Northwestern Gauge Theory & Representation Theory

Note Title 5/10/2009

Goal: describe how gauge theory captures a variety of topics in representation theory.

Plan:
- 2d gauge theory & reps of finite groups
- 3d gauge theory & reps of real & complex Lie groups
- 4d gauge theory & the geometric Langlands program (reps of loop groups, quantum groups etc)

– demonstrate applications of the powerful results & techniques developed in the lectures by Jacob Lurie & Bertrand Toën.

What is gauge theory?

physical theory in which fundamental objects (fields) are connections on principal G-bundles on spacetime M.

Classical gauge theory - describe spaces of connections satisfying classical equations of motion (flatness, or Yang-Mills equations) [won't concern us!]

Quantum gauge theory - a quantum field theory in which we study the collection of all connections (with a weighting given by a Yang-Mills type action) by attaching $\mathbb{C}$-linear invariants.

Topological gauge theory - study "coarse" features depending only on the topology of spacetime M (ie not requiring a metric, conformal structure etc)

<u>Toy model</u> 2d gauge theory with finite gauge group G [variant of Dijkgraaf-Witten theory]

Geometry: to any 0, 1, 2 manifold M we consider the space of gauge fields

$$\mathcal{M}_G(M) = \{ G\text{-bundles on } M \} / \sim$$

[automatically carry flat connections!]

$$= \{ G\text{-Galois covers of } M \} / \sim$$

$\binom{M \text{ connected,}}{\text{pointed}}$ $$= \{ \pi_1(M) \longrightarrow G \} / \text{conjugation}$$

— we'll have to keep track of symmetries / stabilizers:

$\mathcal{M}_G(M)$ is a finite orbifold — finite set of points $\coprod \overset{\bullet}{\circlearrowleft}_{\Gamma_i}$ with finite groups attached.

- $\mathcal{M}_G(pt) = \overset{\bullet}{\circlearrowleft}_G = BG$
- $\mathcal{M}_G(S^1) = \dfrac{G}{G} = \coprod_{\substack{[g] \\ \text{conj classes}}} B Z_G(g)$
- $\mathcal{M}_G(\Sigma_g) = \left\{ \begin{matrix} A_1, \dots A_g \\ B_1, \dots B_g \end{matrix} \in G : \prod [A_i, B_i] = 1 \right\} \Big/ G$

2d gauge theory $\underset{\in\,\mathbb{C}}{Z(M)} = \text{“}\int_{\substack{\text{fields on}\\ M}} e^{-S(\varphi)}\, d\varphi\text{”}$

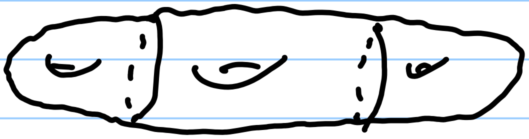
- calculate volume of $\mathrm{Fields}(M)$ with measure $e^{-S(\varphi)}$.

Our case: $Z_G(\Sigma) = \# \mathcal{M}_G(\Sigma)$:

weighted number of G-bundles on Σ

$$= \sum_{\substack{\{P\} \text{ isom}\\ \text{classes}}} \frac{1}{|\mathrm{Stab}\, P|}$$

Locality in Σ: calculate $Z_G(\Sigma)$ via cut & paste



: on manifold with boundary need to specify boundary conditions to get well defined path integral

$\varphi_0 \in \mathrm{Field}(\partial M) \longmapsto$

$$Z(M)(\varphi_0) = \int_{\varphi|_{\partial M} = \varphi_0} e^{-S(\varphi)}\, d\varphi$$

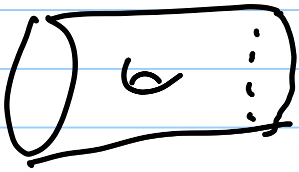
$\Rightarrow Z(M) \in \mathrm{Functions}(\mathrm{Fields}(\partial M)) =: Z(\partial M)$

... ie assign a vector space $Z(\partial M)$ to codimension one manifolds.

: Dually given a function $f \in Fun(Fields(\partial M))$ - ie a weighted collection of boundary conditions - can perform path integral

$$Z(M) : Z(\partial M^{op}) \longrightarrow \mathbb{C}$$

$$f \longmapsto \int f(\varphi|_{\partial M}) e^{-S(\varphi)} d\varphi$$



: More generally have a correspondence

$$\begin{tikzcd} & Fields(M) \arrow[dl, "\pi_{in}"'] \arrow[dr, "\pi_{out}"] & \\ Fields(\partial M_{in}) & & Fields(\partial M_{out}) \end{tikzcd}$$

$$Z(\partial M_{in}) \xrightarrow{Z(M)} Z(\partial M_{out})$$

$$f \longmapsto \left\{ \varphi_{out} \mapsto \int_{\varphi|_{\partial_{out}} = \varphi_{out}} f(\varphi_{in}) e^{-S(\varphi)} d\varphi \right\}$$

$$= \pi_{out *}(\pi_{in}^{*} f \cdot e^{-S})$$

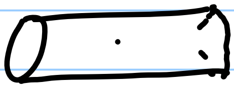
Our finite setting: integrals are finite sums, e^{-S} just means count bundles weighted by automorphisms. Key: functions = measures: can pullback, multiply & pushforward

$$Z_G(S^1) = \text{Functions}\,(\text{Fields}(S^1))$$

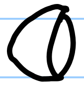
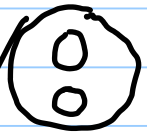
$$= \mathbb{C}\left[\frac{G}{G}\right] = \mathbb{C}[G]^G$$

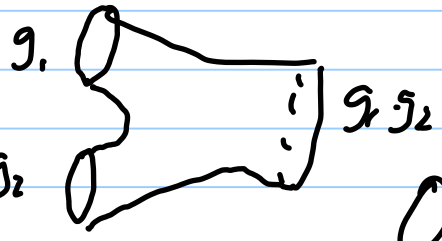
<u>class functions</u> on the group G.

Operations on $Z_G(S^1)$:



$Z_G(\text{cylinder}) = \text{id}$ (physically: Hamiltonian $= 0$ .... ie we're studying quantum mechanics of the vacuum!)

- $Z($  $) = \delta_{\underline{1}} \in \mathbb{C}[G]^G$
- $Z($  $) = \text{eval}_{\underline{1}} : \mathbb{C}G^G \xrightarrow{tr} \mathbb{C}$.
- $Z($  $) =$ convolution $\mathbb{C}G^G \otimes \mathbb{C}G^G \to \mathbb{C}G^G$



: set graph of multiplication map

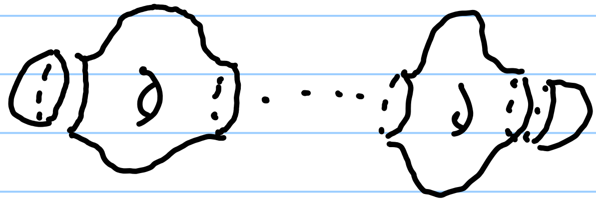
Operator picture: $\mathbb{C}G^G$  $\mathbb{C}G^G$

$\mathbb{C}G^G \subset \mathbb{C}G$ group algebra,
with multiplication induced by pushforward
of measures $\underset{f\quad h}{G\times G} \longrightarrow \underset{f*h}{G}$ $\quad \delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2}$

- In fact $\mathbb{C}G^G$ is a commutative Frobenius algebra: $\mathrm{tr}(f*h)$ is a nondegenerate $= Z($  $)$ inner product
  - & this structure determines $Z_G(\Sigma)$ for all 1,2-manifolds.

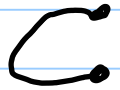
To solve the system: [complete integrability]
Simultaneously diagonalize the local operators ⟷ spectral decomposition

⇒ calculate Spec $\mathbb{C}G^G$
$=$ { homomorphisms $\longrightarrow \mathbb{C}$ }
$=$ { joint eigenvalues } $= \hat{G}$
$=$ { irred characters of G }



$$\rightsquigarrow Z(\Sigma) = \sum_{\substack{\chi \in \text{irred} \\ \text{chars of } G}} \left(\frac{|G|}{\dim\chi}\right)^{2g-2}$$ "Mass formula" [Frobenius]

Codimension 2 :



Before we assigned a vector space (functions on Fields(N)) to an $n-1$ manifold. If N has a boundary $\longrightarrow$ need to fix boundary values on fields to get a vector space!

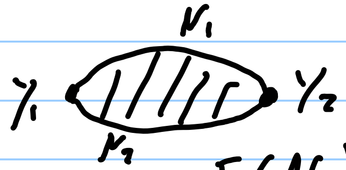
$$\rightsquigarrow Z(N) : \{\text{fields on } \partial N\} \longrightarrow \text{vector spaces}$$

ie $Z(N)$ can be considered as a vector bundle (or more singular $\mathbb{C}$-linear sheaf) on Fields(∂N)

$$\begin{tikzcd} & \text{Fields}(N) \arrow[dl] \arrow[dr] & \\ \text{Fields}(\partial N_{in}) & & \text{Fields}(\partial N_{out}) \end{tikzcd}$$

$$Z(N) : \text{Vect}(\text{Fields}_{in}) \longrightarrow \text{Vect}(\text{Fields}_{out})$$

$$V \longmapsto \pi_{out\,*}\Big(\pi_{in}^{*} V \otimes \underset{\text{weighting}}{\underset{\|}{\text{“}e^{-S}\text{”}}}\Big)$$



$$\Rightarrow \begin{tikzcd} & F(N_1) \arrow[dl] \arrow[dr] \arrow[dd, Rightarrow, "F(M)"] & \\ F(Y_1) & & F(Y_2) \\ & F(N_2) \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} Z(N_1) \arrow[d, Rightarrow, "Z(M)"] \\ Z(N_2) \end{tikzcd}$$

axiomatics in Jacob's talks...

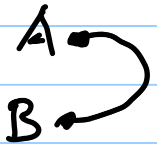
Our case: $Z_G(\bullet) = \mathrm{Vect}\,(\overset{\bullet}{\circlearrowleft}_G)$

$= \mathrm{Vect}(BG) = \mathrm{Rep}_{\mathbb{C}}(G)$ representations

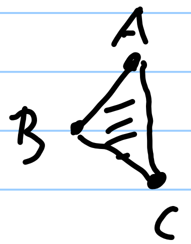
$= \mathrm{Mod}(\mathbb{C}G)$ modules for the group algebra

$Z_G(\emptyset) = \mathrm{Vect}$: covers case of closed (n-1)...

examples:

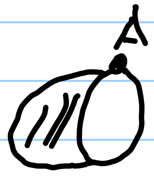


$\longmapsto \mathrm{Hom}_{\mathrm{Rep}\,G}(A,B)$



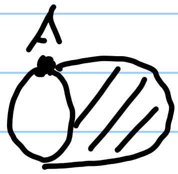
$\mathrm{Hom}(A,B) \otimes \mathrm{Hom}(B,C) \longrightarrow \mathrm{Hom}(A,C)$

$\longmapsto \mathrm{Hom}_G(A,A) = \mathrm{End}_G(A)$



$\mathbb{C} \longrightarrow \mathrm{End}_G A$ unit

$1 \longmapsto \mathrm{Id}_A$



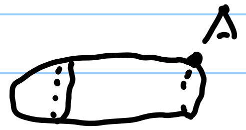
$\mathrm{End}_G\, A \xrightarrow{tr} \mathbb{C}$:

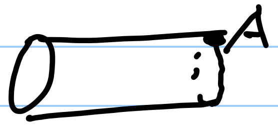
have a canonical trace on endomorphisms of any representation!

$\mathrm{Id}_A \longmapsto \dim A$

ie Rep G is not just any category, it is a "Frobenius" or Calabi-Yau$_0$ category:

reflects fact that $\mathbb{C}G$ is a (noncommutative) Frobenius algebra, with $tr = eval_1 : \mathbb{C}G \to \mathbb{C}$ nondegenerate trace.

More refined version: 



$\mathbb{C}G^G \longrightarrow End_G A$:

$\mathbb{C}G^G$ acts as symmetries of any G-representation!

In fact $\mathbb{C}G^G = Z(\mathbb{C}G)$

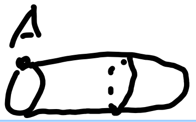
$$= End\, Id_{Mod(\mathbb{C}G)}$$

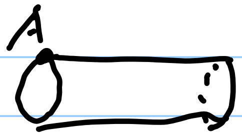
center of the group algebra
= endomorphisms of the identity of Rep G. More algebraically:

$$\mathbb{C}G^G = Hom_{\mathbb{C}G-\mathbb{C}G}(\mathbb{C}G, \mathbb{C}G)$$

endomorphisms of $\mathbb{C}G$ as a bimodule over itself.

— toy model for Hochschild cohomology

 $\Rightarrow$

Dually:  $\mathrm{End}_G A \xrightarrow{tr} \mathbb{C}G^G$

(universal) trace with values in $\mathbb{C}G^G$

$$\mathrm{Id}_A \longmapsto \mathrm{char}_G(A)$$

character of the representation!

In fact $\mathbb{C}G^G = \mathbb{C}G \otimes_{\mathbb{C}G-\mathbb{C}G} \mathbb{C}G$

= target of universal map $\mathbb{C}G \xrightarrow{t} V$ with trace property $t(ab) = t(ba)$

- toy model for Hochschild homology.

Hecke algebras

Natural source of representations:

$K \subset G$ subgroup $\Rightarrow V_{G,K} = \mathbb{C}[G/K]$

$= \mathrm{Ind}_K^G \mathbb{C} \quad \in \mathrm{Rep}_G$

Hecke algebra $\mathcal{H}_{G,K} := \mathrm{End}_G V_{G,K}$

$V_{G,K}$ represents the functor of K-invariants:

$$\mathrm{Hom}_G(V_{G,K}, W) = W^K \circlearrowleft \mathcal{H}_{G,K}$$
$$\varphi \longmapsto \varphi(\delta_{[K]})$$

so $\mathcal{H}_{G,K} = \mathrm{End}(\;)^K$

In particular $\mathcal{H}_{G,K} = \mathrm{Hom}(V_{G,K}, V_{G,K})$

$$= V_{G,K}^{K} = \mathbb{C}[G/K]^{K} = \mathbb{C}[K\backslash G/K]$$

- subalgebra of group algebra $\mathbb{C}G$ consisting of K- biinvariant functions.

$$K\backslash G \times_K G/K \longrightarrow K\backslash G/K$$

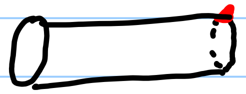
$$\|$$

TFT picture:

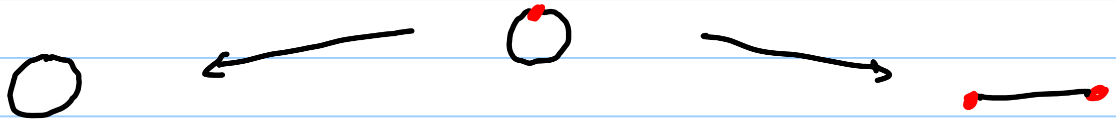
: G-bundles on interval with K reductions at the ends

$$\mathcal{H}_{G,K}\text{-mod} \simeq \text{Reps of } G \text{ generated by } K\text{-invariant vectrs}$$

$$\simeq \langle \text{reps appearing in } V_{G,K} \rangle$$



: fundamental correspondence



$$\begin{tikzcd}
\frac{G}{G} & \frac{G}{K} \arrow[l] \arrow[r] & K\backslash G/K \\
\mathbb{C}G^{G} \arrow[rr, leftrightarrow] & & \mathcal{H}_{G,K} \\
\mathrm{char}(V_{G,K}) & 1_{\frac{K}{K}} \arrow[l, mapsto, "\int"'] & 1_{\mathrm{diag}} \arrow[l, "\pi^*"']
\end{tikzcd}$$

Gauge theory for complex groups

We'd like to replace finite groups by complex (reductive) groups, like $\mathbb{C}^\times$, $GL_n\mathbb{C}$, $SO_n\mathbb{C}$ etc.

There are many variants of G-gauge theory... we'll attempt to linearize the moduli spaces

$$\mathcal{M}_G(M) = \{ G \text{ local systems on } M \}$$
$$= \{ \pi_1(M) \longrightarrow G \} / \sim$$

$$\mathcal{M}_G(S^1) = \frac{G}{G} \underset{\text{open dense}}{\supset} H^{rs}/W$$

collection of conjugacy classes - generically parametrized by eigenvalues

$$\mathcal{M}_G(\Sigma_g) = \left\{ \begin{matrix} A_1, \dots A_g \\ B_1, \dots B_g \end{matrix} \in G^{2g} : \prod [A_i, B_i] = 1 \right\} / G$$

eg $\mathcal{M}_g(\Sigma_1)$ = commuting pairs in $G/\sim$

... interesting singular affine varieties mod action of G.

How do we "count points" on $\mathcal{M}_G(\Sigma_g)$ or "integrate functions" on $\mathcal{M}_G(S^1)$?

- One solution: look over varying finite fields $\mathbb{F}_q \Longrightarrow$ 2d TFT depending on q....
  (cf work of Hausel-Rodriguez-Villegas)

Algebraic geometry (Weil conjectures) teaches us that numbers of points over finite fields are avatars of the cohomology of the underlying variety, & more generally functions on $\mathbb{F}_q$ points are avatars of sheaves...

$$\rightsquigarrow$$

- another solution: categorify!
  numbers $\rightsquigarrow$ vector spaces
  (such as cohomologies)
  vector spaces $\rightsquigarrow$ categories
  (such as categories of sheaves)

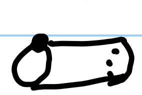
Physics motivation: compactification or dimensional reduction

$$Z \text{ n-dim TFT} \rightsquigarrow Z_{S^1} \text{ (n-1) dim TFT}$$

$$Z_{S^1}(M) = Z(M \times S^1) = \dim M \times S^1$$

(in Jacob's terminology)

dim of a vector space is a number
dim of a category is a vector space —
its <u>Hochschild homology</u> $HH_*(\mathcal{C}) \ni \mathrm{char}(F)$ $F \in \mathcal{C}$
character = charge of F



To linearize we'll need a good theory of functions/measures on G, $\frac{G}{G}$ etc with similar formal properties (pullback, product, pushforward).

Our setting: replace vector spaces of functions by dg categories of sheaves.
(cf Bertrand's lectures)

Have many variants — analogs of classical function spaces!

I. X variety $\rightsquigarrow$ $Q(X)$ dg category of quasicoherent sheaves on X: basic examples are (algebraic) vector bundles (finite or infinite rank). More general objects come from kernels & cokernels (or complexes) of maps between bundles.

More formally $Q(X)$ is defined by

1. assignment Spec $R \longmapsto$ Mod (R) chain complexes of R-modules, with quasi isomorphisms inverted.

2. descent: calculate $Q(X)$ from $Q(U)$ sheaves on a cover $U \to X$ with gluing data on double overlaps $U \times_X U$, triple overlaps $U \times_X U \times_X U$, & so on...

This definition extends to stacks: for us $G \circlearrowright X$ variety $\Rightarrow$ $Q(X/G) =$ G-equivariant sheaves on X

II. $X \rightsquigarrow \mathcal{D}(X)$ dg category of $\mathcal{D}$-modules on X.

Basic objects: vector bundles with flat connection. More general objects: quasicoherent sheaves with flat connection
$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1, \ \nabla^2 = 0 \iff$$
$$T \otimes \mathcal{F} \to \mathcal{F} \text{ extends to } \mathcal{D} \otimes \mathcal{F} \to \mathcal{F}$$

General definition of $\mathcal{D}$-modules parallel:

1. $\operatorname{Spec} R \rightsquigarrow$ modules for $\mathcal{D}_R$ diffops with R-coefficients
2. extend by gluing.

One key motivation for $\mathcal{D}$-modules: tight analogy with classical functions/distributions:

$$f \in C^\infty(X) \quad \text{or} \quad \delta \in C^{-\infty}(X)$$

$$\mapsto \mathcal{D} \cdot f \subset C^\infty(X) \qquad \mathcal{D} \cdot \delta \subset C^{-\infty}(X)$$

Span under algebraic diffop is a $\mathcal{D}$-module. In good cases ("holonomic"), the $\mathcal{D}$-module captures f / δ up to finite dimensional ambiguity:

$\mathbb{A}^1$ example: $e^{\lambda x} \rightsquigarrow \mathcal{D}/\mathcal{D}(\partial-\lambda)$ $\Big\}$ recover up to scalar!

$\delta_\lambda \rightsquigarrow \mathcal{D}/\mathcal{D}(x-\lambda)$

$\mathcal{D} \circlearrowleft \begin{array}{l} x \longrightarrow \partial \\ \partial \longrightarrow -x \end{array}$ gives $\mathcal{D}(\mathbb{A}^1) \longleftrightarrow \mathcal{D}(\mathbb{A}^1)$

Fourier transform, not on functions but on <u>systems</u> of diffeqs...

D-modules are also closely related to the "counting points over $\mathbb{F}_q$" approach, via the Riemann-Hilbert correspondence: the sheaves of <u>solutions</u> of these diffeqs are the characteristic 0 analogs ($\frac{\text{constructible}}{\text{perverse}}$ sheaves) of the sheaves appearing in the Weil conjectures story...

Algebra: Vect is a symmetric monoidal ($\otimes$) category
dgCat is a " " $(\infty,1)$-category

∴ means we can talk about (associative & commutative) algebra objects in dgCat, & modules over these algebras, etc - analog of all of basic algebra (Lurie - DAG II - III)

$Q(X)$ & $D(X)$ are both commutative algebras in dgCat, with multiplication given by $\otimes$ of bundles (or sheaves)

Also have notion of <u>pullback</u> of bundles (or bundles w/ flat connection)

$\Rightarrow \pi : X \to Y$ induces $\pi^* : Q(Y) \to Q(X)$
$\pi^! : D(Y) \to D(X)$

& <u>pushforward</u>: calculate cohomology (coherent or de Rham) along fibers

$\Rightarrow \quad \pi_* : Q(X) \longrightarrow Q(Y)$
$\pi_* : D(X) \longrightarrow D(Y)$

[all operations are <u>derived</u>]

$\Rightarrow$ integral transforms:

$K \in Q(X \times Y)$ (or $D(X \times Y)$)

gives $\mathcal{F} \in Q(X) \longmapsto K * \mathcal{F} \in Q(Y)$

$$= \pi_{Y*}(\pi_X^* \mathcal{F} \otimes K)$$

$$\text{“} = \int \mathcal{F}(x) K(x,y)\, dx \text{”}$$

$$\begin{tikzcd}
 & & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & K \\
\mathcal{F} & X & & Y
\end{tikzcd}$$

$Q(X)$ case: equivalences given this way are known as Fourier-Mukai transforms.

Note these are the kind of operators we need to define “path integrals”

$$\begin{tikzcd}
 & \mathrm{Fields}(M) \arrow[dl] \arrow[dr] & \\
\mathrm{Fields}(\partial_{in}) & & \mathrm{Fields}(\partial_{out})
\end{tikzcd}$$

$K \longleftrightarrow e^{-S}$

Gluing/ Composition

$$\begin{tikzcd}
\mathrm{Fields}(M_1 \coprod_N M_2) \arrow[d, equal] \\
\mathrm{Fields}(M_1) \times_{\mathrm{Fields}(N)} \mathrm{Fields}(M_2)
\end{tikzcd}$$

$$\begin{tikzcd}
 & & \mathrm{Fields}(M_1) \times_{\mathrm{Fields}(N)} \mathrm{Fields}(M_2) \arrow[dl] \arrow[dr] & & \\
 & \mathcal{F}(M_1) \arrow[dl] \arrow[dr] & & \mathcal{F}(M_2) \arrow[dl] \arrow[dr] & \\
\mathcal{F}(\partial_{in}) & & \mathcal{F}(N) & & \mathcal{F}(\partial_{out})
\end{tikzcd}$$

Finite analog: X, Y finite sets $\Rightarrow$

$$Fun(X \times Y) = Fun(X) \otimes Fun(Y)$$
$$= Y \text{ by } X \text{ matrices}$$
$$\simeq Hom(Fun(X), Fun(Y))$$

[note $Fun(X) = Fun(X)^*$ via canonical inner product $f \cdot g = \sum f(x) g(x)$]

Relative version:

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

$$Fun(X \underset{Z}{\times} Y : \text{pairs with same image}) =$$
$$Fun(X) \underset{Fun(Z)}{\otimes} Fun(Y)$$

(impose constraint algebraically)

$$= Z\text{-block diagonal matrices} \quad \begin{pmatrix} \boxed{z_1} & & 0 \\ & \boxed{z_2} & \\ 0 & & \boxed{z_3} \end{pmatrix}$$
$$= Hom_{Fun(Z)}(Fun(X), Fun(Y))$$

Theorem: Functors, tensors & integral transforms

$X \to Z \leftarrow Y$ "perfect stacks" (eg schemes, most char 0 stacks)

- $Q(X) \underset{Q(Z)}{\otimes} Q(Y) \simeq Q(X \underset{Z}{\times} Y) \simeq Fun_{Q(Z)}(Q(X), Q(Y))$

[Toën for X, Y schemes, $Z = pt.$, BZ-Francis-Nadler in general ... note fiber product is derived]

- $\mathcal{D}(X) \underset{\mathcal{D}(Z)}{\otimes} \mathcal{D}(Y) \simeq \mathcal{D}(X \underset{Z}{\times} Y) \simeq Fun_{\mathcal{D}(Z)}(\mathcal{D}(X), \mathcal{D}(Y))$

for schemes [BZ-Nadler]

3d gauge theory: starting from the point.

Recall for Γ finite, $\mathrm{Rep}_{\mathbb{C}}\Gamma = \mathrm{Mod}\,\mathbb{C}\Gamma$

- more generally $\mathrm{Rep}_k\Gamma = \mathrm{Mod}\,k\Gamma$

For infinite groups one specifies different classes of representations (smooth, continuous, measurable, locally constant, ...) by writing as modules for different variants of group algebra ($C_c(G)$, $L^1(G)$, ...)

Categorified setting G affine algebraic group we'll look for different notions of G-(dg) categories: naively want a functor $a_g : \mathcal{C} \to \mathcal{C}$ $\forall g \in G$ with coherences relating $a_{g_1 g_2}$ with $a_{g_1} \circ a_{g_2}$ etc..

& some notion of algebraicity or flatness in g

$\Rightarrow$ $QG, \mathcal{D}G$: qcoherent / flat group algebra of G

- monoidal dg category via convolution

$\mu : G \times G \to G$

$$\mathcal{F} * \mathcal{G} = \mu_*(\pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{G})$$

$Z(pt)$: can consider $\{G\text{-categories}\}$ of the appropriate kind:

Vect Γ-mod (Γ finite),

QG-mod or $\mathcal{D}G$-mod

(algebraic G-cats) (flat/smooth G-cats)

Source of examples: $G \circlearrowright X$ G-variety

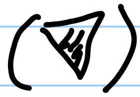
$\Rightarrow$ $Q(X)$ is an algebraic G-category

$\mathcal{D}(X)$ is a smooth G-category

via $G \times X \longrightarrow X$

$QG \otimes Q(X) \longrightarrow Q(x)$ etc.

Prime example $B \subset G$ Borel subgroup

eg  $\subset GL_n$.

G/B = the flag variety of G

(GL_n: full flags in $\mathbb{C}^n$)

$Q(G/B)$ is a quasicoherent G-category

$\mathcal{D}(G/B)$ is a flat G-category.

Theorem (Beilinson-Bernstein)

$\Gamma : \mathcal{D}(G/B) \longrightarrow \mathfrak{g}\text{-mod}$ (via $\mathfrak{g} = \Gamma(G/B, T)$)

is an equivalence to $\mathfrak{g}$-mods with trivial infinitesimal character (action of $Z(U\mathfrak{g})$)

... ie up to an extra parameter (which is easy to correct, at least generically)

$$\mathcal{D}(G/B)_\lambda \simeq \mathfrak{g}\text{-mod}_\lambda \qquad \lambda \in \mathfrak{h}^*/W \text{ generic}$$

* G-action on RHS via conjugation: conjugate $\mathfrak{g}$ action on a rep V to get a new $\mathfrak{g}$-module.

What are the symmetries of this G-category?

$$\mathcal{D}(G) \circlearrowright \mathcal{D}(G/B) \circlearrowleft \mathcal{D}(B\backslash G/B) = \mathcal{H}$$

<u>finite Hecke category</u>.

Action on $\mathfrak{g}$-reps $\Leftrightarrow$ classical <u>intertwiners</u> for representations.

B orbits on G/B = Schubert cells $\longleftrightarrow W$ Weyl group,
each is contractible.

$K(\mathcal{H}) = \mathbb{Z}W$, group algebra of Weyl group

So $\mathcal{H}$ has "bases" (on level of K-group) given by different ways of extending the trivial flat bundle $\mathbb{C}_w$ on each orbit ["simples $i_{!*}\mathbb{C}_w$, T_w = standards $i_*\mathbb{C}_w$, costandards $i_!\mathbb{C}_w$"]

- the study of $\mathcal{H}$ is <u>Kazhdan-Lusztig theory</u>.

Actions of $\mathcal{H}$ on a category $\mathcal{C}$:
give not Weyl actions, but <u>braid group</u> actions - T_{s_i} (simple reflections) don't square to 1
but satisfy $T_{s_i}T_{s_j}T_{s_i} = T_{s_j}T_{s_i}T_{s_j}$ $\quad \underset{i}{\bullet}\!\!-\!\!\!-\!\!\underset{j}{\bullet}$ etc

$\rightsquigarrow$ role in Khovanov link homology ....

Examples of $\mathcal{H}$-modules : "subs" of $\mathcal{D}(G/B)$

$\mathcal{D}(K \backslash G/B) \underset{B\text{-}B}{\simeq}$ Harish-Chandra $(\mathfrak{g}, K)$-modules :

$K \subset G$

central objects in rep theory

[eg K symmetric $\Rightarrow$ reps of real forms of G !]

Cobordism hypothesis : Try different assignments for $Z(pt)$, see how far up they extend...

$\mathrm{Vect}\,\Gamma$-modules $\Rightarrow$ 3d TFT Z_Γ

$\mathcal{Q}G$-modules $\Rightarrow$ 2d TFT [BZ-Nadler - Francis] $Z_G^{\mathcal{Q}}$

$\mathcal{D}G$-modules $\Rightarrow$ 1d TFT (!), but

$\mathcal{H}$-modules $\Rightarrow$ 2d TFT [BZ-Nadler] $\mathcal{X}_G$ (character theory)

From physics POV, all are part of 3d gauge theory, some better behaved than others... [roughly : would-be $\mathcal{D}G$ theory the union of λ-twisted versions of $\mathcal{H}$-theory over $\lambda \in \mathfrak{h}^*/W$]

Structures of categorified rep theory

Study modules $Q(G/H)$ $H<G$, endomorphisms $= Q({}_H\backslash G/H)$ Hecke category – eg $H=G$,

$$\mathrm{End}_G(\mathrm{Vect}) = Q(BG) = \mathrm{Rep}\, G.$$

Morita theory:

X finite set, $\mathrm{Fun}(X\times X)$ = algebra of square matrices is Morita equivalent to $\mathbb{C}$:

$$\mathrm{Mod}(\mathrm{Fun}(X\times X)) \simeq \mathrm{Mod}(\mathbb{C}) = \mathrm{Vect}$$

$X \to Y \Rightarrow \mathrm{Fun}(X \times_Y X)$ algebra of block diagonal matrices Morita equivalent to $\mathrm{Fun}(Y)$:

$$\mathrm{Mod}(\mathrm{Fun}\, X\times_Y X) \simeq \mathrm{Fun}(Y)\text{-mod} = \mathrm{Vect}_Y.$$

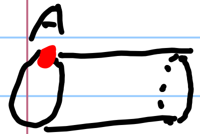
BZ-Francis-Nadler: version for Q(perfect stacks): $\forall H$, $Q({}_H\backslash G/H)$ is Morita equivalent to QG (same notion of <u>dualizable</u> modules)

So any of the modules $Q(G/H)$ generates the whole representation theory!

[special case: $\mathrm{Vect}\,\Gamma \leftrightarrow \mathrm{Vect}\, K\backslash\Gamma/K$, theorem of Müger & Ostrik]

- Very false for $\mathcal{D}$: $\mathrm{Mod}\,\mathcal{D}G$ & $\mathrm{Mod}\,\mathcal{H}$ different...

$Z(S^1)$ has two dual roles (for Z defined on surfaces):

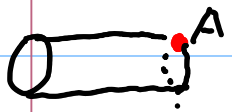


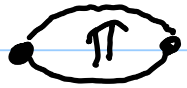
:

- $Z(S^1) = \dim\{Z(\bullet) = \mathrm{Mod}\,\mathcal{A}\}$

  = Hochschild homology (or abelianization) of group algebra, $\mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{op}} \mathcal{A}$

  $Z(S^1)$ carries characters/charges of $A \in Z(\bullet)$



- $Z(S^1) =$ endomorphisms of $Id_{Z(\bullet)}$ 

  = Hochschild cohomology or center of $\mathcal{A}$

  $= \mathrm{Hom}_{\mathcal{A}\otimes\mathcal{A}^{op}}(\mathcal{A},\mathcal{A})$

  – $Z(S^1)$ acts on every $A \in Z(\bullet)$.

Classical version: Drinfeld center of a monoidal category $\mathcal{C}$ =

$Z(\mathcal{C})$: $\mathrm{Hom}_{\mathcal{C}\otimes\mathcal{C}^{op}}(\mathcal{C},\mathcal{C}) =$

$$\{F \in \mathcal{C} + F * \_ \xrightarrow{\sim} \_ * F\}$$

central structure.

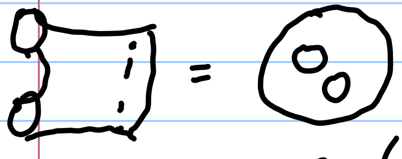
$Z(\mathrm{Vect}\,\Gamma) = \mathrm{Vect}\frac{\Gamma}{\Gamma}$ class bundles

$$= \prod_{[g]} \mathrm{Rep}\, Z_\Gamma(g)$$

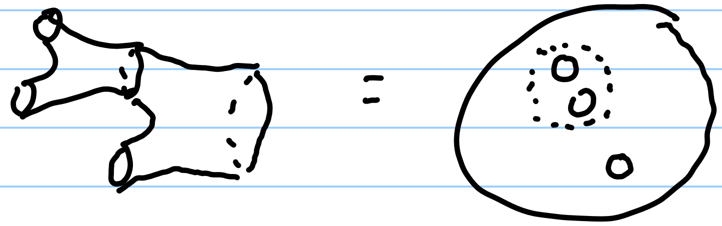
Theorem (BZ-Francis-Nadler)

$Q(\frac{G}{G})$ = center & dim (HH^* & HA_*) of $Q(G)$, & of $Q({}_H\backslash G/H)$ $\forall H < G$ (affine)

Note $Z(S^1)$ carries a braided tensor product (E_2 multiplication):



binary operations labelled by pairs of discs in a larger disc, & compositions parametrized by gluing / "picture-in-picture"



$\mathcal{F} * \mathcal{G} \xrightarrow{\sim} \mathcal{G} * \mathcal{F}$:

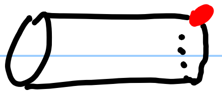
doesn't square to the identity but get braid group action on $\mathcal{F} * \mathcal{F} * \dots * \mathcal{F}$

Also dim $Z(\cdot) = Z(S^1)$ carries action of Diff S^1. making it a ribbon category; monodromy automorphism for every $\mathcal{F} \in Z(S^1)$

$\mathrm{Vect}\frac{\Gamma}{\Gamma} \ni V$ $\quad V|_{[g] \simeq \bullet/Z_\Gamma(g)} \in \mathrm{Rep}\, Z_\Gamma(g)$,

but $g \in \mathrm{Center}(Z_\Gamma(g)) \rightsquigarrow$ gives a canonical automorphism !

D-module case: study diagram



$$\begin{tikzcd}
 & \frac{G}{B} \arrow[dl, "\pi"'] \arrow[dr, "\delta"] & \\
\frac{G}{G} & (G \times G/B)/G & B\backslash G/B \\
{[g]} & (g,B) \arrow[l, mapsto] \arrow[r, mapsto] & (B, g\cdot B)
\end{tikzcd}$$

rel posn in W

Restrict to $1 \in W \longleftrightarrow g\cdot B = B$, ie $g \in B$:

$$\begin{tikzcd}
 & \tilde{G}/G = \{(g,B) : g \in B\}/G \arrow[dl, "s"'] \\
\frac{G}{G} & \cup \\
\cup & T^*G/B \text{ (or rather affine version)} \arrow[dl] \\
\frac{N}{G} &
\end{tikzcd}$$

Grothendieck-Springer simultaneous resolution

Whole diagram is union of W-twisted versions of this!

Note π is a W-Galois cover over the dense open subset $H/W \subset \frac{G}{G}$ (ordering of eigenvalues of g !)

Basic object in the theory: <u>the Springer sheaf</u>.

$\mathcal{S} \in \mathcal{D}\left(\frac{G}{G}\right)$:

- Harish-Chandra's G-invariant system of differential equations satisfied by the G-invariant distributions on G, arising as characters of [admissible, ∞-dim] representations of G .... explicitly given by $\left\{ L \cdot K = 0, \quad L \in \Gamma(G, \mathcal{D})^{G \times G} \right\}$

  — used to show characters are analytic functions with prescribed singularities!

- $\mathcal{S} = S_* \mathbb{C}_{\tilde{G}}$ pushforward of constant sheaf on Springer resolution

  $= \pi_* \delta^* T_1$ horocycle transform of unit in Hecke category

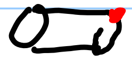
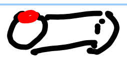
.... $\mathcal{S}|_{G^{rss}}$ is a twisted version of $\mathbb{C}W$

<u>Def</u> (Lusztig) A <u>character sheaf</u> on G is a $\mathcal{D}$-module in the image of the "horocycle" transform $\pi_* \delta^*$ [simple constituents thereof...]

— geometric avatars of characters of finite groups $G(\mathbb{F}_q)$ ∀q ....

BZ-Nadler: character sheaves are characters of $\mathcal{H}$-modules!

<u>Theorem</u> (BZ-Nadler) $HH^*(\mathcal{H}) \simeq HH_*(\mathcal{H})$
$= $ image of $\pi_* \delta^*$ in $\mathcal{D}(\frac{G}{G})$:

in fact have 2d TFT, $\chi_G(\bullet) = \mathcal{H}$-modules
& $\chi_G(S^1) =$ character sheaves
(&   maps given by diagram above)

- ξ is the character of $\mathcal{D}(G/B)$ as a G-category, or of $\mathcal{H}$ as an $\mathcal{H}$-module!

Theorem (Beilinson-Ginzburg-Soergel)

$$\mathcal{D}(B\backslash G/B) \simeq \mathcal{D}({}_{B^\vee}\vdots\, G^\vee \vdots\, {}_{B^\vee})$$

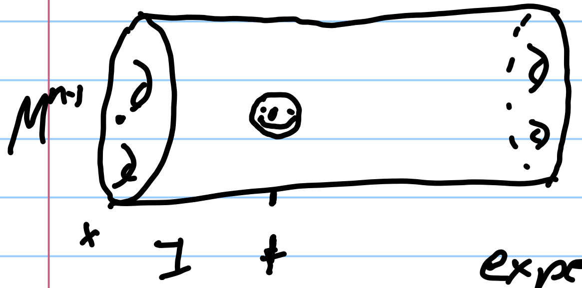
<u>Corollary</u> Langlands duality for 2d TFTs

$$\chi_G \simeq \tilde{\chi}_{G^\vee}$$

- in particular character sheaves for G & $G^\vee$ identified,

4d gauge theory & geometric Langlands

Local operators Key structure in QFT:
make measurements on fields near a point



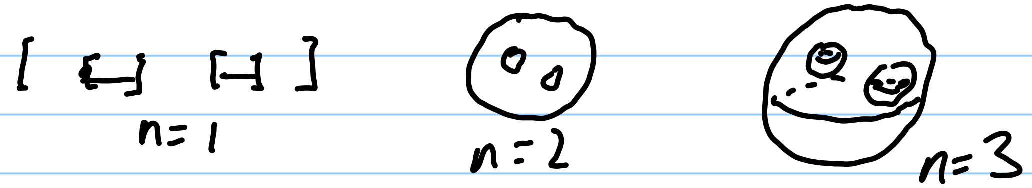
$\langle \mathcal{O} \rangle_M =$

$\int \mathcal{O}(\varphi) e^{-S(\varphi)} D\varphi$

expectation value of measurement
$\mathcal{O}$ of φ at point x & time t

How to formalize? $\mathcal{O} \in Z(S^{n-1})$
functional on values of φ in punctured neighborhood of
(x,t) : $Z(S^{n-1}_{x,t}) \otimes Z(M) \longrightarrow Z(M)$

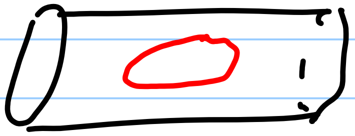
In general $Z(S^{n-1})$ has a product structure
from pair of pants : E_n multiplication



- more & more commutative as n increases

& $Z(M^{n-1})$ is a module.

Loop operators:



Given loop

$\Rightarrow Z(S^{n-2} \times S^1)$ acts on $Z(M)$

$= \dim Z(S^{n-2})$ (& likewise for other submanifolds)

Order operators : $\mathcal{O}(\varphi)$ = some measurement of φ. In gauge theory have Wilson loops : L a loop

$W_{R,L}$ = trace of holonomy along loop L in representation R

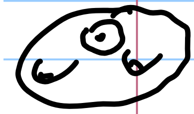
$$\langle W_{R,L}\rangle = \int W_{R,L}(\varphi)\, e^{-S(\varphi)}\, D\varphi$$

Disorder operators : impose a particular type of singularity on fields at a point/loop (ie insert characteristic function of fields with given singularity)

- eg in codim 2 can require connection to have prescribed monodromy.
- changes domain of path integral!

- eg 2d gauge theory, $C \subset G$ conjugacy class $\Rightarrow$ "disorder operator"

$$1_C \in Z_G(S^1) = \mathbb{C}\frac{G}{G}$$



$\langle 1_C\rangle_\Sigma$ = # G-covers with monodromy in C around a given marked point.

4d gauge theory I The B-model B_G.

- 4d analog of our 3d theory Z_G

Space of gauge fields still $\mathcal{M}_G = G$ local systems

$$B_G(N^3) = R\Gamma(\mathcal{M}_G(N^3), \mathcal{O})$$

$$B_G(\Sigma) = \mathcal{O}(\mathcal{M}_G(\Sigma))$$ dg category of coherent sheaves

[$B_G(S^1) = \mathcal{O}(\frac{G}{G})$ - modules = G-equivariant sheaves of dg categories over G]

Loop operators:

$\mathcal{M}_G(S^2) =$  $\frac{\cdot}{G} \times_{\frac{G}{G}} \frac{\cdot}{G} \simeq \frac{\cdot}{G}$

(dg correction!...)

$$B_G(S^2) = \mathcal{O}(\mathcal{M}_G(S^2)) \simeq \mathrm{Rep}\, G$$

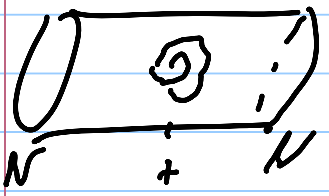
(dg corrected)

$B_G(S^2 \times S^1) = \dim \mathrm{Rep}\, G = \underline{\mathrm{Rep}\, G}$

$\underline{\text{representation ring}}$.

⇒ loop operators are Wilson operators $W_{L,R}$, "measure holonomy in rep R"

- Γ finite (so theory extends to 4-manifolds) & $L \subset N^3 \Rightarrow$ function $W_{L,R}$ on fields on $N^3 \times (\text{time})$, $\rho \mapsto \operatorname{tr} \operatorname{hol}_L(\rho_R)$

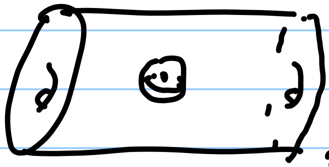


$\Rightarrow$ operator

$$W_{L,R,t} : Z(N) \longrightarrow Z(N)$$

sum over gauge fields weighted by $\operatorname{tr} \operatorname{hol}_L(\rho_R)$.

- Σ surface



$\Rightarrow$ action of E_3 category $\mathcal{B}_G(S^2) \simeq \operatorname{Rep} G$

for every $x \in \Sigma$:

$W_{x,R}$ vector bundle on $\mathcal{M}_G(\Sigma \times I) = \mathcal{M}_G(\Sigma)$
$= \operatorname{Loc}_G(\Sigma)$

$\rho \longmapsto$ fiber of associated bundle ρ_R at point x.

$W_{L,R} : \mathcal{O}(\operatorname{Loc}_G \Sigma) \circlearrowleft$ given by $\otimes V_{L,R}$.

– ie have huge "commutative algebra" acting on $\mathcal{O}(\operatorname{Loc}_G \Sigma)$: $\bigotimes_{x \in \Sigma} \operatorname{Rep} G$

[Kapustin-Witten]

Another 4d gauge theory A_G:

not quite topological, depends on some extra holomorphic structure, but we'll treat formally the same....

closer to Z_G^D. Motivation:

replace rep theory of G / $G(\mathbb{F}_q)$ by rep theory of LG [analog of p-adic groups]

$$[A_G(S^1) \sim \mathcal{D}(LG)\text{-modules}....]$$

eg $L\mathfrak{g}$-rep $\in A_G(S^1)$
or $\hat{\mathfrak{g}}$-reps

Σ now algebraic curve / Riemann surface

Fields now holomorphic G-bundles on Σ

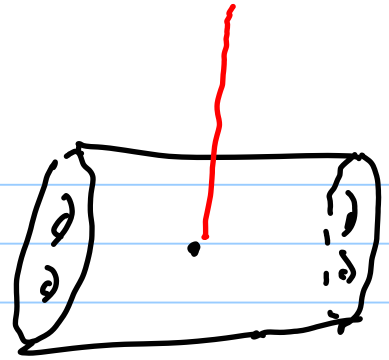
$Bun_G(\Sigma)$ ... harder to describe explicitly!

eg $G = GL_1$, $Bun_G \Sigma = Pic\, \Sigma$

$A_G(\Sigma) = \mathcal{D}(Bun_G \Sigma)$ D-modules on Bun_G

$A_G(N^3) \sim$ de Rham cohomology of moduli of monopoles [sols of Bogomolnyi eqs] ... G bundles with connection satisfying eqn: for $\Sigma \times I$ this says bundle is holomorphic on Σ & complex structure constant in time....

<u>Local operators</u>

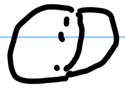


't Hooft operators : introduce singularity in (disorder) bundle : P undergoes some transformation as we cross singular point. 4d : introduce magnetic monopole (gerbe in $N^3 \times$ time) along L.

$\mathcal{M}_G(S^2)$ = possible local geometries of singularity of gauge fields

= $\mathrm{Bun}_G(S^2)$: set theoretically (Grothendieck-Birkhoff)

$\longleftrightarrow \{\mathbb{C}^* \longrightarrow G\}/\sim$



$\longleftrightarrow \mathrm{Hom}(\mathbb{C}^*, T)/W$

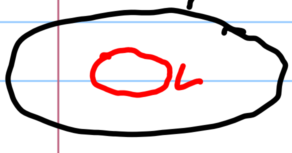
$\longleftrightarrow \mathrm{Hom}(T^\vee, \mathbb{C}^*)/W$

$\longleftrightarrow$ Irreps of <u>Langlands dual group</u> $G^\vee$.

$A_G(S^2 \times S^1) = \underline{\mathrm{Rep}}\, G^\vee$ representation ring

— "possible charges of G-magnetic monopoles"

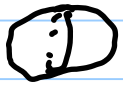
Morally : $R^\vee \in$ irrep $G^\vee$ $\langle H_{R^\vee, L} \rangle =$



M^4

$$\int_{\text{fields}(R^\vee, L)} e^{-S} \, d\varphi$$

What is $A_G(S^2) = \mathcal{D}(\mathrm{Bun}_G S^2)$?

 $\mathcal{D}(LG_+\backslash LG/LG_+) \circlearrowleft \mathcal{D}(LG_+\backslash LG/LG_+)$

spherical Hecke category $\qquad$ $\overset{\|}{\mathcal{H}_{sph}}$

Hecke operators: $LG_+\backslash LG/LG_+ \longleftrightarrow$ irrep $G^\vee$
labels possible relative positions of two G-bundles at a point
[just as $B\backslash G/B \longleftrightarrow W$ labels same for flags...] ie all ways to modify a bundle at a point.

Theorem (Mirković-Vilonen; Lusztig, Drinfeld, Ginzburg) Bezrukavnikov-Finkelberg, Gaitsgory-Lurie

$$\mathcal{H}_{sph} \simeq \mathcal{O}(\mathrm{Loc}_{G^\vee}(S^2))$$

as E_3 categories

[abelian heart: Rep $G^\vee$
+ some dg enhancement]

$\rightsquigarrow$ 't Hooft operators on $A_G(\Sigma)$ labelled by representations of $G^\vee$!

-- $R^\vee \in \mathrm{Rep}\, G^\vee$, $H_{R^\vee} \in \mathcal{H}_{sph}$

$$H_{R^\vee,x} : \mathcal{D}(\mathrm{Bun}_G \Sigma) \circlearrowleft$$

$$\mathcal{F} \longmapsto H_{R^\vee,x} * \mathcal{F}(P) = \int_{\substack{\eta : P' \simeq P \\ \text{away from } x}} \mathcal{F}(P')\, H_{R^\vee}(P, P', \eta)$$

Geometric Langlands Program:

Spectrally decompose $\mathcal{D}(\mathrm{Bun}_G \Sigma)$ under action of $\bigotimes'_{x \in \Sigma} \mathrm{Rep}(G^\vee)$ Hecke operators. ... ie diagonalize commuting action of local operators $A_G(S^2) \circlearrowright A_G(\Sigma)$.

Geometric Satake: local operators in A_G & $B_{G^\vee}$ coincide!

$\Rightarrow$ Strong indication the theories should coincide....

Geometric Langlands Conjecture:

$$\begin{array}{cccc} & \mathcal{D}(\mathrm{Bun}_G \Sigma) & \simeq & \mathcal{O}(\mathrm{Loc}_{G^\vee}\Sigma) \\ & \circlearrowleft & & \circlearrowleft \\ x\in\Sigma & \mathcal{H}_{sph} & \simeq & \mathrm{Rep}\, G^\vee \end{array}$$

..... needs various modifications to not be false for trivial reasons

– Special case of

Montonen-Olive Electric-Magnetic S-duality:

$$A_G \simeq B_{G^\vee} \quad \text{isomorphism of field theories}$$

Spectral decomposition:

Wilson operators are "diagonal matrices" on $\mathrm{Loc}_{G^\vee}\Sigma$: for any $V\in \mathrm{Loc}_{G^\vee}\Sigma$, skyscraper $\mathcal{O}_V$ is an eigenobject for $W_{x,R^\vee}$:

$$\mathcal{O}_V \otimes W_{x,R^\vee} = W_{x,R^\vee}\big|_{\{V\}} = (V_x)_{R^\vee} \otimes \mathcal{O}_V$$

fiber at x of associated bundle $V_{R^\vee}$.