Introduction to D-Modules

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The Geometric Langlands Correspondence

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Warnings 1) It is not a historical account and no name will be given althought one should mention at least Sato, Kashiwara, Bernstein and many others.

2) For simplicity we will work on the holomorphic category although most of it can be done algebraically.

1) Differential equations

Example 1 Local differential equations

- $U \subseteq \mathbb{C}^n$ an open subset; $(z_1, \dots, z_n)$ coordinates
- $\mathcal{O}(U)$ space of holomorphic functions on U.
- $D_i = \frac{\partial}{\partial z_i}$; if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ is a multiindex

  then $|\alpha| = \sum \alpha_i$; $D^\alpha = D_1^{\alpha_1} \circ \dots \circ D_n^{\alpha_n}$
- A differential operator of order k is

$$P = \sum_{|\alpha| \leq k} f_\alpha D^\alpha$$

- A homogeneous system of linear differential equations of order k is

$$\boxed{\sum_{j=1}^{m} P_{ij}(u_j) = 0 \quad i=1,\dots,l}$$

- We want to study differential equations from an algebraic point of view and unleash all the power of sheaf theory.
- For instance, we say that two systems of differential equations are equivalent if "they have the same solutions".

How to recognize equivalent systems?

Example 2 Global differential equations

- X complex manifold of dimension n
- E a vector bundle; ∇ a holomorphic connection:
  $\nabla : \mathcal{E} \to \Omega^1(\mathcal{E})$ $\mathbb{C}$-linear + Leibnitz rule.
- The equation $\nabla s = 0$ is a global system of equations of order 1.
- ∇ can be extended to $\nabla : \Omega^k(\mathcal{E}) \to \Omega^{k+1}(\mathcal{E})$ by
  $$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \otimes \nabla s$$
- ∇ is flat iff $\nabla^2 = 0$
- If ∇ is flat the solutions to $\nabla s = 0$ form a local system

2) The Weyl algebra (local version)

$U \subseteq \mathbb{C}^n$; $\underline{D(U) = \mathcal{O}(U)[D_1, \dots, D_n]}$ is a non commutative ring.

Caution: We have to distinguish between left and right $D(U)$-modules.

- $\left(\sum_{j=1}^{\ell} P_{ij}(u_j) = 0\right)_{i=1,\dots,m}$ a system of equations

  $P = (P_{ij})$ matrix of operators

- We have a morphism of left $D(U)$ modules

$$D(U)^m \xrightarrow[\cdot P]{} D(U)^\ell$$

- Let M_P be the cokernel of $\cdot P$. It is the $\mathcal{D}(U)$-module associated to P.
- Let S be a space where $\mathcal{D}(U)$ acts (eg $\mathcal{O}(U)$) then

$$\mathrm{Hom}_{\mathcal{D}(U)}(M_P, S) = \{u = (u_1, \dots, u_\ell) \in S^\ell \mid P \cdot u = 0\}$$

is the space of solutions of our system.

- Therefore P and Q are equivalent iff

$$M_P \cong M_Q$$

3) The Weyl algebra (Global version)

- X a complex manifold of dimension n
- $\mathcal{O}_X$ sheaf of holomorphic functions
- $f \in \mathcal{O}_X(U)$ defines an operator $\hat{f} \circlearrowright \mathcal{O}_X(U)$
- <u>Definition</u> A differential operator of order k is a linear operator

$$D : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$$

Such that

$$[\hat{f}_k, \dots [\hat{f}_1, [\hat{f}_0, D]\dots] = 0$$

$$\forall f_0, \dots, f_k \in \mathcal{O}_X(U)$$

- The Differential operators of order k form a sheaf $\mathcal{D}_X^{(k)}$.
  We write
$$\mathcal{D}_X = \bigcup_k \mathcal{D}_X^{(k)}$$
- A $\mathcal{D}_X$-Module is a sheaf of modules over $\mathcal{D}_X$
- Remark
  - $\mathcal{D}_X^{(0)}(U) = \mathcal{O}_X(U)$
  - $\mathcal{D}_X^{(1)}(U) / \mathcal{D}_X^{(0)}(U) = \Theta_X$ the sheaf of tangent fields.
  - $\mathcal{D}_X$ is generated by $\mathcal{D}_X^{(1)}$.

Basic examples

1) $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module

2) $\omega_X = \Omega^n_X$ is a right $\mathcal{D}_X$-module with:

$$\omega \cdot g = g\omega \qquad g \in \mathcal{O}_X,\ \omega \in \omega_X$$

$$\omega \cdot \xi = -L_\xi \omega \qquad \xi \in \Theta_X,\ \omega \in \omega_X$$

3) $\mathcal{E}$ a vector bundle, ∇ a connection. Put

$$\nabla : \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \Omega^1(\mathcal{E}) \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

$$s \otimes m \longmapsto \nabla s \otimes m + \sum_{i=1}^{n} s\, dz_i \otimes D_i . m$$

is a map of right $\mathcal{D}_X$ modules

We consider the dual map

$$\mathcal{H}om_{\mathcal{D}_X}(\Omega^1(\mathcal{E})\otimes_{\mathcal{O}_X}\mathcal{D}_X, \mathcal{D}_X) \xrightarrow{\nabla^*} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{D}_X, \mathcal{D}_X)$$

is a map of <u>left</u> $\mathcal{D}_X$-modules.

Then $\check{\mathcal{E}} = \mathrm{Coker}(\nabla^*)$ is the $\mathcal{D}_X$-module associated to the system of equations given by ∇.

4) $\mathcal{E}$ vector bundle, ∇ a flat connection

$\mathcal{E}$ has a structure of left $\mathcal{D}_X$ module:

$$g \cdot s = g s$$

$$\xi \cdot s = \nabla_\xi s$$

The fact that ∇ is flat allows us to extend it to a $\mathcal{D}_X$-action.

Question: What is the relationship between

$\mathcal{E}$ and $\check{\mathcal{E}}$.

4) Basic properties and operations

- $\mathcal{D}_X$ is $\mathcal{O}_X$ quasicoherent
- $\mathcal{D}_X$ is locally noetherian (X manifold)
- we denote by $\mathcal{M}(\mathcal{D}_X)$ the category of left $\mathcal{D}_X$ modules
- $\mathcal{M}(\mathcal{D}_X)$ has enough injectives and finite homological dimension ($\leq 2\dim X$).
- we denote $D(\mathcal{D}_X) = D(\mathcal{M}(X))$ the derived category of bounded complexes of $\mathcal{D}_X$-mod
- $D_{coh}(\mathcal{D}_X) \subseteq D(\mathcal{D}_X)$ complexes with coherent coh.

Some operations:

- left - right translation:

$$\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{D}_X)^R$$
$$\mathcal{M} \mapsto \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$$

with action induced by :

$$\xi . \, w \otimes m = -L_\xi w \otimes m - w \otimes \xi . m$$

- Usual $Hom_{\mathcal{D}_X}, \underset{\mathcal{D}_X}{\otimes}$. For instace if $\mathcal{M}$ is a left $\mathcal{D}_X$ module

$Hom_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is a right $\mathcal{D}_X$ mod.

- Given a map $f : X \to Y$ there are functors

$$f^* : D(\mathcal{D}_Y) \to D(\mathcal{D}_X)$$
$$f_* : D(\mathcal{D}_X) \to D(\mathcal{D}_Y)$$
$$f^! : D(\mathcal{D}_Y) \to D(\mathcal{D}_X)$$
$$f_! : D(\mathcal{D}_X) \to D(\mathcal{D}_Y)$$

And there is a duality functor (contravariant)

$$\mathbb{D} : D(\mathcal{D}_X) \to D(\mathcal{D}_X)$$

$$\mathbb{D}\mathcal{M} = (R)\mathcal{H}om_{\mathcal{D}_X^R}(\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}, \mathcal{D}_X)[n]$$

<u>Example</u>: $(\mathcal{E}, \nabla)$ v.b. with flat connection: $\check{\mathcal{E}} = \mathbb{D}\mathcal{E}$

- Given a $\mathcal{D}_X$-module $\mathcal{M}$, the sheaf of solutions is

$$Sol(\mathcal{M}) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X) \in D(Sh(X))$$

- The de-Rham functor $DR: D(\mathcal{D}_X) \to D(Sh(X))$

$$DR(\mathcal{M}) = \mathcal{H}om_{\mathcal{D}_X}(\mathbb{D}(\mathcal{M}), \mathcal{O}_X)[n]$$

<u>Example</u>:

$$DR(\mathcal{E}) = Sol(\check{\mathcal{E}})[n] = \mathcal{E}^{\nabla}[n].$$

5 The Riemann–Hilbert Correspondence I

Hilbert 21 problem: Given a Riemann surface (non compact) there exists regular differential equations with prescribed monodromy?

For the moment forget about regularity.

Easy R-H correspondence:

X connected complex manifold; $x_0 \in X$

$\pi_1(X, x_0)$ fundamental group

$$\left\{ \begin{array}{c} \text{f.d reps of} \\ \pi_1(X, x_0) \end{array} \right\} / \simeq \;\longleftrightarrow\; \left\{ \begin{array}{c} \text{f.d. local} \\ \text{systems} \end{array} \right\} / \sim \;\longleftrightarrow\; \left\{ \begin{array}{c} \text{f.d. vector b.} \\ \text{with flat conect} \end{array} \right\} / \sim$$

$$\longrightarrow \{ \mathcal{D}_X\text{-Modules} \} / \sim$$

¿ Can we identify the image of this map?

Definition. A $\mathcal{D}_X$-module is called $\mathcal{O}$-coherent if it is coherent as a $\mathcal{O}_X$-module.

Theorem. Any $\mathcal{O}$-coherent $\mathcal{D}_X$-module $\mathcal{M}$ is locally free as $\mathcal{O}_X$-module and the action of $\mathcal{D}_X$ induces a flat connection.

We obtain a correspondence:

$$\left\{ \begin{matrix} \text{f.d reps of} \\ \pi_1(X, x_0) \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \mathcal{O}\text{-coherent} \\ \mathcal{D}_X\text{-modules} \end{matrix} \right\}$$

This is not what we really want.

1) What about regularity?

2) The category of local systems is not closed under π_*, π^*, $\pi_!$, $\pi^!$.

For instance if $U \overset{j}{\hookrightarrow} X$ is an open immersion, and E is a local system on U, what to do with $j_! E$ as a sheaf on X.

Definition: A sheaf $\mathcal{F}$ on X is constructible if there is a stratification of X by locally closed subspaces $\{Y_\lambda\}_{\lambda \in \Lambda}$ such that $\mathcal{F}|_{Y_\lambda}$ is a local system for all λ.

- Put $D_{cons}(X) = D_{cons}(Sh(X))$ the derived category of bounded complexes of sheaves on X whose cohomology sheaves are constructibles
- there are functors π_* π^* $\pi_!$ $\pi^!$ $\mathbb{D}$ on $D_{cons}(X)$.
- Can we identify a subcategory of $D(\mathcal{D}_X)$ that is equivalent (under DR) with $D_{cons}(X)$?

6) Holonomic $\mathcal{D}_X$-modules

$\mathcal{D}_X$ is a filtered sheaf of algebras by the order of the operator.

The associated sheaf of graded rings is

$$Gr_k \mathcal{D}_X = F_k\mathcal{D}_X / F_{k-1}\mathcal{D}_X$$

$Gr_* \mathcal{D}_X$ is a commutative ring

$$Gr_* \mathcal{D}_X = Sym^*(\Theta_X)$$

Hence $\underline{Spec}\,(Gr_* \mathcal{D}_X) = T^*(X)$

The sections of $Gr_* \mathcal{D}_X$ can be identified as functions on $T^*(X)$ polinomic along fibres.

Definition $\mathcal{M}$ a $\mathcal{D}_X$-module. A filtration of $\mathcal{M}$ is a collection $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{M}$ such that

$$\mathcal{M} = \bigcup_k \mathcal{F}_k \; ; \quad \mathcal{D}_X^{(k)} \mathcal{F}_j \subseteq \mathcal{F}_{j+k}.$$

A filtration is called good iff

- $\mathcal{F}_j$ is $\mathcal{O}_X$-coherent $\forall j$
- $\exists\, j \gg 0$ s.t. $\mathcal{D}_X^{(1)} \mathcal{F}_j = \mathcal{F}_{j+1}$

A $\mathcal{D}_X$ module has a good filtration iff it is $\mathcal{D}_X$-coherent.

Let $(\mathcal{M}, F)$ a coherent $\mathcal{D}_X$-module with a good filtration. Then $\mathrm{Gr}_\bullet \mathcal{M}$ is a module over $\mathrm{Gr}_\bullet \mathcal{D}_X$. We define the characteristic variety:

$$ch(\mathcal{M}) = V(\mathrm{Ann}(\mathrm{Gr}_\bullet \mathcal{M})) \subseteq T^*X.$$

<u>Proposition</u> $ch(\mathcal{M})$ does not depend on the good filtration. If $\mathcal{M} \neq 0$ then $\dim ch(\mathcal{M}) \geq n$.

<u>Definition</u>: $\mathcal{M}$ is holonomic if $\dim(ch(\mathcal{M})) = n$.

"Idea": Holonomic $\longleftrightarrow$ Integrable

Example $(\mathcal{E}, \nabla)$ vector bundle with connection. Then

$\check{\mathcal{E}} \neq 0$ and holonomic $\iff$ ∇ is flat.

- Define $D_{hol}(\mathcal{D}_X) \subseteq D_{coh}(\mathcal{D}_X)$ the derived category of bounded complexes with holonomic cohomology.

Proposition 1) The functors $\pi_*, \pi^*, \pi_!, \pi^!, \mathbb{D}$ conserve D_{hol}.

2) $\pi_! = \mathbb{D}\pi_*\mathbb{D}$ $\qquad \pi^* = \mathbb{D}\pi^!\mathbb{D}$.

3) There is a canonical morphism of functors

$$\pi_! \longrightarrow \pi_*$$

that is an isomorphism if π is proper

4) $\pi_!$ is left adjoint to $\pi^!$

π^* is left adjoint to π_*

5) if π is smooth $\pi^! = \pi^* [2(\dim Y - \dim X)]$

$(\pi : Y \to X)$

7 Regular singularities

Let C be a Riemann surface, C_+ a regular compactification, $x \in C_+ \setminus C$, t a local parameter

$$D = \frac{\partial}{\partial t} \qquad d = t \cdot D$$

$\mathcal{D}_C^{\vee}$ the sheaf of subalgebras of D_{C_+} generated by d. $\mathcal{D}_C^{\vee}$ does not depend on the choice of t.

<u>Definition</u> 1) A $\mathcal{O}$-coherent $\mathcal{D}_C$-module $\mathcal{M}$ is regular at x if $i_*(\mathcal{M})$ is a union of $\mathcal{O}$-coherent $\mathcal{D}_C^{\vee}$-modules.

2) A $\mathcal{O}$-coherent $\mathcal{D}_C$-module $\mathcal{M}$ is regular if it is regular at every point of $C_+ \setminus C$.

3) A Holonomic $\mathcal{D}_C$-module is said to have regular singularities (RS) if $\exists\, \mathcal{U}$ dense open subset such that $\mathcal{M}|_{\mathcal{U}}$ is regular

4) A Holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is said to have regular singularities if its restriction to any curve has RS.

We have categories $RS(X) \subseteq Hol(X)$ and $D_{RS}(\mathcal{D}_X) \subseteq D_{Hol}(\mathcal{D}_X)$: the derived category of complexes with RS cohomology.

<u>Theorem</u>: 1) $RS(X)$ is closed under subquotients and extensions.

2) The functors π_* π^* $\pi_!$ $\pi^!$ $\mathbb{D}$, send D_{RS} to D_{RS}.

8) The Riemann-Hilbert correspondence

<u>Theorem</u>: 1) $DR(D_{hol}(X)) \subseteq D_{cons}(X)$.

2) DR commutes with $\pi^*, \pi_*, \pi^!, \pi_!, \mathbb{D}$

3) DR gives an equivalence of categories

$$D_{RS}(X) \longleftrightarrow D_{cons}(X)$$

<u>Question</u>: $D_{RS}(x)$ has a full subcategory $RS(X)$: Holonomic $\mathcal{D}_X$-modules with regular singularities as complexes in degree 0.

What is the image of $RS(X)$ under DR?

<u>Definition</u> 1) An element $\mathcal{F}^\bullet$ of $D_{cons}(X)$ satisfies $(*)_{cons}$ if for every locally closed embedding $Y \xhookrightarrow{i} X$ there exists an open subset $U \subseteq Y$ such that $i^!(\mathcal{F}^\bullet)|_U$ has locally constant cohomology concentrated in degree $\geq -\dim Y$.

2) A perverse sheaf is a complex $\mathcal{F}^\bullet$ in $D_{cons}(X)$ such that $\mathcal{F}^\bullet$ and $D(\mathcal{F}^\bullet)$ satisfy $(*)$

Let $\mathrm{Perv}(X)$ be the full subcategory of $D_{cons}(X)$ of perverse sheaves.

<u>Theorem</u>. DR induces an equivalence of categories

$$RS(X) \xleftrightarrow{DR} \mathrm{Perv}(X).$$