BARCELONA 29-V-09 (1)

GEOMETRIC LANGLANDS PROGRAM

(Frenkel notes: hep-th/0512172)
- Dennis Gaitsgory web page
- Beilinson and Drinfeld: "Quantiz. of Hitch. integr. system and Hecke eigensheaves"

Def: G-local system on a curve X is a pair (P, ∇)
P: holom. ppal G-bdle; ∇: holo. connect.

(Equiv. C^∞-ppal G-bdle with flat connection)

G complex reductive group $T \subset G$ max torus

$$\Big(\underset{\text{roots}}{\Delta} \subset \underset{\mathrm{Hom}(T,\mathbb{C}^*)}{\Lambda} \subset \mathfrak{t}^* \;,\; \Delta^\vee \subset \underset{\mathrm{Hom}(\mathbb{C}^*,T)}{\Lambda^\vee} \subset \mathfrak{t} \Big)$$

$${}^LG \; \big(\Delta_L \subset \Lambda_L \subset \mathfrak{t}^*_L \;,\; \Delta^\vee_L \subset \Lambda^\vee_L \subset \mathfrak{t}_L \big)$$

Langlands dual if $\exists$ isom $\varphi: \mathfrak{t} \to \mathfrak{t}^*_L$

$$\begin{array}{ccc} \mathfrak{t} & \longrightarrow & \mathfrak{t}^*_L \\ \cup & & \cup \\ \Lambda^\vee & \longleftrightarrow & \Lambda_L \\ \cup & & \cup \\ \Delta^\vee & \longleftrightarrow & \Delta_L \end{array}$$

Examples

G	${}^LG$
Gl_n	Gl_n
SL_n	PSL_n
SO_{2n}	SO_{2n}
SO_{2n+1}	Sp_n

Better def:

${}^LG$:= Tannaka group of $G[[t]]$-equiv. $\mathcal{D}$-modules on $G((t))/G[[t]]$

i.e.: $\mathrm{Rep}({}^LG) \xrightarrow{\cong} Sph_G$

$$\underset{\substack{\text{rep. with max}\\ \text{weight } \lambda}}{V^\lambda} \longmapsto IC_\lambda$$

Conj: For each irreducible ${}^LG$-local system on X, there exists a nonzero $\mathcal{D}$-module on Bun_G which is a Hecke eigensheaf (regular holonomic, irred.)

$$\mathrm{Loc}_{{}^LG} X \qquad\qquad \mathcal{D}\text{-modules on } \mathrm{Bun}_G$$

which is a Hecke eigensheaf

$$E \longmapsto K_E$$

$$\mathcal{H}_G = \{P, P', x, \beta : P|_{X-x} \xrightarrow{\cong} P'|_{X-x}\}$$

$$\begin{array}{ccc} & \mathcal{H}_G & \\ \swarrow^{\alpha} & & \searrow^{\beta} \\ \mathrm{Bun}_G \ni P' & & (x, P) \in X \times \mathrm{Bun}_G \end{array}$$

$$T^\lambda(K_E) = E_{\rho_\lambda} \boxtimes K_E$$

$$T^\lambda(K_E) \overset{\cdot\cdot}{=} R\alpha_*\left(\beta^*(K_E) \otimes IC_\lambda\right)$$

$$\mathcal{H}_G \xrightarrow{} X, \qquad \text{fiber} = L_x G \backslash G(\mathcal{K}) \times_{G(\mathcal{O})} G(\mathcal{K})/G(\mathcal{O})$$

Particular case: $G = \mathbb{C}^*$ $\qquad X \longrightarrow J(X) = H^1(X, \mathcal{O})/H^1(X, \mathbb{Z})$

$H_1(X) \cong H_1(J(X)) = H^1(X, \mathbb{Z})$

$H_1(X) = \pi_1^{ab}(X)$, $\quad H_1(J(X)) = \pi_1(J(X)) = H_1(X, \mathbb{Z}) \cong H^1(X,\mathbb{Z})$

cup prod unimodular

$$\mathrm{Loc}_1 X \longrightarrow \mathrm{Loc}_1 J(X) \subset \mathcal{D}\text{-mod on } J(X)$$
$$E \longmapsto F_E$$

1) $i^* F_E \cong E$

2) $m^* F_E = F_E \boxtimes F_E \qquad m : \mathrm{Pic} \times \mathrm{Pic} \to \mathrm{Pic}$

Derived version of Geometric Langlands Conjecture

$$
\begin{tikzcd}
D(\mathrm{Loc}_{^LG} X, \mathcal{O}) \arrow[r, leftrightarrow] \arrow[d, squiggly, "\text{deforms}"'] & D(\mathrm{Bun}_G, \mathcal{D}) \arrow[d, squiggly, "\text{classical limit}"] \\
D(T^*\mathrm{Bun}_{^LG} X, \mathcal{O}) & D(T^*\mathrm{Bun}_G X, \mathcal{O})
\end{tikzcd}
$$

Hitchin maps:

$$
\begin{tikzcd}
T^*\mathrm{Bun}_{^LG} X \arrow[d] \arrow[r, phantom, "\text{dual fibers}"] & T^*\mathrm{Bun}_G X \arrow[d] \\
\bigoplus_{1}^{\mathrm{rk}\, G} H^0(X, \Omega^{\otimes d_i}) \arrow[r, phantom, "="] & \bigoplus_{1}^{\mathrm{rk}\, G} H^0(X, \Omega^{\otimes d_i})
\end{tikzcd}
$$

(Recent related work: Ngo Bao Chau "Fibration de Hitchin et endoscopie"
Invent. 2006, math/0406599

Quantum integrable system on Y

Ring homom. $\underset{\substack{\text{comm.}\\\text{alg.}}}{h : A} \longrightarrow \Gamma(Y, \mathcal{D}_Y)$

(in fact, we must consider twisted $\mathcal{D}$-modules with a line bdle L $\mathcal{D}_Y^L = \mathcal{D}iff(L, L)$)

The tangent bundle T_Y has a Lie algebra str. This Lie bracket extends uniquely to a Poisson bracket $\{\cdot,\cdot\}$ on the sheaf of rings $\mathrm{Sym}\, T_Y$ by Leibniz $\{f, gh\} = \{f, g\}h + \{f, h\}g$

· Classical integrable system

$$h^{cl} : \underset{\substack{\text{comm.}\\\text{ring}}}{A^{cl}} \longrightarrow \Gamma(Y, \mathrm{Sym}\, T_Y)$$

s.t. $\{h^{cl}(a_1), h^{cl}(a_2)\} = 0 \quad \forall a_1, a_2 \in A^{cl}$

Def: h quantization of h^{cl} if A filtered, h comp. with filtration, $\mathrm{gr} A \cong A^{cl}$, and

$$\begin{array}{ccc} A^{cl} & \xrightarrow{h^{cl}} & \Gamma(Y, \mathrm{Sym}\, T_Y) \\ \wr\!\wr & & \uparrow \\ \mathrm{gr} A & \xrightarrow{\mathrm{gr}\, h} & \mathrm{gr}\, \Gamma(Y, \mathcal{D}_Y) \end{array}$$

$\mathcal{D}$-module associated to h quant. int. system.

$$\begin{array}{l} \mathrm{Spec}\, A \\ \quad \Psi \\ \quad \sigma \longmapsto \mathcal{F}_\sigma := \mathcal{D}_Y \otimes_A A/\sigma \\ \text{maximal ideal} \end{array}$$

Example 1:

$$A = \mathbb{C}[\partial_1, \dots, \partial_m] \xrightarrow[\text{inclusion}]{h} \Gamma(Y, \mathcal{D}_Y) = \mathbb{C}[x_1, \dots, x_m, \partial_1, \dots, \partial_m]$$

$$Y = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_m]$$

$$\begin{array}{ccc} \operatorname{Spec} A & \longrightarrow & \mathcal{D}\text{-modules on } Y \\ \psi & & \\ (\partial_1 - a_1, \dots, \partial_m - a_m) & \longmapsto & \mathcal{D}_Y \otimes_A A/m \end{array}$$

This is the $\mathcal{D}$-module generated by one generator ξ and with relation: $\partial_i \xi = a_i \xi$

This corresponds to the ~~tan~~ trivial line bdle with flat connection given by $d + \sum a_i dx_i$

Example 2: X = curve, $Y = J$ Jacobian $= H^1(X, \mathcal{O}_X)/\Lambda$

$A = \mathrm{Sym}\, H^1(X, \mathcal{O}_X)$

$$h : \mathrm{Sym}\, \underbrace{H^1(X, \mathcal{O}_X)}_{T_J \text{ at } 0} \longrightarrow \Gamma(J, \mathcal{D}_J)$$

$$\begin{gathered} \mathrm{Spec}\, \mathrm{Sym}\, H^1(X, \mathcal{O}_X) \\ \| \\ H^1(X, \mathcal{O}_X)^\vee \\ \| \\ \omega \in H^0(X, \Omega_X) \\ \| \end{gathered}$$

Local systems
where underlying
line bdle is $\mathcal{O}_X$

$(d + \omega)$

<u>Oper</u>: For $H = GL_n$, it is a triple

$$\Big(\underset{\text{vbr. bdle}}{E}, \underset{\text{connection}}{\nabla : E \to E \otimes \Omega}, \underset{\text{"full" flag } (\mathrm{rk}\, E_i = i)}{0 = E_0 \subset E_1 \subset \cdots \subset E_n = E} \Big)$$

such that
1) $\nabla(E_i) \subset E_{i+1} \otimes \Omega_X$
2) $\nabla_i : E^i \xrightarrow{\cong} E^{i+1} \otimes \Omega$

<u>Example</u>: $n = 2$, $\det(E) = \mathcal{O}$ $\quad 0 \to L \to E \to L^{-1} \to 0$

with $L \cong L^{-1} \otimes \Omega$, $\Rightarrow L^2 = \Omega$

The isom. class of E is uniquely determined:

$\mathrm{Ext}^1(L^{-1}, L) = H^1(L^2) = H^1(K) \cong \mathbb{C}$

This is the maximally unstable indecomposable vbr. bdle of rank 2

<u>Def</u>: $\underset{\in \mathbb{C}}{\lambda}$-connection. It is an operator

$$\nabla^\lambda : E \to E \otimes \Omega$$

s.t. $\nabla^\lambda(fm) = \lambda\, df \otimes m + f\, \nabla^\lambda(m)$

If $\lambda \neq 0$, $\frac{1}{\lambda} \nabla^\lambda$ is a usual connection

If $\lambda = 0$, it is an $\mathcal{O}_X$-linear homom $M \to M \otimes \Omega_X$

<u>Def</u>: Classical oper is ~~any ope~~ a λ-oper with $\lambda = 0$

<u>Prop</u>: $\mathrm{gr}\, \mathcal{O}(Op_{{}^LG}) = \mathcal{O}(Op^{cl}_{{}^LG})$

<u>Example</u>: $Op^{cl}_{sl_2} \cong H^0(\Omega_X^{\otimes 2})$

Op_{sl_2} is affine space with vbr. bdle $H^0(\Omega_X^{\otimes 2})$

$$\begin{tikzcd} \Omega^{1/2} \arrow[r] & E \arrow[r] \arrow[dr] & \Omega^{-1/2} \arrow[d] \\ & & \Omega^{3/2} \arrow[r, "i"] & E \otimes \Omega \end{tikzcd}$$

<u>Hitchin map</u>: $T^*\mathrm{Bun}_G \longrightarrow \mathrm{Hitch}_G(X) := \Gamma(X, \Omega_X \times_{\mathbb{G}_m} \mathfrak{g}^*/\mathrm{Ad}\, G)$

$\mathfrak{g}^*/\!/\mathrm{Ad}\, G := \mathrm{Spec}\, \underbrace{(\mathrm{Sym}\, \mathfrak{g})^G}_{\text{free ring by Chevalley}} = \mathrm{Spec}\, k[P_1, \dots, P_n] \qquad d_i = \deg P_i$

$T^*\mathrm{Bun}_G \ni (P, \gamma)$ $\quad \gamma \in H^1(X, \Omega_X \otimes P \times_G \mathfrak{g}^*) \to H^1(X, \Omega_X \times_{\mathbb{G}_m} (\mathfrak{g}^*/\mathrm{Ad}\, G)) = H^0(X, \bigoplus_{i=1}^{n} \Omega^{d_i})$

<u>Lemma</u>: $\mathrm{Hitch}_G(X) = \mathrm{Op}^{cl}_{{}^LG}(X)$ (and $\mathrm{Op}_{{}^LG}(X)$ is an affine space with vb sp. $\mathrm{Op}^{cl}_{{}^LG}(X)$)

Define the Classical integrable system using the Hitchin map:

$$\begin{array}{ccc} h^{cl}: \mathcal{O}(\mathrm{Op}^{cl}_{{}^LG}(X)) & \longrightarrow & \Gamma(\mathrm{Bun}_G, \mathrm{Sym}\, T_{\mathrm{Bun}_G}) \\ \| & & \| \\ \mathcal{O}(\mathrm{Hitch}_G X) & \longrightarrow & \mathcal{O}(T^*\mathrm{Bun}_G, \mathcal{O}_{T^*\mathrm{Bun}_G}) \end{array}$$

Beilinson and Drinfeld construct a quantization:

$$h: \mathcal{O}(\mathrm{Op}_{{}^LG}(X)) \longrightarrow \Gamma(\mathrm{Bun}_G, \mathcal{D}^L_{\mathrm{Bun}_G})$$

The fact that this is a quant. of h^{cl} is used to prove that K_E is nonzero, and to estimate its singular support