Geometric Langland Conj.
(Beilinson-Drinfeld)
$\pi_1(X) \to G$

← Langlands (Number Th.)

← Fourier

← Physics EM duality
Olive-Montonen

FOURIER

$$G = \begin{cases} S^1 \\ \mathbb{R} \end{cases} \qquad \underset{\substack{\| \\ \text{char} \\ \chi : G \to U(1)}}{G^\vee} = \begin{cases} \mathbb{Z} & e^{inx} \quad n \in \mathbb{Z} \\ \mathbb{R} & e^{i\omega x} \quad \omega \in \mathbb{R} \end{cases}$$

$$\begin{tikzcd} & G^\vee \times G \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ G^\vee & & G \end{tikzcd} \qquad \chi(f, x) = \chi_f(x) \text{ univ. character.}$$

$$\mathbb{F} : \mathrm{Fun}(G^\vee) \longrightarrow \mathrm{Fun}(G)$$

"Fun" = L^2, Tempered distributions

$$f \longmapsto \pi_{2*}(\pi_1^*(f) \cdot \chi)$$

$$\underset{\text{points}}{\delta_\lambda(t)} \longleftrightarrow e^{i\lambda x} \text{ characters}$$

$$\begin{matrix} \text{mult.} \\ \cdot g \,/\, \cdot e^{i\lambda t} \,/\, \cdot t \end{matrix} \longleftrightarrow \begin{matrix} \text{conv} \,/\, \text{transl} \,/\, \text{diff} \\ * \hat{g} \quad T_\lambda \quad \frac{d}{dx} \end{matrix}$$

$\boxed{\text{Generalize "Function space" to } D(A, \mathcal{O})}$

A function generates an $\mathcal{O}_X$-module $\overset{\text{leap}}{\rightsquigarrow}$ use all coh. $\mathcal{O}_X$-modules
We need "integration" π_{2*} $\rightsquigarrow$ better use $D(A, \mathcal{O})$

$$\underset{\substack{\| \\ V/\Lambda}}{A} = \text{Abelian var.} \quad \mu : A \times A \to A \qquad \underset{\substack{\| \\ \bar{V}^*/\Lambda^*}}{A^\vee} = \left(\begin{matrix} L \\ \downarrow \\ A \end{matrix}, \mu^* L \xrightarrow{\sim} L \boxtimes L \right)$$

$$\begin{tikzcd} & A^\vee \times A \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ A^\vee & & A \end{tikzcd} \qquad \leftarrow \mathcal{P} \text{ Poincaré bundle}$$

Thm (Mukai)

$$\mathbb{F} : D(A^\vee, \mathcal{O}) \xrightarrow{\sim} D(A, \mathcal{O})$$

$$\mathcal{F} \longmapsto \pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{P})$$

$$\mathcal{O}_{[L]} \longmapsto L$$

Convolution $\qquad$ tensor prod.

$$\mathcal{F} * \mathcal{G} := \mu_*(\mathcal{F} \boxtimes \mathcal{G}) \qquad \otimes$$

$$\mathcal{O}_L * \mathcal{O}_M = \mathcal{O}_{L \otimes M} \longleftrightarrow L \otimes M$$

$\mathcal{D}$-modules

$\mathcal{D}_X$ = associat. algebra generated by alg. fcns $\mathcal{O}_X$ and vector fields T_X with relations

$$\partial f - f\partial = f'$$

$$\partial_1\partial_2 - \partial_2\partial_1 = [\partial_1, \partial_2]$$

What are $\mathcal{D}$-modules?

- Generalized fcns

$$f \longmapsto \mathcal{D}\cdot f \subset \mathrm{Fun}\, X \quad \text{(left) } \mathcal{D}\text{-module}$$

Examples $\mathbb{A}^1$ $\mathcal{D}_{\mathbb{A}^1} = \mathbb{C}\langle x, \partial_x\rangle / \partial x - x\partial = 1$ Weyl algebra

$$\mathcal{D}\cdot e^{\lambda x} \cong \mathcal{D}/\mathcal{D}(\partial_x - \lambda) \text{ “}= \mathbb{C}[x]\cdot e^{\lambda x}\text{”}$$

$$\mathcal{D}\cdot \delta_\lambda \cong \mathcal{D}/\mathcal{D}(x-\lambda) \text{ “}= \mathbb{C}[\partial]\cdot \delta_\lambda\text{”}$$

Weyl algebra has involution $x \longmapsto \partial_x$
$\partial_x \longmapsto -x$

Therefore $\mathcal{D}_{\mathbb{A}^1}\text{-mod} \longleftrightarrow \mathcal{D}_{\mathbb{A}^1}\text{-mod}$
“$e^{\lambda x}$” $\longleftrightarrow$ “$\delta_\lambda(t)$”

- Vector bdle with flat connection
$\updownarrow$
relations in $\mathcal{D}$ are satisfied

Another choice for "Fun":
derived cat. of (left) $\mathcal{D}$-modules

A = ab. var

$$\begin{array}{l} A^{\natural} \quad \text{"flat geometric characters"} \\ \big\downarrow \text{forget comm.} \quad \left(\begin{smallmatrix} L \\ \downarrow \\ A \end{smallmatrix}, \nabla, \mu^* L \cong L \boxtimes L\right) \\ A^{\vee} \end{array}$$

<u>Thm</u>: (Laumon, Rothstein)

$$D(A^{\natural}, \mathcal{O}) \xrightarrow{\sim} D(A, \mathcal{D})$$

<u>Remark</u>: This is a "quantization" of Mukai

$$\begin{tikzcd} A^{\vee} \times H^0(A, \Omega^1_A) \arrow[dr, "A^{\vee}"] & & T^{\vee}A \cong A \times T_0^{\vee}A \arrow[dl, "A"'] \\ & H^0(A, \Omega^1_A) = T_0^{\vee}A & \end{tikzcd}$$

Fiberwise Fourier-Mukai gives

$$\begin{tikzcd} D(A^{\vee} \times H^0(A, \Omega^1), \mathcal{O}) \arrow[r, "\sim"] \arrow[d, squiggly, "\text{deform}"] & D(T^{\vee}A, \mathcal{O}) \arrow[d, squiggly, "\text{quantize}"] \\ D(A^{\natural}, \mathcal{O}) \arrow[r, "\sim", "\text{Laumon-Rothstein}"'] & D(A, \mathcal{D}) \end{tikzcd}$$

Suppose $A = J(X)$ Jacobian of Riemann surface X

Fourier-Mukai
$$\begin{matrix} \mathcal{P} \\ \wr \\ J^\vee \times J \end{matrix} \qquad [L] \in J^\vee \quad \mathcal{O}_{[L]} \rightsquigarrow \begin{matrix} L \\ \wr \\ J \end{matrix}$$
$$D(J^\vee, \mathcal{O}) \xrightarrow{\sim} D(J, \mathcal{O})$$

Abel Jacobi $AJ_{x_0} : X \longrightarrow J(X)$ induces $\pi_1(X)^{ab} = H_1(X, \mathbb{Z}) \xrightarrow{\sim} \pi_1(J)$

Jac X is self-dual: $J^\vee = Jac(Jac\, X) \xrightarrow[AJ^*_{x_0}]{} J = Jac(X)$

Relative Fourier-Mukai

$$\begin{tikzcd}
J^\vee \times H^0(J, \Omega_J) \arrow[dr, "J^\vee"'] & & J \times T_0^\vee J = T^\vee J \arrow[dl, "J"] \\
& H^0(J, \Omega) = T_0^\vee J &
\end{tikzcd}$$

$$\begin{matrix} D(J^\natural, \mathcal{O}) & \xrightarrow[\text{Laumon-Rothstein}]{\sim} & D(J, \mathcal{D}) \\ \| & & \\ Loc_{\mathbb{C}^*} J & & \\ \text{Abel-Jacobi } \| & & \\ Loc_{\mathbb{C}^*} X & & \end{matrix}$$

$E \in Conn_{\mathbb{C}^*} X$ (thought of as $\mathcal{O}_{[E]}$, skyscraper on $Conn_{\mathbb{C}^*} X$)

Abel-Jacobi $X \hookrightarrow J(X)$. since $\pi_1^{ab} X = \pi_1^{ab} J(X) = \pi_1 J(X)$

there is a uniquely def. flat bundle K_E on $J(X)$

Think of this as a $\mathcal{D}$-module on $J(X)$

G complex reductive group $T \subset G$ max. torus

$$\Big(\underset{\text{roots}}{\Delta} \subset \underset{Hom(T,\mathbb{C}^*)}{\Lambda} \subset \mathfrak{t}^*, \quad \Delta^\vee \subset \underset{Hom(\mathbb{C}^*,T)}{\Lambda^\vee} \subset \mathfrak{t}\Big)$$

$${}^LG \quad \big(\Delta_L \subset \Lambda_L \subset \mathfrak{t}^*_L, \quad \Delta^\vee_L \subset \Lambda^\vee_L \subset \mathfrak{t}_L\big)$$

Langlands dual if $\exists$ isom

$$\begin{array}{ccc} \varphi: \mathfrak{t} & \longrightarrow & \mathfrak{t}^*_L \\ \cup & & \cup \\ \Lambda^\vee & \longleftrightarrow & \Lambda_L \\ \cup & & \cup \\ \Delta^\vee & \longleftrightarrow & \Delta_L \end{array}$$

Examples

G	${}^LG$	
GL_n	GL_n	A
SL_n	PSL_n	A
SO_{2n}	SO_{2n}	D
SO_{2n+1}	Sp_{2n}	$C \leftrightarrow D$
$Spin_{2n+1}$	$Sp_{2n}/\mathbb{Z}/2$	$C \leftrightarrow D$
$Spin_{2n}$	$SO_{2n}/\mathbb{Z}/2$	D

Mirković-Vilonen

${}^LG$:= Tannaka group of $G[[t]]$-equiv $\mathcal{D}$-modules on $G((t))/G[[t]]$

i.e. $Rep({}^LG) \xrightarrow{\cong} Sph_G$

$V^\lambda \longmapsto IC_\lambda$

rep. with maximal weight λ

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Beilinson-Drinfeld: Opers G
Frenkel-Gaitsgory-Vilonen: Irred. GL_n conn.

<u>Conj</u>: For each irred. ${}^LG$-local system on X (Riemann surf.) there exists a nonzero $\mathcal{D}$-module on Bun_G which is a Hecke eigensheaf (regular irred. holonomic)

$$\mathrm{Loc}_{{}^LG} X \qquad\qquad \mathcal{D}\text{-modules on } \mathrm{Bun}_G$$

$$E \longmapsto K_E$$

$$\mathcal{H}_G = \{ P, P', x, \beta : P|_{X-x} \xrightarrow{\cong} P'|_{X-x} \}$$

$$\begin{array}{ccc} & \mathcal{H}_G & \\ \swarrow^{\alpha} & & \searrow^{\beta} \\ \mathrm{Bun}_G \ni P' & & (x,P) \in X \times \mathrm{Bun}\, G \end{array}$$

$$T^\lambda(K_E) = E_{\rho_\lambda} \boxtimes K_E$$

$$T^\lambda(K_E) = R\alpha_* (\beta^*(K_E) \otimes IC_\lambda)$$

Derived version of Geometric Langlands Conj.

$$\begin{array}{ccc} D(\mathrm{Loc}_{{}^LG} X, \mathcal{O}) & \xrightarrow{\sim} & D(\mathrm{Bun}_G, \mathcal{D}) \\ \Big\uparrow \text{deforms} & & \Big\uparrow \text{quant.} \\ D(T^*\mathrm{Bun}_{{}^LG} X, \mathcal{O}) & & D(T^*\mathrm{Bun}_G X, \mathcal{O}) \end{array}$$

Hitchin maps:

$$\begin{array}{ccc} T^*\mathrm{Bun}_{{}^LG} X & & T^*\mathrm{Bun}_G X \\ \downarrow & & \downarrow \\ \bigoplus H^0(X, \Omega^{\otimes d_i}) & = & \bigoplus_{1}^{\mathrm{rk}\, G} H^0(X, \Omega^{\otimes d_i}) \end{array}$$

choosing invariant bilinear pairing on $\mathfrak{g}$

$$T^\vee \mathrm{Bun}_n X = \{(E, \theta \in H^0(\mathrm{End}\,E \otimes K))\} = \mathrm{Higgs}_n X$$

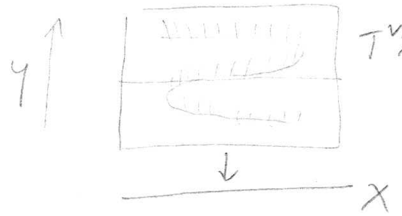
E is an $\mathcal{O}_X$-module

$$E \to E \otimes K_X$$

$$T \otimes E \to E$$

$$\mathrm{Sym}\,T \otimes E \to E$$

E becomes an $\mathcal{O}_{T^\vee X}$-module, denoted $\mathcal{E}$



$E = \pi_* \mathcal{E}$ (forget $\mathcal{O}_{T^\vee X}$-mod str.)

Characteristic polyn. of θ

$$y^n + a_1(t) y^{n-1} + a_2(t) y^{n-2} + \dots + a_n(t)$$ eq. of supp $\mathcal{E}$
Spectral curve

$$a_i \in H^0(K^{\otimes i}) \qquad i = 1, \dots, n$$

$$\begin{array}{c} T^\vee \mathrm{Bun}_n X \\ \big\downarrow h \quad \text{Hitchin map} \\ B = \bigoplus H^0(K^i) \end{array} \supset B^{reg}$$
(Y smooth)

$h^{-1}(Y \text{ smooth}) = J(X)$

$h^{-1}(0) = \mathrm{Bun}_n X \cup$ other components $=:$ nilpotent cone

Derived version of Geometric Langlands conj.

$$\begin{tikzcd}
D(\mathrm{Loc}_{{}^LG} X, \mathcal{O}) \arrow[r, "\sim"] & D(\mathrm{Bun}_G, \mathcal{D}) \\
D(T^*\mathrm{Bun}_{{}^LG} X, \mathcal{O}) \arrow[u, "\text{deforms}"] & D(T^*\mathrm{Bun}_G X, \mathcal{O}) \arrow[u, "\text{quant.}"']
\end{tikzcd}$$

$$\begin{tikzcd}
T^\vee \mathrm{Bun}_{{}^LG} \arrow[dr] & & T^\vee \mathrm{Bun}_G \arrow[dl] \\
A^\vee \arrow[u, phantom, "\subset"] \arrow[dr] & B = H^0(\Omega^{d_i}) & A \arrow[u, phantom, "\cup"] \arrow[dl] \\
& B^{reg} \arrow[u, phantom, "\cup"] &
\end{tikzcd}$$

1) For $G = GL_n$, the fibers over B^{reg} are Jacobians, which are self dual, therefore $A = A^\vee$

2) The duality between fibrations has been checked for reductive groups (Hitchin, Thaddeus-Hausel, Donagi-Pantev)

3) EM duality $\nabla \cdot E = 0$ $\quad \nabla \cdot B = 0$ $\quad (E, B) \mapsto (B, -E)$

$\nabla \times E = -\frac{\partial B}{\partial t}$ $\quad \nabla \times B = \frac{\partial E}{\partial t}$

Olive-Montonen Yang-Mills

Kapustin-Witten SUSY $N=1$ TQFT