline{C}_{h_V(\theta)}} \otimes \zeta^{\otimes -(n-1)}$ which has degree zero and so admits holomorphic connections. With this in mind we set

$$L := L' \otimes \pi_V^* p_C^* \zeta^{\otimes -(n-1)}.$$

To see that ∇ induces a relative holomorphic connection D on L we will need the following fact.

Let $\theta \in H^0(C, \mathrm{ad}(V) \otimes \omega_C)$ be a Higgs field and let

$$\left(p : \overline{C} \to C, N \in \boldsymbol{\mathcal{P}ic}^{n(n-1)(g-1)}(\overline{C}) \right)$$

be the associated spectral data. Suppose that $\overline{C}$ is smooth and that $p : \overline{C} \to C$ has simple ramification. Let $R \subset \overline{C}$ denote the ramification divisor. Then there is a canonical isomorphism of affine spaces

$$\tau : \begin{pmatrix} \text{holomorphic} \\ \text{connections} \\ \text{on } V \end{pmatrix} \longrightarrow \begin{pmatrix} \text{meromorphic connections on } N \\ \text{with logarithmic poles along } R \\ \text{and residue } \left(-\dfrac{1}{2}\right) \end{pmatrix}$$

Indeed, since $(\overline{C}, N)$ is built out from (V, θ) via the spectral construction we have that $p_* N = V$. Away from the ramification divisor N is both a subbundle in $p^* V$ and a quotient bundle of $p^* V$. Furthermore if ∇ is a holomorphic connection on V, the pullback $p^*\nabla$ is a holomorphic connection on $p^* V$ and so on $\overline{C} - R$ we get a holomorphic connection on N given by the composition

$$N \to p^* V \xrightarrow{p^*\nabla} p^* V \otimes \Omega^1_{\overline{C}} \to N \otimes \Omega^1_{\overline{C}}. \tag{3.2}$$

On all of $\overline{C}$ the composition (3.2) can be viewed as a meromorphic connection on N with pole along R. The order of the pole and the residue of this meromorphic connection can be computed locally near the ramification divisor R. Since p has simple ramification, in an appropriate local (formal or analytic) coordinate centered at a point $r \in R$ the map p can be written as $z \mapsto z^2$. The image of this local chart in C is, say an analytic disk $\mathbb{D} \subset C$ centered at a branch point. Over $\mathbb{D}$ the covering $\overline{C} \to C$ splits into $n-1$ connected components: $p^{-1}(\mathbb{D}) = \mathbb{D}_0 \coprod \mathbb{D}_1 \coprod \ldots \coprod \mathbb{D}_{n-2}$ where $p_0 := p_{|\mathbb{D}_0} : \mathbb{D}_0 \to \mathbb{D}$ is the two sheeted ramified cover given by $p_0(z) = z^2$ and $p_i := p_{|\mathbb{D}_i} : \mathbb{D}_i \to \mathbb{D}$ are one sheeted components for $i = 1, \ldots, n-2$. Over $\mathbb{D}$ the bundle V will then split into a direct sum of a rank two piece V_0 and a rank $n-2$ piece W. For the calculation of the polar part of the connection D near this point only the rank two piece V_0 of the bundle is relevant since upon restriction to $\mathbb{D}_0$ the natural adjunction morphisms $p^*V \to N$ and $N \to p^!V = p^*V \otimes \mathcal{O}_{\overline{C}}(R)$ factor through $p_0^*V_0 \to N$ and $N \to p_0^*V_0 \otimes \mathcal{O}_{\mathbb{D}_0}(r)$ respectively.

Thus we focus on the covering of disks $p_0 : \mathbb{D}_0 \to \mathbb{D}$, $p_0(z) = z^2$, where z, w are the coordinates on $\mathbb{D}_0$ and $\mathbb{D}$ respectively. we denote the covering involution of this map by σ, i.e. $\sigma : \mathbb{D}_0 \to \mathbb{D}_0$, $\sigma(z) = -z$.

Without a loss of generality we may assume that $N_{|\mathbb{D}_0}$ has been trivialized. This induces a trivialization of $V_0 = p_{0*}\mathcal{O}_{\mathbb{D}_0}$: the frame of this trivialization consists of $\mathbf{e}_+, \mathbf{e}_- \in \Gamma(\mathbb{D}, V_0)$, where $\mathbf{e}_+$ is a frame for the subsheaf of σ-invariant sections of V_0, and $\mathbf{e}_-$ is a frame for the subsheaf of σ-anti-invariant sections of V. concretely, if we use the canonical identification $\Gamma(\mathbb{D}, V_0) = \Gamma(\mathbb{D}_0, \mathcal{O})$, the section $\mathbf{e}_+$ corresponds to $1 \in \Gamma(\mathbb{D}_0, \mathcal{O})$ and the section $\mathbf{e}_-$ corresponds to $z \in \Gamma(\mathbb{D}_0, \mathcal{O})$. Since $\nabla : V_0 \to V_0 \otimes \Omega^1_{\mathbb{D}}$ is a holomorphic connection on V_0, we will have that in the trivialization given by the frame $(\mathbf{e}_+, \mathbf{e}_-)$ it is given by

$$\nabla = d + A(w)dw, \qquad A(w) \in \mathrm{Mat}_{2\times 2}(\Gamma(\mathbb{D}, \mathcal{O})).$$

But the change of frame $(\mathbf{e}_+, \mathbf{e}_-) \to (\mathbf{e}_+, \mathbf{e}_-)\exp(-\int A(w)dw)$, transforms the connection $d + A(w)dw$ into the trivial connection d and since holomorphic changes of frame do not affect the polar behavior it suffices to check that the connection $\tau(\nabla)$ on $\mathbb{O}_{\mathbb{D}_0}$ induced from d by (3.2) has a logarithmic pole at $z = 0$ with residue $(-1/2)$.

By (3.2) we have that the meromorphic connection $\tau(\nabla)$ on $\mathcal{O}_{\mathbb{D}_0}$ is defined as the composition

$$\mathcal{O}_{\mathbb{D}_0^\times} \longrightarrow (p_0^*V_0)_{|\mathbb{D}_0^\times} \xrightarrow{p_0^*\nabla} (p_0^*V_0)_{|\mathbb{D}_0^\times} \otimes \Omega^1_{\mathbb{D}_0^\times} \longrightarrow \Omega^1_{\mathbb{D}_0^\times}$$

where the first and third maps are induced from the adjunction maps $\mathcal{O}_{\mathbb{D}_0^\times} \to p^!p_*\mathcal{O}_{\mathbb{D}_0^\times} = p^*p_*\mathcal{O}_{\mathbb{D}_0^\times} = p^*V_0$ and $p^*V_0 = p^*p_*\mathcal{O}_{\mathbb{D}_0^\times} \to \mathcal{O}_{\mathbb{D}_0^\times}$.

To compute $\tau(\nabla)$ choose a smal subdisk $U \subset \mathbb{D}_0$, s.t. $0 \not\in U$. Then $p_0^{-1}(p_0(U)) = U \coprod \sigma(U)$, and the restriction of p_0 to U and $\sigma(U)$ induces an identification

$$\Gamma(U, p^*V) \cong \Gamma(U, \mathcal{O}) \oplus \Gamma(\sigma(U), \mathcal{O}) \tag{3.3}$$

so that adjunction morphisms $\Gamma(U, \mathcal{O}_U) \to \Gamma(U, p^*V_0)$ and $\Gamma(U, p^*V_0) \to \Gamma(U, \mathcal{O}_U)$ become simply the inclusion $\mathbf{i}$ and projection $\mathbf{p}$ for this direct sum decomposition.

Now the connection p_0^* on the bundle $p_0^*V_0$ has a flat frame $(p_0^*\mathbf{e}_+, p_0^*\mathbf{e}_-)$. In terms of the decomposition (3.3) we have

$$p_0^*\mathbf{e}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad p_0^*\mathbf{e}_- = \begin{pmatrix} z \\ -z \end{pmatrix}$$

Now if $f(z) \in \Gamma(U, \mathcal{O})$ we get

$$\begin{aligned}
\tau(\nabla)(f) &= \mathbf{p} \circ p_0^* \nabla \circ \mathbf{i}(f) = \mathbf{p} \circ p_0^* \nabla \begin{pmatrix} f(z) \\ 0 \end{pmatrix} \\
&= \mathbf{p} \circ p_0^* \nabla \left[\frac{f(z)}{2} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{f(z)}{2z} \cdot \begin{pmatrix} z \\ -z \end{pmatrix} \right] \\
&= \mathbf{p} \left[\frac{f'(z)}{2} dz \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\frac{f'(z)}{2z} - \frac{f(z)}{2z^2} \right) dz \cdot \begin{pmatrix} z \\ -z \end{pmatrix} \right] \\
&= \mathbf{p} \left[\begin{pmatrix} f'(z) - \frac{1}{2} \frac{f(z)}{z} \\ \frac{1}{2} \frac{f(z)}{z} \end{pmatrix} dz \right] \\
&= f'(z) dz - \frac{1}{2} f(z) \frac{dz}{z}.
\end{aligned}$$

Hence $\operatorname{Res}_{z=0}(\tau(\nabla)) = -1/2$ as claimed.

Next suppose we are given a line bundle $\mathscr{L}$ on a variety X and a trivalizing open cover $\{U_\alpha\}$ for $\mathscr{L}$ with local frames $\mathbf{e}_\alpha \in \Gamma(U_\alpha, \mathscr{L})$. Let $g_{\alpha\beta} \in \Gamma(U_{\alpha\beta}, \mathcal{O}^\times)$ be the transition functions for these frames: $\mathbf{e}_\alpha = \mathbf{e}_\beta g_{\beta\alpha}$. In particular any section of $\mathscr{L}$ is given by a collection $\{s_\alpha\}$ of locally defined holomorphic functions $s_\alpha \in \Gamma(U_\alpha, \mathcal{O})$ satisfying $s_\alpha = g_{\alpha\beta} s_\beta$, and a connection $\nabla : \mathscr{L} \to \mathscr{L} \otimes \Omega^1_X$ is given by connection one forms $a_\alpha \in \Gamma(U_\alpha, \Omega^1_X)$ satisfying $a_\alpha - a_\beta = d \log g_{\beta\alpha}$. If $\varrho \in \mathbb{Q}$ is a fixed rational number and if $\mathfrak{s} \in \Gamma(X, \mathcal{L}^{\otimes \varrho})$ is a global section in some rational power of $\mathscr{L}$, then we can choose a trivializing cover $\{U_\alpha\}$ for $\mathscr{L}$ which is also a trivializing cover for $\mathcal{L}^{\otimes \varrho}$ and such that the transition functions for $\mathcal{L}^{\otimes \varrho}$ in appropriately chosen local frames are all branches $g_{\alpha\beta}^{\varrho}$ of the ϱ-th powers of the transition functions $g_{\alpha\beta}$ for $\mathscr{L}$ and the section $\mathfrak{s}$ is represented by a collection $\{\mathfrak{s}_\alpha\}$ of locally defined holomorphic functions satisfying $\mathfrak{s}_\alpha = g_{\alpha\beta}^{\varrho} \mathfrak{s}_\beta$. Taking $d \log$ of both sides of this last identity we get that

$$\left(-\frac{1}{\varrho} d \log \mathfrak{s}_\alpha \right) - \left(-\frac{1}{\varrho} d \log \mathfrak{s}_\beta \right) = d \log g_{\beta\alpha}.$$

In other words the collection $\{d - (1/\varrho) d \log \mathfrak{s}_\alpha\}$ gives a meromorphic connection on $\mathcal{L}$ with pole along the divisor $\mathfrak{s} = 0$. Furthermore if the divisor of $\mathfrak{s}$ is smooth this connection has residue $-1/\varrho$.

Now consider the line bundle $N \otimes p^* \zeta^{\otimes -(n-1)}$ on $\overline{C} = \overline{C}_{h_V(\theta)}$. As explained above the connection ∇ on V induces a meromorphic connection $\tau(\nabla)$ on N with a logarithmic pole along R and residue $(-1/2)$. On the other hand we have $\mathcal{O}_{\overline{C}}(R) = p^* \zeta^{\otimes 2(n-1)}$. Let $\mathfrak{s}$ be a holomorphic section of $p^* \zeta^{\otimes 2(n-1)}$ that vanishes on R. Then by the discussion in the previous paragraph $\mathfrak{s}$ induces a meromorphic connection $\mathfrak{D}$ on the line bundle $p^* \zeta^{\otimes -(n-1)}$: if we trivialize $p^* \zeta^{\otimes -(n-1)}$ on an open $U \subset \overline{C}$ and if $\mathfrak{s}$ is represented by a holomorphic function $\mathfrak{s}_U \in \Gamma(U, \mathcal{O})$ in this trivialization, then in the same trivialization $\mathfrak{D} := d + \frac{1}{2} d \log(\mathfrak{s}_U)$. By construction $\mathfrak{D}$ has logarithmic poles along R and residue $1/2$. Therefore the tensor product connection $\tau(\nabla) \otimes \mathrm{id} + \mathrm{id} \otimes \mathfrak{D}$ is a holomorphic connection on $N \otimes p^* \zeta^{\otimes -(n-1)} = L_{|\overline{C}_{h_V(\theta)} \times \{\theta\}}$.

The whole construction makes sense relatively over $T^\vee_V \boldsymbol{\mathcal{B}un}$ and so by varying $\theta \in T^\vee_V \boldsymbol{\mathcal{B}un}$ we get a relative holomorphic connection D on L, uniquely characterized by the property that its restriction to the slice $\overline{C}_{h_V(\theta)} \times \{\theta\}$ is equal to $\tau(\nabla) \otimes \mathrm{id} + \mathrm{id} \otimes \mathfrak{D}$.

Now we can use (L, D) as an input for the GL_1-version of the geometric Langlands correspondence. More precisely, applying the construction from Example 2.5 to the local system (L, D) and along the smooth fibers of π_V we get a relative rank one local system $(\widetilde{L}, \widetilde{\nabla})$ on the part of

$$\mathbf{Pic}\left(\overline{\mathcal{C}}_V / C \times T^\vee_V \boldsymbol{\mathcal{B}un}\right) = \mathbf{Higgs} \times_B T^\vee_V \boldsymbol{\mathcal{B}un}$$

sitting over $B - \Delta$. If we push this forward $(\widetilde{L}, \widetilde{\nabla})$ to $\mathbf{Higgs}$ we get a meromorphic local system (M, δ), where M is a holomorphic vector bundle on $\mathbf{Higgs}$, and δ is a meromorphic connection on M acting along the fibers of $\mathsf{h} : \mathbf{Higgs} \to B$.

The bundle M can be described explicitly. Let $(E, \psi) \in \mathbf{Higgs}$ be any point, then the fiber of M at (E, ψ) is given by

$$M_{(E,\psi)} = \bigoplus_{\substack{\theta \in T^\vee_V \boldsymbol{\mathcal{B}un} \\ h(\theta) = h(\psi)}} \mathscr{P}_{(L_\theta, N_\psi)},$$

where $\mathscr{P} \to \boldsymbol{\mathcal{P}ic}^0(\overline{C}_{h(\psi)}) \times \mathbf{Pic}(\overline{C}_{h(\psi)})$ is the standard Poincare line bundle, L_θ is the restriction of L to the slice $\overline{C}_{h(\theta)} \times \{\theta\}$, and $N_\psi \in \mathbf{Pic}(\overline{C}_{h(\psi)})$ is the line bundle corresponding to (E, ψ) via the spectral correspondence.

Remark 3.2. • *The above approach will give rise to a geometric Langlands correspondence if we can find a way to convert the* ${}_{\mathrm{ab}}H^i$*-eigen module* (M, δ) *on* $\mathbf{Higgs}$ *to an* H^i*-eigen module on* $\mathbf{Bun}$*. To do this we can take several routes: we can either average the* (M, δ) *over all* $\theta \in T^\vee_V \boldsymbol{\mathcal{B}un}$*, or use deformation quantization as in [Ari02, Ari05], or use Simpson's non-abelian Hodge theory [Sim91, Sim92, Sim97] as we will do in the remainder of the paper.*

• *To set up the previous construction for an arbitrary group* G *we need to establish a duality between* $\boldsymbol{\mathcal{H}iggs}_0$ *and* ${}^L\mathbf{Higgs}$*. This was done in [DP06] and we will review it in section 4.*

• *The correspondences* $\mathbf{Higgs}^i$ *and* ${}_{\mathrm{ab}}\mathbf{Higgs}^i$ *can be related geometrically:* ${}_{\mathrm{ab}}\mathbf{Higgs}^i$ *is the total space of the relative conormal bundle of* $\mathbf{Higgs}^i \subset \mathbf{Bun} \times \mathbf{Bun} \times C$ *over* C*.*

Chapter 4

The classical limit

In this chapter we review the construction of the Fourier-Mukai functor FM appearing in the decomposion (2.3) or equivalently in step (2) of the six step process in chapter 5.1.

4.1 The classical limit conjecture

Fix a curve C and groups G, ${}^{L}G$. The moduli stacks of Higgs bundles arise naturally in an interesting limiting case of conjecture (GLC): the so called *classical limit*.

On the local system side of (GLC) the passage to the limit is based on Deligne's notion of a z-connection [Sim97] which interpolates between the notions of a local system and a Higgs bundle. A z-connection is by definition a triple (V, ∇, z), where $\pi : V \to C$ is a principal G-bundle on C, $z \in \mathbb{C}$ is a complex number, and ∇ is a differential operator satisfying the Leibnitz rule up to a factor of z. Equivalently, ∇ is a z-splitting of the Atiyah sequence for V:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{ad}(V) \arrow[r] & \mathcal{E}(V) \arrow[r, "\sigma"] & T_C \arrow[r] \arrow[l, bend left, "\nabla"] & 0. \end{tikzcd}$$

Here $\mathrm{ad}(V) = V \times_{\mathrm{ad}} \mathfrak{g}$ is the adjoint bundle of V, $\mathcal{E}(V) = (\pi_* T_V)^G$ is the Atiyah algebra of V, $\sigma : \mathcal{E}(V) \to T_C$ is the map induced from $d\pi : T_V \to \pi^* T_C$, and ∇ is a map of vector bundles satisfying $\sigma \circ \nabla = z \cdot \mathrm{id}_{T_C}$.

When $z = 1$ a z-connection is just an ordinary connection. More generally, when $z \neq 0$, rescalling a z-connection by z^{-1} gives again an ordinary connection. However for $z = 0$ a z-connection is a Higgs bundle. In this sense the z-connections give us a way of deforming a connection into a Higgs bundle. In particular the moduli space of z-connections can be viewed as a geometric 1-parameter deformation of $\boldsymbol{\mathcal{L}oc}$ parametrized by the z-line and such that the fiber over $z = 1$ is $\boldsymbol{\mathcal{L}oc}$, while the fiber over $z = 0$ is $\boldsymbol{\mathcal{H}iggs}_0$ the stack of Higgs bundles with trivial first Chern class. Using this picture we can view the derived category $D_{\mathrm{coh}}(\boldsymbol{\mathcal{H}iggs}_0, \mathcal{O})$ as the $z \to 0$ limit of the category $D_{\mathrm{coh}}(\boldsymbol{\mathcal{L}oc}, \mathcal{O})$.

On the ${}^{L}\mathbf{Bun}$ side the limit comes from an algebraic deformation of the sheaf of rings $\mathcal{D}$ of differential operators on ${}^{L}\boldsymbol{\mathcal{B}un}$. More precisely $\mathcal{D}$ is a sheaf of rings which is filtered by the filtration by orders of differential operators. Applying the Rees construction [Ree56, Ger66, Sim91] to this filtration we get a flat deformation of $\mathcal{D}$ parametrized by the z-line and such that the fiber of this deformation at $z = 1$ is $\mathcal{D}$ and the fiber at $z = 0$ is the symmetric algebra $S^\bullet T = \operatorname{gr}\mathcal{D}$ of the tangent bundle of ${}^L\mathbf{Bun}$. Passing to categories of modules we obtain an interpretation of $D_{\mathrm{coh}}({}^L\mathbf{Bun}, S^\bullet T)$ as the $z \to 0$ limit of $D_{\mathrm{coh}}({}^L\mathbf{Bun}, \mathcal{D})$. Since ${}^L\mathbf{Higgs}$ is the cotangent stack of ${}^L\mathbf{Bun}$ the category $D_{\mathrm{coh}}({}^L\mathbf{Bun}, S^\bullet T)$ will be equivalent to $D_{\mathrm{coh}}({}^L\mathbf{Higgs}, \mathcal{O})$ and so we get a limit version of the conjecture (GLC) which predicts the existence of a canonical equivalence of categories

$$\text{(clGLC}_0\text{)} \qquad \boxed{\mathfrak{cl}_0 : D_{\mathrm{coh}}(\boldsymbol{\mathcal{H}iggs}_0, \mathcal{O}) \xrightarrow{\cong} D_{\mathrm{coh}}({}^L\mathbf{Higgs}, \mathcal{O})}$$

which again sends structure sheaves of points to eigensheaves of a classical limit version of the Hecke functors.

We also expect that the equivalence $\mathfrak{cl}_0$ extends to an equivalence

$$\text{(clGLC)} \qquad \boxed{\mathfrak{cl} : D_{\mathrm{coh}}(\boldsymbol{\mathcal{H}iggs}, \mathcal{O}) \xrightarrow{\cong} D_{\mathrm{coh}}({}^L\boldsymbol{\mathcal{H}iggs}, \mathcal{O})}$$

which again sends structure sheaves of points to eigensheaves of a classical limit version of the Hecke functors.

The construction of the classical limit of the Hecke functors is somewhat involved. The first step is to notice that the spectral correspondence (see e.g. [Don95]) gives an equivalence of the abelian category of quasi-coherent sheaves on ${}^L\mathbf{Higgs} = \operatorname{tot}\left(T^\vee_{{}^L\mathbf{Bun}}\right)$ with the abelian category of Ω^1-valued quasi-coherent Higgs sheaves on ${}^L\mathbf{Bun}$, that is with the abelian category of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a quasi-coherent sheaf on ${}^L\mathbf{Bun}$ and $\varphi : \mathcal{E} \to \mathcal{E} \otimes \Omega^1$ is an $\mathcal{O}$-linear map satisfying $\varphi \wedge \varphi = 0$. In particular we can view $D_{\mathrm{coh}}({}^L\mathbf{Higgs}, \mathcal{O})$ as a full subcategory of the derived category $D\mathsf{Higgs}({}^L\mathbf{Bun})$ of quasi-coherent Higgs sheaves on ${}^L\mathbf{Bun}$.

Since the Hecke functors were defined as integral transforms for $\mathcal{D}$ modules on ${}^L\mathbf{Bun}$ we can use the Higgs sheaf interpretation of $D_{\mathrm{coh}}({}^L\mathbf{Higgs}, \mathcal{O})$ and define the classical limit of the Hecke functor as an integral transform for Higgs sheaves. There is one missing ingredient for such a definition however: we must specify a specialization of the kernel $\mathcal{D}$-module ${}^LI^{\mu,x}$ to a quasi-coherent Higgs sheaf on the Hecke stack ${}^L\mathbf{Hecke}^{\mu,x}$. For this one can use the same process that we used to define the classical limit of the right hand side of (GLC), namely the Rees deformation of a filtered object to its associated graded.

More precisely, suppose that we can find a quasi-coherent sheaf ${}^L\mathbb{I}^{\mu,x}$ on ${}^L\boldsymbol{\mathcal{H}}\mathbf{ecke}^{\mu,x} \times \mathbb{C}$ so that:

- ${}^L\mathbb{I}^{\mu,x}$ is a module over the Rees sheaf for the sheaf of differential operators on ${}^L\boldsymbol{\mathcal{H}}\mathbf{ecke}^{\mu,x}$;
- The restriction of ${}^L\mathbb{I}^{\mu,x}$ to ${}^L\boldsymbol{\mathcal{H}}\mathbf{ecke}^{\mu,x} \times \{1\}$ is isomorphic to ${}^LI^{\mu,x}$ as a $\mathcal{D}$-module.

Typically such an extension ${}^L\mathbb{I}^{\mu,x}$ of ${}^LI^{\mu,x}$ will come from choosing a good filtration on ${}^LI^{\mu,x}$, since for any good filtration we can take ${}^L\mathbb{I}^{\mu,x}$ to be the Rees module associated with the filtration. Thus one strategy for finding the classical limit will be to equip ${}^LI^{\mu,x}$ with a functorial good filtration.

The restriction ${}^L\mathfrak{I}^{\mu,x} := {}^L\mathbb{I}^{\mu,x}/z \cdot {}^L\mathbb{I}^{\mu,x}$ of ${}^L\mathbb{I}^{\mu,x}$ to ${}^L\boldsymbol{\mathcal{H}}\mathbf{ecke}^{\mu,x} \times \{0\}$ is then naturally a module over the associated graded ring $\mathsf{gr}\,\mathcal{D}$ ($\cong \mathrm{Sym}^\bullet T$), i.e. it is a Higgs sheaf on ${}^L\boldsymbol{\mathcal{H}}\mathbf{ecke}^{\mu,x}$ which can be viewed as the *classical limit* Hecke kernel. This immediately gives rise to a *classical limit* Hecke functor

$$
{}^L\mathfrak{H}^{\mu,x} : \quad
\begin{array}{ccc}
D_{\mathrm{coh}}\left({}^L\boldsymbol{\mathcal{H}iggs}, \mathcal{O}\right) & \longrightarrow & D_{\mathrm{coh}}\left({}^L\boldsymbol{\mathcal{H}iggs}, \mathcal{O}\right) \\
\cap & & \cap \\
D\mathsf{Higgs}\left({}^L\boldsymbol{\mathcal{B}un}\right) & \longrightarrow & D\mathsf{Higgs}\left({}^L\boldsymbol{\mathcal{B}un}\right) \\
 & & \\
(\mathcal{E},\varphi) & \longmapsto & q_!^{\mu,x}\left((p^{\mu,x})^*(\mathcal{E},\varphi)\otimes({}^L\mathfrak{I}^{\mu,x})\right),
\end{array}
\tag{4.1}
$$

In general one expects that the correct filtration on the intersection cohomology sheaf ${}^LI^{\mu,x}$ comes from mixed Hodge theory. By definition ${}^LI^{\mu,x}$ is the middle perversity extension of the trivial rank one local system from the smooth locus of ${}^L\mathbf{Bun}$. In particular, Saito's theory [Sai90] implies that ${}^LI^{\mu,x}$ has a canonical structure of a mixed Hodge module. It is natural to expect that the Hodge filtration of this mixed Hodge module will provide the correct classical limit of ${}^LI^{\mu,x}$. This is trivially the case for minuscule μ's and suggests the following

Definition 4.1. *The* **classical limit Hecke kernel** ${}^L\mathfrak{I}^{\mu,x}$ *is the associated graded of* ${}^LI^{\mu,x}$ *with respect to the Hodge filtration in Saito's mixed Hodge module structure on* ${}^LI^{\mu,x}$.

Using the classical limit Hecke kernel we can now define the classical limit Hecke functor by the formula (4.1).

In the next section we explain the main result of [DP06], which asserts that away from the discriminant in the Hitchin base, there exists a Fourier-Mukai kernel on $\boldsymbol{\mathcal{H}iggs} \times ({}^L\boldsymbol{\mathcal{H}iggs})$ which gives an equivalence of $D_{\mathrm{coh}}(\boldsymbol{\mathcal{H}iggs}, \mathcal{O})$ with $D_{\mathrm{coh}}({}^L\boldsymbol{\mathcal{H}iggs}, \mathcal{O})$ which transforms structure sheaves of points to eigen sheaves for the abelianized Hecke correspondences. This statement is in fact equivalent to the classical limit conjecture (clGLC) due to a forthcoming work of Arinkin and Bezrukavnikov who establish an isomorphism between the commutative algebra of classical limit Hecke functors and the commutative algebra of abelianized Hecke functors.

4.2 Duality of Hitchin systems

The classical limit conjecture (clGLC) can be viewed as a self duality of Hitchin's integrable system: Hitchin's system for a complex reductive Lie group G is dual to Hitchin's system for the Langlands dual group ${}^LG$. This statement can be interpreted at several levels:

- First, a choice of an invariant bilinear pairing on the Lie algebra $\mathfrak{g}$, induces an isomorphism between the bases of the Hitchin systems for G and ${}^LG$, interchanging the discriminant divisors.
- The general fiber of the neutral connected component $\underline{\mathbf{Higgs}}_0$ of Hitchin's system for G is an abelian variety. We show that it is dual to the corresponding fiber of the neutral connected component ${}^L\underline{\mathbf{Higgs}}_0$ of the Hitchin system for ${}^LG$.

- The non-neutral connected components $\underline{\mathbf{Higgs}}_\alpha$ form torsors over $\underline{\mathbf{Higgs}}_0$. According to the general philosophy of [DP08], these are dual to certain gerbes. In our case, we identify these duals as natural gerbes over ${}^L\underline{\mathbf{Higgs}}_0$. The gerbe $\boldsymbol{\mathcal{H}iggs}$ of G-Higgs bundles was introduced and analyzed in [DG02]. This serves as a universal object: we show that the gerbes involved in the duals of the non-neutral connected components $\underline{\mathbf{Higgs}}_\alpha$ are induced by $\boldsymbol{\mathcal{H}iggs}$.
- More generally, we establish a duality over the complement of the discriminant between the gerbe $\boldsymbol{\mathcal{H}iggs}$ of G-Higgs bundles and the gerbe ${}^L\boldsymbol{\mathcal{H}iggs}$ of ${}^LG$-Higgs bundles, which incorporates all the previous dualities.
- Finally, the duality of the integrable systems lifts to an equivalence of the derived categories of $\boldsymbol{\mathcal{H}iggs}$ and ${}^L\boldsymbol{\mathcal{H}iggs}$. As a corollary we obtain a construction of eigensheaves for the abelianized Hecke operators on Higgs bundles.

To elaborate on these steps somewhat note that the Hitchin base B and the universal cameral cover $\widetilde{\mathscr{C}} \to C \times B$ depend on the group G only through its Lie algebra $\mathfrak{g}$. The choice of a G-invariant bilinear form on $\mathfrak{g}$ determines an isomorphism $\boldsymbol{l} : B \to {}^LB$ between the Hitchin bases for the Langlands-dual algebras $\mathfrak{g}$, ${}^L\mathfrak{g}$. This isomorphsim lifts to an isomorphism ℓ of the corresponding universal cameral covers. (These isomorphisms are unique up to automorphisms of $\widetilde{\mathscr{C}} \to C \times B$: There is a natural action of $\mathbb{C}^\times$ on B which also lifts to an action on $\widetilde{\mathscr{C}} \to C \times B$. The apparent ambiguity we get in the choice of the isomorphisms $\boldsymbol{l}, \ell$ is eliminated by these automorphisms.)

The next step [DP06] is to show that the connected component P_b of the Hitchin fiber $h^{-1}(b)$ over some general $b \in B$ is dual (as a polarized abelian variety) to the connected component ${}^LP_{\boldsymbol{l}(b)}$ of the corresponding fiber for the Langlands-dual system. This is achieved by analyzing the cohomology of three group schemes $\overline{\mathcal{T}} \supset \mathcal{T} \supset \mathcal{T}^0$ over C attached to a group G. The first two of these were introduced in [DG02], where it was shown that $h^{-1}(b)$ is a torsor over $H^1(C, \mathcal{T})$. The third one $\mathcal{T}^0$ is their maximal subgroup scheme all of whose fibers are connected. It was noted in [DG02] that $\overline{\mathcal{T}} = \mathcal{T}$ except when $G = \mathrm{SO}(2r+1)$ for $r \geq 1$. Dually one finds [DP06] that $\mathcal{T} = \mathcal{T}^0$ except for $G = \mathrm{Sp}(r)$, $r \geq 1$. In fact, it turns out that the connected components of $H^1(\mathcal{T}^0)$ and $H^1(\overline{\mathcal{T}})$ are dual to the connected components of $H^1({}^L\overline{\mathcal{T}}), H^1({}^L\mathcal{T}^0)$, and we are able to identify the intermediate objects $H^1(\mathcal{T}), H^1({}^L\mathcal{T})$ with enough precision to deduce that they are indeed dual to each other.

Finally we extend the basic duality to the non-neutral components of the stack of Higgs bundles. The non-canonical isomorphism from non-neutral components of the Hitchin fiber to P_b can result in the absence of a section, i.e. in a non-trivial torsor structure [HT03, DP08]. In general, the duality between a family of abelian varieties $A \to B$ over a base B and its dual family $A^\vee \to B$ is given by a Poincare sheaf which induces a Fourier-Mukai equivalence of derived categories. It is well known [DP08, BB07, BB09] that the Fourier-Mukai transform of an A-torsor A_α is an $\mathcal{O}^*$-gerbe ${}_\alpha A^\vee$ on $A^\vee$. Assume for concretness that G and ${}^LG$ are semisimple. In this case there is indeed a natural stack mapping to $\underline{\mathbf{Higgs}}$, namely the moduli stack $\boldsymbol{\mathcal{H}iggs}$ of semistable G-Higgs bundles on C. Over the locus of stable bundles, the stabilizers of this stack are isomorphic to the center $Z(G)$ of G and so over the stable locus $\boldsymbol{\mathcal{H}iggs}$ is a gerbe. The stack $\boldsymbol{\mathcal{H}iggs}$ was analyzed in [DG02]. From [DP08] we know that every pair $\alpha \in \pi_0(\underline{\mathbf{Higgs}}) = \pi_1(G), \beta \in \pi_1({}^LG) = Z(G)^\wedge$ defines a $U(1)$-gerbe ${}_\beta\underline{\mathbf{Higgs}}_\alpha$ on the connected component $\underline{\mathbf{Higgs}}_\alpha$ and that there is a Fourier-Mukai equivalence of categories $D^b({}_\beta\underline{\mathbf{Higgs}}_\alpha) \cong D^b({}_\alpha{}^L\underline{\mathbf{Higgs}}_\beta)$. In our case we find that all the $U(1)$-gerbes ${}_\beta\underline{\mathbf{Higgs}}_\alpha$ are induced from the single $Z(G)$-gerbe $\boldsymbol{\mathcal{H}iggs}$, restricted to component $\underline{\mathbf{Higgs}}_\alpha$, via the homomorphisms $\beta : Z(G) \to U(1)$. These results culminate in [DP06, Theorem B], which gives a duality between the Higgs gerbes $\boldsymbol{\mathcal{H}iggs}$ and ${}^L\boldsymbol{\mathcal{H}iggs}$. The key to the proof is our ability to move freely among the components of $\boldsymbol{\mathcal{H}iggs}$ via the abelianized Hecke correspondences.

Chapter 5

Non-abelian Hodge theory

5.1 Results from non-abelian Hodge theory

Non Abelian Hodge theory, as developed by Donaldson, Hitchin, Corlette, Simpson [Don87, Hit87a, Cor88, Sim92, Cor93, Sim97], and many others, establishes under appropriate assumptions the equivalence of local systems and Higgs bundles. A richer object (harmonic bundle or twistor structure) is introduced, which specializes to both local systems and Higgs bundles. This is closely related to Deligne's notion of a z-connection (see chapter 4.1): at $z = 1$ we have ordinary connections (or local systems), while at $z = 0$ we have Higgs bundles. Depending on the exact context, these specialization maps are shown to be diffeomorphisms or categorical equivalences. Originally Corlette and Simpson proved the non-abelian Hodge theorem for projective manifolds:

Theorem [Cor88, Sim92, Cor93, Sim97] *Let $(X, \mathcal{O}_X(1))$ be a smooth complex projective variety. Then there is a natural equivalence of dg $\otimes$-categories:*

$$\mathsf{nah}_X : \begin{pmatrix} \text{finite rank } \mathbb{C}\text{-local} \\ \text{systems on } X \end{pmatrix} \longrightarrow \begin{pmatrix} \text{finite rank } \mathcal{O}_X(1)\text{-semistable} \\ \text{Higgs bundles on } X \text{ with} \\ ch_1 = 0 \text{ and } ch_2 = 0 \end{pmatrix}$$

Remark 5.1. *(a) Here by a* Higgs bundle *we mean a pair (E, θ) where E is a vector bundle on X, and $\theta : E \to E \otimes \Omega^1_X$ is an $\mathcal{O}_X$-linear map satisfying $\theta \wedge \theta = 0$. A Higgs bundle (E, θ) is $\mathcal{O}_X(1)$-semistable if for every θ-invariant subsheaf $\mathcal{F} \subset E$ we have*

$$\chi(\mathcal{F} \otimes \mathcal{O}(n))/\operatorname{rk}(\mathcal{F}) \leq \chi(E \otimes \mathcal{O}(n))/\operatorname{rk}(E), \text{ for } n \gg 0.$$

(b) We can also consider Higgs sheaves. These are by definition pairs $(\mathcal{F}, \theta)$ where $\mathcal{F}$ is a coherent or quasi-coherent sheaf on X, and $\theta : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$ is an $\mathcal{O}_X$-linear map satisfying $\theta \wedge \theta = 0$. We will write $\mathsf{Higgs}_{\text{qcoh}}(X)$ (respectively $\mathsf{Higgs}(X)$) for the abelian category of Ω^1_X-valued Higgs sheaves $(\mathcal{F}, \theta)$ with quasi-cherent (respectively coherent) $\mathcal{F}$.

In our considerations we will make a frequent use of a special version of the abelianization procedure discussed in chapter 3 which is specifically adapted to work with Higgs sheaves with coefficients in bundles of holomorphic 1-forms:

The spectral correspondence for Ω^1-valued Higgs sheaves:
Consider the total space $Y = \mathrm{tot}(\Omega^1_X)$ of the cotangent bundle to X and let $\pi : Y \to X$ be the natural projection. The projection π is an affine map which induces equivalences of abelian categories

$$\pi_\sharp : \mathsf{QCoh}(Y) \xrightarrow{\cong} \mathsf{Higgs}_{\mathrm{qcoh}}(X) \tag{5.1}$$

which also restricts to an equivalence

$$\pi_\sharp^{\mathrm{fin}/X} : \mathsf{Coh}^{\mathrm{fin}/X}(Y) \xrightarrow{\cong} \mathsf{Higgs}(X). \tag{5.2}$$

Here $\mathsf{Coh}^{\mathrm{fin}/X}(Y)$ denotes the category of all coherent sheaves proper over X (and hence finite.

Explicitly $\pi_\sharp$ sends a sheaf $\mathcal{E}$ on Y to the pair $(\mathcal{F}, \theta)$ where $\mathcal{F} = \pi_*\mathcal{E}$, $\theta : \mathcal{F} \to \mathcal{F}$ is the pushforward of the natural map $(\bullet) \otimes \lambda : \mathcal{E} \to \mathcal{E} \otimes \pi^*\Omega^1_X$, and $\lambda \in \Gamma(Y, \pi^*\Omega^1_X)$ is the tautological section.

The functor $\pi_\sharp$ and its restriction $\pi_\sharp^{\mathrm{fin}/X}$ have natural inverces:

$$\begin{tikzcd}
\mathsf{Higgs}_{\mathrm{qcoh}}(X) \arrow[r, "\mathsf{SC}_X"] & \mathsf{QCoh}(Y) \\
\mathsf{Higgs}(X) \arrow[u, phantom, "\cup"] \arrow[r, "\mathsf{SC}_X^{\mathrm{fin}/X}"'] & \mathsf{Coh}^{\mathrm{fin}/X}(Y) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

called the **spectral construction functors**.

The spectral construction functor SC_X is built in two steps:

(i) Given a Higgs sheaf $(\mathcal{F}, \theta)$ use the contraction with the Higgs field

$$(\theta \llcorner \bullet) : T_X \otimes \mathcal{F} \to \mathcal{F}$$

to endow $\mathcal{F}$ with a structure of a module over $S^\bullet T_X$.

(ii) Use the fact that Y is the spectrum of $S^\bullet T_X$ over $X4$ and apply the $(\bullet)^\sim$ functor to the $S^\bullet T_X$-module $\mathcal{F}$ to get the sheaf $\mathsf{SC}_X(\mathcal{F}, \theta)$.

For quasi-projective varieties, the one dimensional analogue of the Corlette-Simpson theorem was settled by Simpson twenty years ago [Sim90]. The open case in higher dimension had to await the recent breakthroughs by Biquard [Biq97], Jost-Yang-Zuo [JYZ07], Sabbah [Sab05], and especially Mochizuki [Moc06, Moc09, Moc07a, Moc07b]. This higher dimensional theory produces an equivalence of parabolic local systems and parabolic Higgs bundles, quite analogous to what is obtained in the compact case. Mochizuki is able to prove a version of the non-abelian Hodge correspondence which allows for singularities of the objects involved:

Theorem [Moc06, Moc09] *Let $(X, \mathcal{O}_X(1))$ be a smooth complex projective variety and let $D \subset X$ be an effective divisor. Suppose that we have a closed subvariety $Z \subset X$ of codimension ≥ 3, such that $X - Z$ is smooth and $D - Z$ is a normal crossing divisor.*

Then there is a canonical equivalence of dg $\otimes$-categories:

$$\mathsf{nah}_{X,D} : \left(\begin{array}{l} \textit{finite rank tame} \\ \textit{parabolic } \mathbb{C}\textit{-local} \\ \textit{systems on } (X, D) \end{array} \right) \longrightarrow \left(\begin{array}{l} \textit{finite rank locally abelian} \\ \textit{tame parabolic Higgs bun-} \\ \textit{dles on } (X, D) \textit{ which are} \\ \mathcal{O}_X(1)\textit{-semistable and satisfy} \\ \textit{parch}_1 = 0 \textit{ and } \textit{parch}_2 = 0 \end{array} \right)$$

Mochizuki requires three basic ingredients for this theorem:

(1) a good compactification, which is smooth and where the boundary is a divisor with normal crossings away from codimension 3;

(2) a local condition: tameness (the Higgs field is allowed to have at most logarithmic poles along D) and compatibility of filtrations (the parabolic structure is locally isomorphic to a direct sum of rank one objects); and

(3) a global condition: vanishing of parabolic Chern classes.

A feature of the non-abelian Hodge correspondence that is specific to the open case is captured in another result of Mochizuki:

Theorem [Moc07a, Moc07b] *Let U be a quasi-projective variety and suppose U has two compactifications*

$$\begin{tikzcd} X & & Y \\ & U \arrow[ul, hook', "\phi"] \arrow[ur, hook, "\psi"'] & \end{tikzcd}$$

where:

- *X, Y are projective and irreducible;*
- *X is smooth and $X - U$ is a normal crossing divisor away from codimension 3;*

Then the restriction from X to U followed by the middle perversity extension from U to Y gives an equivalence of abelian categories:

$$\phi_{*!} \circ \psi^* : \left(\begin{array}{l} \textit{irreducible tame} \\ \textit{parabolic } \mathbb{C}\textit{-local} \\ \textit{systems on } (X, D) \end{array} \right) \longrightarrow \left(\begin{array}{l} \textit{simple } \mathcal{D}\textit{-modules on } Y \textit{ which} \\ \textit{are smooth on } U \end{array} \right)$$

5.2 Using non-abelian Hodge theory

As we mentioned before non-abelian Hodge theory provides a natural approach to constructing the geometric Langlands correspondence $\mathfrak{c}$. The relevance of non-abelian Hodge theory to the problem is already implicit in the work of Beilinson-Drinfeld [BD03] on quantization of Hitchin hamiltonians, in the work of Arinkin [Ari02, Ari08] on the quasi-classical version of the geometric Langlands conjecture, and it the work of Bezrukavnikov-Braverman [BB07] on the Fourier-Mukai interpretation of the correspondence in positive characteristic. The non-abelian Hodge theory approach was brought in the spotlight in the mirror symmetry work of Hausel-Thaddeus [HT03] and several key features of the approach were worked out in the ground breaking work of Kapustin-Witten [KW06] on gauge theory/sigma model duality, in the work of Frenkel-Witten [FW08] on endoscopy, and in our own work [DP06] on the classical limit of the Geometric Langlands Conjecture.

The possibility suggested by these works is that the known (see [DP06] and the discussion in chapter 4) eigensheaf of the abelianized Heckes, which is a Higgs-type object $(\mathcal{E}, \varphi)$, extends by non abelian Hodge theory to a twistor eigensheaf on ${}^{L}\mathbf{Bun}$. The original Higgs sheaf appears at $z = 0$, while at the opposite end $z = 1$ we can expect to find precisely the Hecke eigensheaf postulated by the (GLC).

Recall (see section 2.2 formula 2.3) that our strategy here is to try and decompose the (GLC) map $\mathfrak{c}$ as a composition of two quantization maps $\mathsf{quant}_{\mathbf{Bun}}$, quant_{C} and a Fourier-Mukai transform FM.

As a warmup let us describe these ingredients for $G = GL_n(\mathbb{C}) \cong {}^{L}G$. The first step is to note that every rank n local system $\mathbb{V}$ on C is converted through the non-abelian Hodge correspondence on C into a semi-stable Higgs bundle $(E, \theta) := \mathsf{nah}_C(\mathbb{V})$ of rank n and degree zero on C. On the other hand deformation theory [Fal93, BL94] allows us to identify moduli stack $\boldsymbol{\mathcal{H}iggs}$ (respectively $\mathbf{Higgs}$) with the cotangent stack $T^{\vee}\boldsymbol{\mathcal{B}un}$ (respectively $T^{\vee}\mathbf{Bun}$). Therefore we can view $\mathsf{nah}_C(\mathbb{V})$ as a point in $T^{\vee}\boldsymbol{\mathcal{B}un}$ and hence define the dequantization map[1] quant_C^{-1} by setting

$$\mathsf{quant}_C^{-1}(\mathbb{V}) = \mathcal{O}_{\mathsf{nah}_C(\mathbb{V})},$$

where $\mathcal{O}_{\mathsf{nah}_C(\mathbb{V})}$ denotes the sky-scraper sheaf on $T^{\vee}\boldsymbol{\mathcal{B}un}$ supported at the point $\mathsf{nah}_C(\mathbb{V}) \in T^{\vee}\boldsymbol{\mathcal{B}un}_0 \subset T^{\vee}\boldsymbol{\mathcal{B}un}$.

The Fourier-Mukai transform appearing as the middle term of the formula 2.3 is the abelianized version of (GLC) that we discussed in section 4.2. In the case of $GL_n(\mathbb{C})$ it has a transparent geometric interpretation. In this case the Hitchin base can be described explicitly as

$$B = H^0(\omega_C) \oplus H^0(\omega_C^{\otimes 2}) \oplus \cdots \oplus H^0(\omega_C^{\otimes n}),$$

and the Hitchin map on $T^{\vee}\boldsymbol{\mathcal{B}un}$ sends a Higgs field (E, θ) to the coefficients of the characteristic polynomial $\det(\lambda \,\mathrm{id}_E - \theta)$. as explained before the space B parametrizes cameral covers of C but in this case it also parametrizes n-sheeted spectral covers contained in $\mathrm{tot}(T^{\vee}C)$: for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in B$, the associated spectral cover is the curve

$$\overline{C}_\alpha : \lambda^n + \pi_C^*\alpha_1\lambda^{n-1} + \cdots + \pi_C^*\alpha_{n-1}\lambda + \pi_C^*\alpha_n = 0,$$

[1] The notation here is meant to suggest that the smooth $\mathcal{D}$-module $\mathbb{V}$ on C should be viewed as a quantization of the spectral sheaf on $\mathrm{tot}(T^{\vee}C)$ corresponding to the Higgs bundle (E, θ).

where $\pi_C : \mathrm{tot}(T^\vee C \to C$ is the natural projection, and $\lambda \in H^0(\mathrm{tot}(T^\vee C, \pi_C^*\omega_C)$ is the tautological section. We will write $p_\alpha : \overline{C}_\alpha \to C$ for the restriction of π_C to $\overline{C}_\alpha$.

The fiber $h^{-1}(\alpha)$ of the Hitchin map $h : T^\vee \boldsymbol{\mathcal{B}un} \to B$ (respectively the fiber $\mathsf{h}^{-1}(\alpha)$ of the Hitchin map $\mathsf{h} : T^\vee \mathbf{Bun} \to B$) is the moduli stack (respectively the rigidified moduli spatck) of pure one dimensional sheaves on $\mathrm{tot}(T^\vee C)$ which are supported on $\overline{C}_\alpha$ and are finite of degree n over C. In particular when $\overline{C}_\alpha$ is smooth we get that $h^{-1}(\alpha) = \boldsymbol{\mathcal{P}ic}(\overline{C}_\alpha)$ is the stack of line bundles on $\overline{C}_\alpha$, and that $\mathsf{h}^{-1}(\alpha) = \mathbf{Pic}(\overline{C}_\alpha)$ is the Picard variety of $\overline{C}_\alpha$.

In addition if we choose an integer d and write $\boldsymbol{\mathcal{B}un}_d$ for the stack of vector bundles of degree d on C, then we have that

$$\begin{pmatrix}\text{the fiber of } h_{|T^\vee \boldsymbol{\mathcal{B}un}_d} \\ \text{over } \alpha \in B\end{pmatrix} = \boldsymbol{\mathcal{P}ic}^{d-\mathsf{d}_n}(\overline{C}_\alpha), \tag{5.3}$$

where $\mathsf{d}_n = n(n-1)(g-1)$. Recall also that for smooth curves we have a Poincare sheaf $\mathscr{P}_\alpha \to \boldsymbol{\mathcal{P}ic}(\overline{C}_\alpha) \times \boldsymbol{\mathcal{P}ic}(\overline{C}_\alpha)$ which induces Fourier-Mukai equivalences

$$\begin{array}{ccc} D_{\mathrm{coh}}(\boldsymbol{\mathcal{P}ic}(\overline{C}_\alpha), \mathcal{O}) & \xrightarrow{\mathsf{FM}_\alpha} & D_{\mathrm{coh}}(\boldsymbol{\mathcal{P}ic}(\overline{C}_\alpha), \mathcal{O}) \\ \cup & & \cup \\ D_{\mathrm{coh}}(\boldsymbol{\mathcal{P}ic}^0(\overline{C}_\alpha), \mathcal{O}) & \xrightarrow{\mathsf{FM}_\alpha} & D_{\mathrm{coh}}(\mathbf{Pic}(\overline{C}_\alpha), \mathcal{O}) \end{array}$$

This equivalence makes sense relatively over the open set B^{sm} of B parametrizing smooth spectral covers. In particular if we write $(T^\vee \boldsymbol{\mathcal{B}un})^{\mathrm{sm}} := h^{-1}(B^{\mathrm{sm}})$, $(T^\vee \mathbf{Bun})^{\mathrm{sm}} := \mathsf{h}^{-1}(B^{\mathrm{sm}})$, etc. for the corresponding open sets in the stacks of Higgs bundles, then we can construct a relative Poincare line bundle

$$\mathscr{P} \to (T^\vee \boldsymbol{\mathcal{B}un})^{\mathrm{sm}} \times_{B^{\mathrm{sm}}} (T^\vee \boldsymbol{\mathcal{B}un})^{\mathrm{sm}},$$

which restricts to the standard Poincare line bundle on each fiber. This Poincare line bundle induces a non-trivial Fourier-Mukai auto-equivalence

$$\mathsf{FM} : D_{\mathrm{coh}}((T^\vee \boldsymbol{\mathcal{B}un})^{\mathrm{sm}}, \mathcal{O}) \xrightarrow{\cong} D_{\mathrm{coh}}((T^\vee \boldsymbol{\mathcal{B}un})^{\mathrm{sm}}, \mathcal{O})$$

which in turn restricts to an equivalence

$$D_{\mathrm{coh}}((T^\vee \boldsymbol{\mathcal{B}un}_0)^{\mathrm{sm}}, \mathcal{O}) \xrightarrow{\cong} D_{\mathrm{coh}}((T^\vee \mathbf{Bun})^{\mathrm{sm}}, \mathcal{O}).$$

For the case $G = GL_n(\mathbb{C})$ these equivalences are precisely the classical limit Hecke correspondences (clGLC) and (clGLC_0) discussed in section 4.1. These functors realize for $G = GL_n(\mathbb{C})$ the duality of Hitchin systems discussed in section 4.2 and in particular transport structure sheaves of points into eigensheaves for the abelianized Hecke correspondences.

Remark 5.2. *This setup only gives the classical limit Langlands transforms of Higgs bundles on C for which the associated spectral cover is smooth. The hope here is that the same process will work in full generality but for this one must extend the Poincare sheaf (and prove that it induces a Fourier-Mukai equalivalence) to the full fiber product $T^\vee \boldsymbol{\mathcal{B}un} \times_B T^\vee \boldsymbol{\mathcal{B}un}$. Using some difficult results on the compactified Picard variety (see [EGK02]) this can be done by* *hand for spectral curves that are not too singular (e.g. irreducble with nodes and cusps only). Recently Arinkin [Ari07, Ari10] proved a deep and far reaching strengthening of the self-duality of compactified Jacobians which in particular implies that the Poincare sheaf can be extended to all integral spectral curves.*

Now since $\mathsf{nah}_C(\mathbb{V}) \in T^\vee \boldsymbol{\mathcal{B}un}_0$ it follows that by applying $\mathsf{FM} \circ \mathsf{quant}_C^{-1}$ to $\mathbb{V}$ we will get a complex of coherent sheaves $\mathsf{FM} \circ \mathsf{quant}_C^{-1}(\mathbb{V}) = \mathsf{FM}(\mathcal{O}_{\mathsf{nah}_C(\mathbb{V})}$ on $T^\vee \mathbf{Bun}$. In fact it is easy to see that $\mathsf{FM}(\mathcal{O}_{\mathsf{nah}_C(\mathbb{V})}$ is a sheaf supported on the fiber of the Hitchin map. Indeed, consider $\alpha = h(\mathsf{nah}_C(\mathbb{V}))$, let $\overline{C}_\alpha$ be the corresponding spectral curve, and let

$$\imath_\alpha : \mathrm{Pic}(\overline{C}_\alpha) = \mathsf{h}^{-1}(\alpha) \subset (T^\vee \mathbf{Bun})^{\mathrm{sm}}$$

be the natural inclusion. Since $\mathsf{nah}_C(\mathbb{V}) \in T^\vee \boldsymbol{\mathcal{B}un}_0$ we can view $\mathsf{nah}_C(\mathbb{V})$ as a point[2] in $\boldsymbol{\mathcal{P}ic}^0(\overline{C}_\alpha)$ and hence $\mathsf{nah}_C(\mathbb{V})$ represents a translation ivariant degress zero line bundle $\mathscr{L}_\mathbb{V}$ on $\mathbf{Pic}(\overline{C}_\alpha)$. In these terms the sheaf $\mathsf{FM} \circ \mathsf{quant}_C^{-1}(\mathbb{V})$ is given simply as

$$\mathsf{FM} \circ \mathsf{quant}_C^{-1}(\mathbb{V}) = \imath_{\alpha *}\mathscr{L}_\mathbb{V}.$$

Since by construction $\imath_{\alpha *}\mathscr{L}_\mathbb{V}$ is a coherent sheaf on $T^\vee \mathbf{Bun}$ which is finite over $\mathbf{Bun}$ we can use the projection $\pi : T^\vee \mathbf{Bun} \to \mathbf{Bun}$ to convert this sheaf $\pi_\sharp^{\mathrm{fin}\,/\,\mathbf{Bun}} \imath_{\alpha *}\mathscr{L}_\mathbb{V}$ to a coherent Higgs sheaf on $\mathbf{Bun}$. The discussion in chapter 4 implies that this coherent Higgs sheaf will be an eigensheaf for the classical limit Hecke correspondences.

In view of this it is natural to try and finish the construction of the Langlands duality map $\mathfrak{C}$ by quantizing suitably the Higgs sheaf $\pi_\sharp^{\mathrm{fin}\,/\,\mathbf{Bun}} \imath_{\alpha *}\mathscr{L}_\mathbb{V}$ (in a way compatible with the action of the Hecke operators) to obtain a Hecke eigen $\mathcal{D}$-module $\mathfrak{c}(\mathcal{O}_\mathbb{V})$. The idea is to apply again the non-abelian Hodge correspondence but this time on the stack $\mathbf{Bun}$ and convert this Higgs sheaf to a $\mathcal{D}$-module. Unfortunately for this to work one has to overcome some serious problems:

- The moduli $\mathbf{Bun}$ is not a smooth projective variety. It is a smooth Artin stack of infinite type (but locally of finite type). This complicates things since none of the versions of the non-abelian Hodge theorem work in this generality.

- Even if we can find a modification of the non-abelian Hodge theorem that will apply to $\mathbf{Bun}$, one still nneds to verify that the Higgs sheaf $\pi_\sharp^{\mathrm{fin}\,/\,\mathbf{Bun}} \imath_{\alpha *}\mathscr{L}_\mathbb{V}$ satisfies the non-trivial hypothesis of the theorem. In particular we must check that this Higgs sheaf is stable and has vanishing Chern classes.

Hence we will need a more sophisticated version of the non-abelian Hodge theorem, e.g. Mochizuki's theorem discussed in the previous section. Before we can apply Mochizuki's theory however we will need to recast our problem as a problem about spaces rather than stacks.

[2] Intrinsically, as noted in (5.3), the rank n degree 0 Higgs bundle $\mathsf{nah}_C(\mathbb{V})$ corresponds to a point in $\boldsymbol{\mathcal{P}ic}^{-\mathsf{d}_n}(\overline{C}_\alpha)$. However if we choose a theta characteristic ζ on C, the translation $(\bullet) \otimes \pi^*\zeta$ identifies $\boldsymbol{\mathcal{P}ic}^{-\mathsf{d}_n}(\overline{C}_\alpha)$ with $\boldsymbol{\mathcal{P}ic}^0(\overline{C}_\alpha)$. This process works uniformly over the Hitchin base and in fact works for all groups G. In general it corresponds to a choice of Hitchin section as explained in [DP06].

Recall that even though $\mathbf{Bun}$ is not a stack of finite type, it contains natural open stacks of finite type, namely the moduli stacks

$$\mathbf{Bun}^s \subset \mathbf{Bun}^{ss} \subset \mathbf{Bun}$$

of stable and semi-stable bundles respectively. It is well known that these stacks have coarse moduli-spaces

$$\begin{tikzcd} \mathbf{Bun}^s \arrow[d] & \subset & \mathbf{Bun}^{ss} \arrow[d] \\ \mathfrak{Bun}^s & \subset & \mathfrak{Bun}^{ss} \end{tikzcd}$$

In fact it is known [Ses82] that

- $\mathbf{Bun}^s$ is a smooth quasi-projective variety and the map $\mathbf{Bun}^s \to \mathfrak{Bun}^s$ is an isomorphism;
- for every $d \in \mathbb{Z}$ the component $\mathfrak{Bun}_d^{ss}$ is an irreducible projective algebraic variety which is smooth in codimension four.
- For every $\alpha \in B^{\mathrm{sm}}$ the natural morphism $\pi : T^\vee \mathbf{Bun} \to \mathbf{Bun}$ induces a morphism $\mathbf{Pic}^d(\overline{C}_\alpha) \to \mathbf{Bun}_d$ (given by $\pi((\bullet)\otimes p_\alpha^* \zeta^{(n-1)})$, and if we set $\mathbf{Pic}^d(\overline{C}_\alpha)^s$ for the preimage of $\mathfrak{Bun}_d^s$ we get a (non-proper) morphism of quasi-projective varieties

$$\mathbf{Pic}^d(\overline{C}_\alpha)^s \to \mathfrak{Bun}_d^s. \tag{5.4}$$

Consider now the locus $\mathfrak{Bun}_d^{vs} \subset \mathfrak{Bun}_d^s$ of all very stable vector bundles, i.e. vector bundles that do not admit any nilpotent Higgs fields other than the zero Higgs field [Lau88]. It can be shown [DP09b] that the locus $\mathfrak{Bun}_d^{vs}$ is the largest open subset in $\mathfrak{Bun}_d^s$ over which the morphism (5.4) is proper. In fact, it turns out that (5.4) is finite and flat over $\mathfrak{Bun}_d^{vs}$. This implies that the restriction of the Higgs sheaf $\pi_\sharp^{\mathrm{fin}/\mathbf{Bun}} \imath_{\alpha *}\mathscr{L}_{\mathbb{V}}$ is a Higgs bundle which can alos be computed as the pushforward of the line Higgs bundle[3] $(\mathscr{L}_{\mathbb{V}}, \alpha)$ on $\mathbf{Pic}^d(\overline{C}_\alpha)^s$. In other words our construction produces a vector Higgs bundle on the Zariski open set $\mathfrak{Bun}_d^{vs}$ of the projective algebraic variety $\mathfrak{Bun}_d^{ss}$. This put us in a postion to apply Mochizuki's theorems from the previous section.

Specifically we can look at the configuration of moduli

$$\begin{tikzcd} \mathbf{Bun} & & \widehat{\mathfrak{Bun}^{ss}} \\ & \mathfrak{Bun}^{vs} \arrow[ul, hook', "\phi"] \arrow[ur, hook, "\psi"'] & \end{tikzcd}$$

[3]Here we view

$$\alpha \in H^0(C, \oplus_{i=0}^{n-1} \omega_C^{\otimes i}) = H^0(C, p_{\alpha *}\omega_{\overline{C}_\alpha}) = H^0(\overline{C}_\alpha, \omega_{\overline{C}_\alpha})$$

as a holomorphic one form on $\mathbf{Pic}(\overline{C}_\alpha)$ via the Abel-Jacobi map.

where $\widehat{\mathfrak{Bun}^{ss}} \to \mathfrak{Bun}^{ss}$ is a blow-up such that $\widehat{\mathfrak{Bun}^{ss}} - \mathfrak{Bun}^{vs}$ is a divisor with normal crossings away from codimension three, and such that our vector Higgs bundle on $\mathfrak{Bun}^{vs}$ extends to a tame rabolic Higgs bundle on $\widehat{\mathfrak{Bun}^{ss}}$ satisfying the Mochizuki conditions.

To identify the appropriate blow-up $\widehat{\mathfrak{Bun}^{ss}} \to \mathfrak{Bun}^{ss}$ one can look at the rational map $f_d : \mathbf{Pic}^d(\overline{C}_\alpha) \dashrightarrow \mathbf{Bun}_d^{ss}$ induced from (5.4). Now we can blow-up the source and target of f_d to get a finite morphism $\hat{f}_d : \widehat{\mathbf{Pic}^d}(\overline{C}_\alpha) \to \widehat{\mathfrak{Bun}_d^{ss}}$. If necessary we can blow-up further to make the complement of $\mathfrak{Bun}_d^{vs}$ a divisor with normal crossings. Then $\hat{f}_{d*}(\mathscr{L}_{\mathbb{V}}, \alpha)$ will be a quasi-parabolic Higgs bundle on $\widehat{\mathfrak{Bun}^{ss}}$ with poles along the divisor $D := \widehat{\mathfrak{Bun}_d^{ss}} - \mathfrak{Bun}_d^{vs}$. Next we will have to check that $\hat{f}_{d*}(\mathscr{L}_{\mathbb{V}}, \alpha)$ is tame, locally abelian, and that we can choose the parabolic weights functorially so that the first and second parabolic Chern classes of $\hat{f}_{d*}(\mathscr{L}_{\mathbb{V}}, \alpha)$ vanish.

Once this is accomplished we can apply the quasi-projective non-abelian Hodge correspondence from Mochizuki's theorem to get a parabolic local system

$$(\mathscr{V}, \nabla) = \mathsf{nah}_{\widehat{\mathfrak{Bun}_d^{ss}}, D}\left(\hat{f}_{d*}(\mathscr{L}_{\mathbb{V}}, \alpha)\right)$$

Finally we apply the Mochizuki extension functor to get a $\mathcal{D}$-module $\phi_{*!}\psi^*(\mathscr{V}, \nabla)$ and use functoriality of push-forwards of twistor $\mathcal{D}$-modules to argue that the $\mathcal{D}$-module $\phi_{*!}\psi^*(\mathscr{V}, \nabla)$ has the Hecke eigen property.

The same strategy works in general. The dequantization and Fourier-Mukai functors in this case are essentially the same as above but modified to work for arbitrary groups. Constructing the quantization of a Higgs bundle on **Bun** is still the trickiest part. The situation is essentially non-compact: There is a locus S in the moduli space ${}^L\mathbf{Bun}^s$ of stable bundles along which our Higgs field φ blows up. This can be traced back, essentially, to the difference between the notions of stability for bundles and Higgs bundles. The cotangent bundle $T^\vee({}^L\mathbf{Bun}^s)$ embeds as a Zariski-open in ${}^L\mathbf{Higgs}^s$. If we ignore stability the two are equal: $T^\vee({}^L\mathbf{Bun}) = {}^L\mathbf{Higgs}$. But as moduli of stable objects, there is a locus Un in ${}^L\mathbf{Higgs}$ parametrizing stable Higgs bundles with *unstable* underlying bundle. In order to turn the projection ${}^L\mathbf{Higgs}^s \to {}^L\mathbf{Bun}^s$ into a morphism, Un must be blown up to an exceptional divisor $\widehat{\mathsf{Un}}$. Then the Higgs field part φ of the Hecke eigensheaf $(\mathcal{E}, \varphi)$ on ${}^L\mathbf{Bun}^s$ blows up along the image S of $\widehat{\mathsf{Un}}$.

In current work with C. Simpson [DP09b, DPS09b, DPS09a], we are investigating the possibility of applying non-abelian Hodge theory to the (GLC). The heart of the matter amounts to verification of the Mochizuki conditions: we need to find where the Higgs field blows up, resolve this locus to obtain a normal crossing divisor, lift the objects to this resolution, and verify that the parabolic chern classes of these lifts vanish upstairs. This would provide the crucial third step in the following six step recipe for producing the candidate automorphic sheaf:

$$\begin{tikzcd}
\boxed{G\text{-local system } (V,\nabla) \text{ on } C} \arrow[d, Rightarrow, "(1)"'] \\
\boxed{G\text{-Higgs bundle } (E,\theta) \text{ on } C} \arrow[d, Rightarrow, "(2)"'] \\
\boxed{{}_{\mathrm{ab}}{}^{L}\mathbf{Hecke}\text{-eigensheaf on } {}^{L}\mathbf{Higgs}} \arrow[d, Rightarrow, "(3)"'] \\
\boxed{\begin{array}{l}\text{parabolic Higgs sheaf on } {}^{L}\mathbf{Bun}^{s} \\ \text{satisfying Mochizuki's conditions} \\ \text{(1)-(3)}\end{array}} \arrow[d, Rightarrow, "(4)"'] \\
\boxed{\begin{array}{l}\text{parabolic local system on } {}^{L}\mathbf{Bun}^{s} \\ \text{satisfying Mochizuki's conditions} \\ \text{(1)-(3)}\end{array}} \arrow[d, Rightarrow, "(5)"'] \\
\boxed{\begin{array}{l}\text{ordinary local system on} \\ \text{Zariski open in } {}^{L}\mathbf{Bun}\end{array}} \arrow[d, Rightarrow, "(6)"'] \\
\boxed{\mathcal{D}\text{-module on } {}^{L}\mathbf{Bun}}
\end{tikzcd}$$

Note that all of the other steps in this process are essentially already in place. The functor (1) is given by the Corlette-Simpson non-abelian Hodge correspondence $(E,\theta) = \mathsf{nah}_C(V,\nabla)$ on the smooth compact curve C. The functor (2) sends $(E,\theta) \in \boldsymbol{\mathcal{H}iggs}$ to $\mathsf{FM}(\mathcal{O}_{(E,\theta)})$ where FM is a Fourier-Mukai transform for coherent sheaves on $T^{\vee}\boldsymbol{\mathcal{B}un} = \boldsymbol{\mathcal{H}iggs}$. In fact FM is the integral transform with kernel the Poincare sheaf constructed (away from the discriminant) in [DP06]. This sheaf is supported on the fiber product of the two Hitchin fibrations $h : \boldsymbol{\mathcal{H}iggs}_0 \to B$ and ${}^{L}h : {}^{L}\mathbf{Higgs} \to B$ and we discussed it briefly in section 4.1. The functor (4) is the parabolic non-abelian Hodge correspondence $\mathsf{nah}_{{}^{L}\mathbf{Bun}^{ss},\mathsf{S}}$ of Mochizuki. Here ${}^{L}\mathbf{Bun}^{ss}$ denotes the (rigidified) stack of semistable bundles. Note that here we are applying the first Mochizuki theorem not to a projective variety but to a smooth proper Deligne-Mumford stack with a projective moduli space. In fact Mochizuki's proof [Moc09] works in this generality with no modifications. The functors (5) and (6) are the pullback and middle extension functors applied to the two compactifications ${}^{L}\mathbf{Bun}^{ss} \supset {}^{L}\mathbf{Bun}^{s} \subset {}^{L}\mathbf{Bun}$. In order to conclude that the composition $(6)\circ(5)$ is an equivalence we need a strengthening of Mochizuki's extension theorem which would allow for Y to be an Artin stack which is only locally of finite type [DPS09a].

In the next section we explain some of the issues that one needs to tackle in order to carry out step (3).

Chapter 6

Parabolic Higgs sheaves on the moduli of bundles

To construct the functor (3) we need to convert a translation invariant line bundle $\mathscr{L}$ on the Hitchin fiber into a stable parabolic Higgs sheaf $(\mathcal{E}, \varphi)$ on the moduli of bundles. The strategy is:

- construct a suitable blow-up of the Hitchin fiber which resolves the rational map to ${}^L\mathbf{Bun}^s$;
- pull $\mathscr{L}$ and the taulogical one form on the Hitchin fiber to this blow-up;
- twist with an appropriate combination of the exceptional divisors;
- push-forward the resulting rank one Higgs bundle on the blow-up to ${}^L\mathbf{Bun}^s$ to obtain a quasi-parabolic Higgs sheaf $(\mathcal{E}, \varphi)$ on $({}^L\mathbf{Bun}^s, \mathsf{S})$
- fix parabolic weights for $(\mathcal{E}, \varphi)$ so that $parch_1 = 0$ and $parch_2 = 0$.

In [DP09b, DPS09b] we work out this strategy for $G = GL_2(\mathbb{C})$. The first task here is to understand the divisors $\widehat{\mathsf{Un}}$ and S geometrically.

6.1 Wobbly, shaky, and unstable bundles

A G-bundle E is *very stable* if it has no nonzero nilpotent Higgs fields θ [Lau88]. Very stable bundles are stable [Lau88]. We call a bundle *wobbly* if it is stable but not very stable, and we call a bundle *shaky* if it is in S. A major step towards carrying out our program is the identification of shaky bundles:

Theorem [DP09b] *Let $G = {}^LG = GL_2(\mathbb{C})$. Fix a smooth Hitchin fiber $\mathbf{Higgs}_{\overline{C}}$.*

(a) The rational map $\mathbf{Higgs}_{\overline{C}} \dashrightarrow \mathbf{Bun}^s$ can be resolved to a morphism $\widehat{\mathbf{Higgs}}_{\overline{C}} \to \mathbf{Bun}^s$ by a canonical sequence of blow-ups with smooth centers.

(b) For every translation invariant line bundle $\mathcal{L}$ on $\mathbf{Higgs}_{\overline{C}}$, and for any twist by exceptional divisors of the pullback of $\mathcal{L}$ to $\widehat{\mathbf{Higgs}}_{\overline{C}}$, the polar divisor of the associated quasi-parabolic Higgs sheaf $(\mathcal{E}, \varphi)$ is independent of $\overline{C}$, $\mathscr{L}$, and the twist, and is equal to S.

(c) The shaky bundles are precisely the wobbly ones.

This is in exact agreement with the expected behavior of the Hecke eigensheaf, according to Drinfeld and Laumon [Lau95].

In view of this theorem, the key geometric issue needed for a proof of the GLC along these lines is therefore an analysis of the locus of wobbly bundles and of the sequence of blowups needed to convert it into a normal crossing divisor. For $G = GL_2(\mathbb{C})$ this analysis is carried out in [DP09b].

In specific cases it is possible to work out the moduli spaces, wobbly loci, and Hecke correspondences in great detail. One such case is when the curve is $\mathbb{P}^1$ with n marked points, and the group is $G = GL_2(\mathbb{C})$.

This is an instance of the tamely ramified Geometric Langlands Conjecture, or the Geometric Langlands Conjecture for parabolic local systems and bundles. This natural extension of the GLC is explained beautifully in [Fre08, GW06], and a simple case (elliptic curve with one marked point) is analyzed in [FW08] from a point of view similar to ours. The six step process outlined above applies equally well to the ramified case: in fact, as explained above, our use of non-abelian Hodge theory has the parabolic structures built in even when the initial objects are defined over a compact curve, so there is every reason to expect that our construction should work just as well when the initial object is itself parabolic.

A major surprise is that in the parabolic case, the [DPS09b] characterization of the poles of the parabolic Higgs sheaf $(\mathcal{E}, \varphi)$ on **Bun** needs to be modified. Wobbly bundles are still shaky, but new, non-wobbly components of the shaky locus can arise. These seem to be related to the variation of GIT quotients. In this section we illustrate this new phenomenon in the first non trivial case, $n = 5$. The results will appear in [DPS09b].

There is a large body of work describing the moduli space M_n of semi-stable $GL(2)$ parabolic bundles (or flat $U(2)$ connections) on $\mathbb{P}^1$ with n marked points as well as its cohomology ring, see e.g. [Bau91, Jef94, BR96, BY96]. In several of these references one can find an identification of M_5 as a del Pezzo surface dP_4, the blowup of $\mathbb{P}^2$ at 4 general points. Actually, M_n is not a single object: it depends on the choice of parabolic weights at the n points. For instance [Bau91] if we choose all the parabolic weights to be equal to $1/2$, then the moduli space M_n can be described explicitly as the blow-up of $\mathbb{P}^{n-3}$ at n-points lying on a rational normal curve. The dP_4 description of M_5 holds for the lowest chamber, when the parabolic weights α are positive but small. By working out the GIT picture, we find [DPS09b] that in the case of balanced weights there are actually four chambers, and the corresponding moduli spaces are: dP_4 for $0 < \alpha < \frac{2}{5}$, dP_5 for $\frac{2}{5} < \alpha < \frac{2}{3}$, $\mathbb{P}^2$ for $\frac{2}{3} < \alpha < \frac{4}{5}$, and empty for $\frac{4}{5} < \alpha < 1$. The Hecke correspondence essentially relates the space at level α to the corresponding space at level $1 - \alpha$. The non-abelian Hodge theiry description gives us the flexibility of working in a chamber of our choosing; we choose the self-dual dP_5 chamber at $\frac{2}{5} < \alpha < \frac{3}{5}$.

We find that in the lowest chamber, the shaky locus does agree with the wobbly locus. It consists of the 10 lines on the dP_4, together with 5 additional rational curves, one from each of the five rulings on the dP_4, and all five passing through the same point $p \in dP_4$. In particular, this divisor fails to have normal crossings at p and so is not suitable for the non-abelian Hodge theory approach. As we move to the next chamber, it is precisely the point p that is blown up to produce the dP_5. We check [DPS09b] that the wobbly locus now consists of 15 of the 16 lines on the dP_5 - the proper transforms of the 15 previous components. This is where the new phenomenon first shows up: the shaky locus actually consists of all 16 lines on dP_5. In our self-dual chamber, the shaky divisor has normal crossings, the total space of the Hecke correspondence is smooth, the rational map from the Hitchin fiber to M_5 has a natural resolution producing a parabolic Higgs sheaf of on M_5, and there exist twists and assignments of parabolic weights along the shaky locus that fulfill the Mochizuki conditions from section 5.1. More or less all of this fails on the dP_4 or the $\mathbb{P}^2$ model; in particular, there is no solution to the Mochizuki conditions involving only 15 of the lines. This gives in this case an explicit construction (modulo solving the differential equations inherent in the non-abelian Hodge theory) of the Hecke eigensheaf demanded by the GLC.

6.2 On functoriality in non-abelian Hodge theory

Showing that the $\mathcal{D}$-module we construct on ${}^L\mathbf{Bun}$ in step (6) in section 5.2 is indeed a Hecke eigensheaf depends on having good functorial properties of the non-abelian Hodge correspondence and the Mochizuki extension theorem in the parabolic context. The main task is to define direct images of parabolic objects under fairly general circumstances and to establish their basic properties. The aspects of functoriality needed for our construction in examples are relatively easy to establish, basically because the resolved abelianized Hecke correspondences tend to be finite. Nevertheless, it seems natural to try to establish the functorial behavior in general. We are currently pursuing this in a joint project with C.Simpson [DPS09a].

Through the works of Mochizuki [Moc07a, Moc07b] and Jost-Yang-Zuo [JYZ07] we know that the de Rham cohomology of the $\mathcal{D}$-module extensiom (of the restriction to $X \setminus D$ of) a tame parabolic local system on (X, D) can be calculated directly in terms of L^2 sections with respect to the harmonic metric. In the case of a map to a point, the functoriality we need identifies this also with the cohomology of (the Dolbeault complex associated to) the corresponding parabolic Higgs bundle. Our plan is to establish the general case of functoriality by combining this with an appropriate extension of the techniques of Simpson's [Sim93].

Appendix A

Ind-schemes and smoothness

Recall that an *ind-scheme* is an ind-object $\mathfrak{X}$ in the category $(\mathsf{Sch}/\mathbb{C})$. An ind-scheme is most conveniently described by its functor of points $h_{\mathfrak{X}}$ which by definition is a filtered colimit of closed embeddings of quasi-compact schemes. Specifically

$$h_{\mathfrak{X}} \cong \underset{\alpha\in A}{\underrightarrow{\mathrm{colim}}}\, \mathrm{Hom}(\bullet, X^{\alpha}) : (\mathsf{Sch}/\mathbb{C})^{\mathrm{op}} \to (\mathsf{Set})$$

where A is filtrant poset[1], and for all $\alpha \in A$ we have specified quasi-compact complex schemes X^{α} (not necessarily of finite type), together with closed immersions $X^{\alpha} \subset X^{\alpha'}$, whenever $a \leq \alpha'$. We will write $\mathfrak{X} = \underrightarrow{\mathrm{colim}}_{\alpha\in A} X^{\alpha}$.

An ind-scheme is called *smooth* if it can be represented by a colimit $\mathfrak{X} = \underrightarrow{\mathrm{colim}}_{\alpha\in A} X^{\alpha}$ of smooth X^{α}'s. An ind-scheme is called *formally smooth* if its functor of points satisfies the *infinitesimal lifting property.* This means that for any scheme S and any nilpotent thickening $S \subset T$ the natural map $h_{\mathfrak{X}}(T) \to h_{\mathfrak{X}}(S)$ is surjective. Equivalently [Sha82] an ind-scheme is formally smooth if and only if the local ring at every point of $\mathfrak{X}$ is the completion of the symmetric algebra on the cotangent space at that point. Smooth ind-schemes are automatically formally smooth (see e.g. [Sha82]) but the converse is not true in general.

Two typical examples of formally smooth and reduced ind-schemes (see e.g. [BL94, Tel98] are the algebraic loop group $G(\mathbb{C}((t)))$ associated with a complex reductive group G and the associated affine Grassmanian $\mathcal{G}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$. It turns out that the ind-scheme $\mathcal{G}_G$ is not smooth in fact not even locally smooth. Somewhat surprisingly this is a result in Hodge theory. It is a combination of two theorems concerning the Hodge structures on ind-varieties: The first theorem is a general result of Simpson-Teleman [ST98, Proposition 6.15] according to which a complex ind-variety $\mathfrak{X}$ which satisfies

(a) $\mathfrak{X} = \underrightarrow{\mathrm{colim}}_{n\in\mathbb{N}} X^{n}$ with each X^{n} a complex projective variety;

(b) locally, near every point, $\mathfrak{X}$ is a colimit of smooth analytic varieties;

[1] That is a non-empty set A equipped with a partial order "$\leq$" which is filtrant i.e for every $\alpha, \beta \in A$, there exists $\gamma \in A$ satisfying $\alpha \leq \gamma$ and $\beta \leq \gamma$.

must have a degenerating Hodge-to-de Rham spectral sequence.

Note that the affine flag variety $\mathcal{G}_G$ is a colimit of closures of classical Schubert cells and so automatically satisfies condition (a).

The second theorem is a calculation of Fishel-Grojnowski-Teleman [FGT08] according to which $H^1(\mathcal{G}_G, \Omega^1)$ is isomorphic to the continuous Lie algebra cohomology $H^1(\mathfrak{g}[[t]], \mathfrak{g};\ \mathfrak{g}[[t]]dt)$ for the adjoint action on the coefficients which in turn can be shown [FGT08] to be isomorphic to $\mathbb{C}[[t]]$. Since $H^2(\mathcal{G}_G, \mathbb{C}) = \mathbb{C}$ we see that the Hodge-to-de Rham spectral sequence on $\mathcal{G}_G$ can not degenerate and therefore $\mathcal{G}_G$ will not satisfy (b).

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