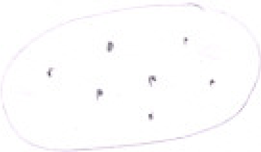
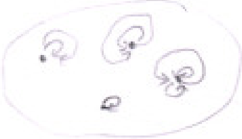
TALKS TIFR MARCH-08

Scheme : Set of points (with some structure : topology, sheaf $\mathcal{O}_X$, loc. isom to Spec A)



ex: $\mathfrak{M}_X$ moduli sch. of vtr. bdles

Alg. stack : Category (with some structure which we will see)



ex: $\mathcal{M}_X$ moduli stack of vtr. bdles

$$\begin{tikzcd}
\mathcal{M} \arrow[r, phantom, "\supset"] & \mathcal{M}^{semist} \arrow[d, "S\text{-equiv.}"] \arrow[r, phantom, "\supset"] & \mathcal{M}^{st} \arrow[d, "\text{forget autom.}"] \\
& \mathfrak{M}^{semist} \arrow[r, phantom, "\supset"] & \mathfrak{M}^{st}
\end{tikzcd}$$

Another example: $G \curvearrowright X$

Quot. as scheme: might not exist,
might need to restr. to semist.

S-equiv.

Quot as stack $[X/G]$

$\mathbb{C}^* \curvearrowright \mathbb{C}^2$ $(\lambda x, \frac{1}{\lambda} y)$

Grothendieck topologies

Topology (classically): is a collection of morph $U \underset{\text{(open incl.)}}{\hookrightarrow} X$ in the topological cat.

Grothendieck topology on a category is a coll. of morph. in that category $f: U \to V$ plays the role of an open set in U.

cover $\{U_i \to X\}$ surj.

Intersection $U_i \times_X U_j \to X$

Site := Category with a Grothendieck topology
Zariski topology $f: U_i \hookrightarrow X$ open inclusion
étale topology $f: U_i \to X$ étale
fppf topology finitely presented flat morph.

Schemes as functor of points

M scheme ⇝ functor of points $\underset{\substack{\parallel \\ F}}{\mathrm{Hom}_S(-,M)} : (Sch/S) \to (Sets)$

$B \mapsto \underset{\substack{\parallel \\ F(B)}}{\mathrm{Hom}_S(B,M)}$

Functor = presheaf of sets in Groth. topol.

A functor is a Sheaf if: Open covering

(Mono) $\{U_i \to U\}$ $X, Y \in F(U)$ $X|_i = Y|_i$ $\forall i \Rightarrow X = Y$

(Glueing) $X_i \in F(U_i)$, $X_i|_{ij} = X_j|_{ij} \Rightarrow \exists X : X|_i = X_i$

$(Sch) \subset (\text{Alg spaces}) \subset (\text{sheaves}) \subset (\text{presheaf})$

loc. Zariski ~~affine~~ loc. étale affine $\quad \overset{\parallel}{\text{space}} \quad \overset{\parallel}{\text{functor}}$

$\underline{\mathcal{M}}'_X(B) =$ isom classes of vbr bdles on $B \times X$ Not a sheaf!

$\underline{\mathcal{M}}_X(B) =$ equiv. class up to pullback by line bdle A sheaf sometimes (for example, for simple sheaves)

Sometimes, $\underline{\mathcal{M}}_X$ is representable (ex: Jacobian, X curve, $(r,d) = 1$)

weaker: M corepresents F if $\exists \phi$

$$\begin{tikzcd} F \arrow[d] \arrow[dr, bend left, "\forall"] & \\ \mathrm{Hom}(-,M) \arrow[r, dashed, "\exists!"'] & \mathrm{Hom}(-,N) \end{tikzcd}$$

M coarse moduli scheme

Correct glueing for families of vbr. bdles

(Glueing) $\{U_i \to U\}$ $\underset{U_i \times X}{\overset{F_i}{\wr}}$ and $\varphi_{ij} : F_i|_{ij} \xrightarrow{\cong} F_j|_{ij}$ s.t. $\varphi_{ij}\varphi_{jk} = \varphi_{ik} \Longrightarrow \exists \underset{U \times X}{\overset{F}{\wr}}$

"KEEP ISOMORPHISMS" !

$$\frac{\text{2-Functors}}{\text{(presheaf)}} \qquad \frac{\text{Stack}}{\text{sheaf of groupoids}}$$

$$\mathcal{F}: (Sch/_S) \longrightarrow (\text{Groupoids}) = \begin{cases} \text{ob: category} \\ \text{1-mor = functors} \\ \text{2-mor = nat. transf.} \end{cases}$$

Example:

$\boxed{\text{ob}}$ $B \longmapsto \mathcal{M}_X(B) = \begin{cases} \text{ob} & V \hookrightarrow B\times X \\ \text{mor} & V \xrightarrow{\cong} V' \end{cases}$ } ← important

$\boxed{\text{1-mor}}$ $B' \to B \longmapsto f^*: \mathcal{M}_X(B') \to \mathcal{M}_X(B)$ functor }

$\boxed{\text{2-mor}}$ $B'' \xrightarrow{g} B' \xrightarrow{f} B$, $\underset{f\circ g}{\curvearrowright}$ $\longmapsto g^* \circ f^* \xrightarrow{\varepsilon_{g,f}} (f\circ g)^*$ ← not important

Sheaf of gruppoids $\{U_i \to U\}$

- (Sheaves of morph.) $X, Y \in \mathcal{F}(U)$, $\varphi_i : X|_i \to Y|_i$ s.t. $\varphi_i|_{ij} = \varphi_j|_{ij} \Rightarrow \exists\, \eta : X \to Y$ s.t. $\eta|_i = \varphi_i$
- (Mono) $X, Y \in \mathcal{F}(U)$, $X \overset{\varphi}{\underset{\psi}{\rightrightarrows}} Y$ s.t. $\varphi_i = \psi_i$, then $\varphi = \psi$
- (Glue of obj.) $X_i \in \mathcal{F}(U_i)$, $\varphi_{ij} : X_j|_{ij} \to X_i|_{ij}$: $\varphi_{ij}\varphi_{jk} = \varphi_{ik} \Rightarrow \exists\, X \in \mathcal{F}(U)$ and $\varphi_i : X|_i \cong X_i$ s.t. $\varphi_{ji} \circ \varphi_i|_{ij} = \varphi_j|_{ij}$

$$\begin{tikzcd}
 & \overset{\text{(not defined yet)}}{(\text{Alg. stacks})} \arrow[r, hook] & \overset{\substack{\text{(tricky def.)} \\ \text{stack} \\ \shortparallel}}{(\text{Sheaf})} \arrow[r, hook] & \overset{\substack{\text{2-functor} \\ \shortparallel}}{(\text{Presheaves of groupoids})} \\
(Sch) \arrow[r, hook] \arrow[ur] & (\text{Alg. spaces}) \arrow[r, hook] \arrow[u, hook] & \underset{\substack{\shortparallel \\ \text{space} \\ \text{sheaf. aca}}}{(\text{Spaces})} \arrow[r, hook] \arrow[u, hook] & \underset{\substack{\shortparallel \\ \text{functor}}}{(\text{Presheaves of sets})} \arrow[u, hook]
\end{tikzcd}$$

geom.

(Now: $\mathcal{F}$ holds <u>all</u> points, i.e. B-valued for any scheme B)

<u>Category fibered over Sch/S</u>

$$\begin{tikzcd}
\mathcal{F} \arrow[d, "p", "\text{structure functor}"'] \\
(Sch/S)
\end{tikzcd}$$

Example: Vr. bdles

Category fibered <u>on grupoids</u>

1)
$$\begin{tikzcd}
\exists X' \arrow[r, dashed, "\exists \phi"] \arrow[d, dashed, mapsto] & X \arrow[d, mapsto] \\
B' \arrow[r, "\forall"] & B
\end{tikzcd}$$
"existence of pullback..."

2) $\forall$
$$\begin{tikzcd}
X_3 \arrow[rr] \arrow[dd, mapsto] & & X_1 \arrow[dd, mapsto] \\
& X_2 \arrow[ur] \arrow[dd, mapsto] & \\
B_3 \arrow[rr] \arrow[dr] & & B_1 \\
& B_2 \arrow[ur] &
\end{tikzcd}$$
$\exists ! \varphi : X_3 \to X_2$ making the diag. comm.

"... [illegible] unique up to unique isom"

<u>Cor</u>: fiber $\mathcal{F}(B)$ is a grupoid

<u>Def</u>: A <u>stack</u> is a cat. fibered on grupoids s.t. $B \mapsto \mathcal{F}(B)$ is a sheaf.

<u>Morph. of stacks</u>: $f : \mathcal{F} \to \mathcal{G}$ functor s.t. $p_{\mathcal{G}} \circ f = p_{\mathcal{G}}$

$$Hom_S(\mathcal{F}, \mathcal{G}) = cat \begin{cases} ob : \text{morph. of stacks} \\ mor : \text{nat. transf.} \end{cases}$$

Comm. diagram
$$\begin{tikzcd}
& \mathcal{G} \arrow[dr, "g"] & \\
\mathcal{F} \arrow[ur, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "\Downarrow \alpha"] & & \mathcal{H}
\end{tikzcd}$$
α isom. of fctors

④

<u>Examples</u>: $\mathcal{M}_X$ $\begin{cases} \text{ob} & V \rightsquigarrow B\times X \\ \text{mor} & \begin{cases} f: B'\to B \\ (f\times id)^* V \xrightarrow{\cong} V' \end{cases} \end{cases}$

$[X/G]$: $\begin{cases} \text{ob:} & \begin{tikzcd} E \arrow[r, "\text{equiv}"] \arrow[d, "G\text{-bdle}"] & X \\ B & \end{tikzcd} \\ \text{mor:} & \begin{tikzcd} E' \arrow[r] \arrow[d] \arrow[rr, bend left] \arrow[dr, phantom, "\square"] & E \arrow[r, "\cong"] \arrow[d] & X \\ B' \arrow[r] & B & \end{tikzcd} \end{cases}$

$\overline{\mathcal{M}}_g$ $\begin{cases} \text{ob:} & \begin{tikzcd} X \arrow[d] \\ B \end{tikzcd} \ \left[\begin{array}{l}\text{stable curve} \\ \text{of genus } g \\ \text{over } B\end{array}\right. \\ \text{mor:} & \begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ B' \arrow[r] & B \end{tikzcd} \end{cases}$

$U \rightsquigarrow (Sch/U)$
scheme associated stack

$\begin{cases} \text{ob} & B \to U \quad \overset{p_U}{\longmapsto} \quad B \to U \to S \\ \text{mor} & \begin{tikzcd} B' \arrow[rr] \arrow[dr] & \circlearrowleft & B \arrow[dl] \\ & U & \end{tikzcd} \end{cases}$

<u>Yoneda</u>: Morphism $U \to \mathcal{F}$ is the same thing as an object $X \in \mathcal{F}(U)$

<u>Fiber product</u>: $f_1: \mathcal{F}_1 \to \mathcal{G}$ $f_2: \mathcal{F}_2 \to \mathcal{G}$

Define $\mathcal{F}_1 \times_{\mathcal{G}} \mathcal{F}_2 = \begin{cases} \text{ob} & (\overset{\mathcal{F}_1(U)}{\overset{\cup\!\!|}{X_1}}, \overset{\mathcal{F}_2(U)}{\overset{\in}{X_2}}, \underset{\text{isom. in } \mathcal{G}}{\underset{\cap}{\alpha}}: f_1(X_1) \to f_2(X_2)) \\ \text{mor} & \begin{array}{c}(X_1, X_2, \alpha)/_U \\ \downarrow \\ (Y_1, X_2, \beta)/_V\end{array} \quad \begin{tikzcd} X_1 \arrow[d, "\phi_1"] \\ Y_1 \end{tikzcd} \ \begin{tikzcd} X_2 \arrow[d, "\phi_2"] \\ Y_2 \end{tikzcd} \ \text{s.t.} \begin{tikzcd} f_1(X_1) \arrow[r, "\alpha"] \arrow[d, "f_1(\phi_1)"'] & f_2(X_2) \arrow[d, "f_2(\phi_2)"] \\ f_1(Y_1) \arrow[r, "\beta"] & f_2(Y_2) \end{tikzcd} \end{cases}$

<u>Representability</u>: $\mathcal{X}$ represented by $\left\{\begin{array}{l}\text{alg. space} \\ \text{scheme}\end{array}\right\}$ if it is assoc. to an $\left\{\begin{array}{l}\text{alg. sp} \\ \text{scheme}\end{array}\right\}$

$f: \mathcal{F} \to \mathcal{G}$ representable if $\begin{tikzcd} U \times_{\mathcal{G}} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \forall\, U \arrow[r] & \mathcal{G} \end{tikzcd}$ $U \times_{\mathcal{G}} \mathcal{F}$ is rep. by alg. space
("fibers are alg. spaces")
$\underbrace{\qquad}_{\text{over schemes}}$

"P" property of morph, local on target, stable under pullback
(e.g. flat, smooth, étale, surj, finite type, ...)

Def: $\underset{\text{representable morph}}{f}$ has "P" if $\forall U \to \mathcal{G}$, the morph $U \times_{\mathcal{G}} \mathcal{F} \to U$ has "P"

Diagonal:

$$\begin{tikzcd} \mathrm{Iso}_U(X_1,X_2) \arrow[r] \arrow[d] & \mathcal{F} \arrow[d, "\Delta_{\mathcal{F}}"] \\ U \arrow[r, "(X_1{,}X_2)"] & \mathcal{F} \times_S \mathcal{F} \end{tikzcd}$$

Prop: $\Delta_{\mathcal{F}}$ representable $\Leftrightarrow$ $\forall \underset{\text{scheme}}{U}$, $U \to \mathcal{F}$ is representable

Def: $^{\text{D-M}}$Algebraic Stack: A stack $\mathcal{F}$ is algebraic if

1) $\Delta_{\mathcal{F}}$ is representable, quasi-compact and separated

(atlas) 2) $\exists$ scheme U, $U \twoheadrightarrow \mathcal{F}$ étale and surj.

Def: Artin Alg. Stack

1) — —

2) $\exists$ scheme U, $U \to \mathcal{F}$ smooth (hence loc. of finite type) and surj.

Prop: D.M. $\Rightarrow$ Δ unramified (Aut(x) discrete and reduced)

Art. $\Rightarrow$ Δ finite type (Aut(x) finite type)

Examples: $\coprod \mathrm{Quot}_m \longrightarrow \mathcal{M}_X$

$X \longrightarrow [X/G]$

"P" local property of schemes (regular, normal, reduced, char p)

Def: $\mathcal{F}$ has "P" $\Leftrightarrow$ the atlas has "P"

<u>Def</u>: <u>Substack</u> $\mathcal{E} \xhookrightarrow{i} \mathcal{F}$ full subcategory s.t.

1. $X \in \operatorname{ob}\mathcal{F}$ is in $\operatorname{ob}\mathcal{E}$, then the same holds for all isom. objects
2. $X \in \mathcal{E}(B) \Rightarrow$ all pullbacks f^*X are in $\mathcal{E}$
3. $X \in \mathcal{E}(B) \Leftrightarrow X|_i \in \mathcal{E}$ $\forall i$ where $\{U_i \to B\}$ is a covering

<u>Open inclusion</u> (closed, loc. closed): i is representable and open (closed, loc. closed)

<u>Def</u>: $\mathcal{F}$ <u>separated</u> iff $\Delta_{\mathcal{F}}$ is universally closed

<u>Criterion</u>: $\forall$ A valuation ring, K field of fractions

$$\begin{tikzcd}
 & & \mathcal{F} \arrow[d, "p"] \\
\operatorname{Spec} K \arrow[r, hook, "i"] \arrow[urr, bend left] & \operatorname{Spec} A \arrow[r] \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] & S
\end{tikzcd}$$

$p \circ g_1 = p \circ g_2$

$\alpha : g_1|_{\operatorname{Spec} K} \xrightarrow{\simeq} g_2|_{\operatorname{Spec} K}$

$\Rightarrow \exists\, \tilde{\alpha} : g_1 \to g_2$ extending α

Def: $f: \mathcal{F} \to \mathcal{G}$ separated if $\forall \underset{\text{affine}}{U} \to \mathcal{G}$, $U \times_{\mathcal{G}} \mathcal{F}$ is a separated stack.

Criterion:

$$\begin{tikzcd} & & \mathcal{F} \arrow[d] \\ \operatorname{Spec} K \arrow[r, "i"] & \operatorname{Spec} A \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r] & \mathcal{G} \end{tikzcd}$$

1) $\beta: p \circ g_1 \xrightarrow{\cong} p \circ g_2$

2) $\alpha: g_1|_{\operatorname{Spec} K} \xrightarrow{\cong} g_2|_{\operatorname{Spec} K}$

3) $f(\alpha) = \beta|_{\operatorname{Spec} K}$

$\Longrightarrow \exists\, \tilde{\alpha}: g_1 \to g_2$ s.t. $\tilde{\alpha}|_{\operatorname{Spec} K} = \alpha$

and $f(\tilde{\alpha}) = \beta$

Def: $\mathcal{F}$ proper if it is separated, finite type and

$$\exists \underset{\substack{\text{proper} \\ \text{scheme}}}{X} \longrightarrow \mathcal{F}$$

surj, representable

Criterion:

$$\forall \quad \begin{tikzcd} & & \mathcal{F} \arrow[d] \\ \operatorname{Spec} K \arrow[r] \arrow[urr, bend left, "g"] & \operatorname{Spec} A \arrow[r] & S \end{tikzcd}$$

$\exists$ K' finite field extn, A' integral closure in K'

$$\begin{tikzcd} & & \mathcal{F} \arrow[dd, bend left] \\ \operatorname{Spec} K' \arrow[r] \arrow[d, "u"] \arrow[urr, bend left, "g \circ u"] & \operatorname{Spec} A' \arrow[d] \arrow[ur, dashed, "\exists"] & \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} A \arrow[r] & S \end{tikzcd}$$

Point

$$\begin{tikzcd} \xi : \operatorname{Spec} K \arrow[r] \arrow[dr] & \mathcal{F} \arrow[d] \\ & S \end{tikzcd}$$

Dimension: Recall: X, Y loc. Noeth. schemes
$f : X \to Y$ flat

Then $\dim_x(X) = \dim_x(f) + \dim_{f(x)}(Y)$

$$\dim_x(f) = \dim_x(f^{-1}(x))$$

Déf: $\underset{\text{atlas}}{X} \xrightarrow{u} \underset{\text{loc. Noeth}}{\mathcal{F}}$ $\quad \dim_\xi(\mathcal{F}) := \dim_x(X) - \dim_x(u)$

Example: $\dim [X/G] = \dim X - \dim G$

$\dim BG = -\dim G$

Quasi-coh sheaf: $\forall\ X \to \mathcal{F}$, S_X sheaf on X

$$\begin{tikzcd} \forall\ X \arrow[r, "f"] \arrow[dr] & Y \arrow[d] \\ & \mathcal{F} \end{tikzcd}, \quad \varphi_f : S_X \xrightarrow{\simeq} f^* S_Y$$

(with cocycle)

Sheaf of differentials (or tangent): easy for D.M.

Cotangent complex for Artin

$\mathfrak{M}^{ss}_X$	$\mathcal{M}_X$
semistable	all
S-equiv.	isomorph
$V \sim V' \otimes p^*L$	$V \cong V'$
$\mathfrak{M}^s$ can be fine moduli space	$\mathcal{M}^s$ is never representable ($\mathrm{Aut} \supset \mathbb{C}^*$)

$$\begin{tikzcd}
{[R^s/GL(N)]} \arrow[r] & {[R^s/PGL(N)]} \\
\mathcal{M}^s_X \arrow[u, phantom, sloped, "\cong"] \arrow[r, "f"] & \mathfrak{M}^s_X \arrow[u, phantom, sloped, "\cong"]
\end{tikzcd}$$

$\dim f = -1 \Rightarrow \dim \mathcal{M}^s_X = \dim \mathfrak{M}^s_X - 1$

$\dim X = 1 \rightsquigarrow \dim \mathcal{M}_C = r^2(g-1)$ $\quad \dim \mathfrak{M}_C = r^2(g-1)+1$

$\mathcal{M}_C$ smooth