ISOMORFISME DE SATAKE GEOMÈTRIC

CSIC
12/10/07

1. Àlgebra de Hecke
2. Isomorfisme de Satake

$\underbrace{\mathbb{F}_q((t))}_{F}, \underbrace{\mathbb{F}_q[[t]]}_{\mathcal{O}}$

~~3. Feixos perversos, Grassmanniana afí, functor-sheaf dictionary~~

4. Geometrització de Satake.
   Conseqüència: construcció canònica de ^LG via dualitat de Tannaka.

5. Hecke eigensheaves, G arbitrari.

Referències

1. Thm. 4.4.4. + 4.4.7. [CM] pg. 594
   §5.1 [F] pg. 56

2. §5.2 [F], pg. 58 (fórmula $\hat{f} = \ldots$)

3.

4. §5.4 ([F], pg. 60, [CM] pg. 599 → convolution product & weight functor

5. §6.1 [F], pg. 64
   ~~[CM] pg.~~

[CM] de Cataldo, Migliorini, Perverse sheaves..., BAMS

[F] E. Frenkel, Lectures on Langlands ..., BAMS

1.) $F = \mathbb{F}_q((t))$ G reductive Lie group

$\mathcal{O} = \mathbb{F}_q[[t]]$

$\mathcal{H} = \mathcal{H}(G(F), G(\mathcal{O})) = \{ f : G(F) \to \mathbb{C}, \text{ supp } f \text{ cpct., bi-inv't} \}$

convolution product:

$$(f_1 \star f_2)(g) = \int_{G(F)} f_1(gh^{-1}) f_2(h)\, dh$$

Haar measure, Vol $G(\mathcal{O}) = 1$.

Example Torus T

$$\mathcal{H}(T(F), T(\mathcal{O})) \overset{\text{algebra isom.}}{\simeq} \mathbb{C}[\check{P}]$$

$\check{P}$ = group of cocharacters of T (= characters of $\check{T}$).

because

$$T(\mathcal{O}) \backslash T(F) / T(\mathcal{O}) \;\; = \;\; T(F)/T(\mathcal{O}) = \{ \check{\lambda}(t) \}_{\check{\lambda} \in \check{P}}$$

For general G, $T \subset G$ max. torus ($\leftarrow$ split G)

$$G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}) \simeq \check{P}_+ \;\; (= \check{P}/W)$$

Hence $\{ c_{\check{\lambda}} \}_{\check{\lambda} \in P_+}$ basis of $\mathcal{H}(G(F), G(\mathcal{O}))$

$c_{\check{\lambda}}$ = char. function of $G(\mathcal{O})\, \check{\lambda}(t)\, G(\mathcal{O})$

Restriction map

$$\mathcal{H}(G(F), G(\mathcal{O})) \to \mathcal{H}(T(F), T(\mathcal{O}))^W$$

is not morphism of algebras!

... but: 2)

Theorem (Satake) ~~$\mathbb{Z}$~~ let $N \subset G$ unipotent group of Borel

The map
$$\mathcal{H}(G(F), G(\mathcal{O})) \longrightarrow \mathbb{C}[\check{P}] \otimes \mathbb{Z}[q^{\pm 1/2}]$$
$$f \longmapsto \hat{f} = \sum_{\check{\lambda} \in \check{P}} \left(q^{\langle \rho, \check{\lambda} \rangle} \int_{N(F)} f(n \cdot \check{\lambda}(t))\, dn \right) e^{\check{\lambda}}$$

is an injective morphism of algebras, ~~and~~ and it induces an isomorphism
$$\mathcal{H}(G(F), G(\mathcal{O})) \otimes \mathbb{Z}[q^{\pm 1/2}] \xrightarrow{\simeq} \mathbb{C}[\check{P}]^W \otimes \mathbb{C}[q^{\pm 1/2}]$$

Observation (Langlands)
$$\mathbb{C}[\check{P}]^W \simeq \mathrm{Rep}({}^L G).$$

4) Recall that

$$\{ c_{\check\lambda} = \chi_{G(\mathcal{O})\check\lambda(t)G(\mathcal{O})} \}_{\check\lambda \in \check P_+}$$

is basis of $\mathcal{H}(G(F), G(\mathcal{O}))$

Under Satake isomorphism, if $\check\lambda \in \check P_+$,

$$\mathrm{Rep}({}^L G) \ni [V_{\check\lambda}] \mapsto H_\lambda = q^{-\langle \check\rho, \check\lambda \rangle} c_{\check\lambda} + \sum_{\substack{\check\mu \in \check P_+ \\ \check\mu < \check\lambda}} a_{\lambda\mu} c_\mu$$

$a_{\lambda\mu} \in \mathbb{Z}_+[q]$.

$$\left.\begin{array}{l} \lambda := \check\lambda \\ \mu := \check\mu \\ \vdots \end{array}\right\} \in \check P_+$$

Interpretation of $a_{\lambda\mu}$ (~~Kazhdan~~ Lusztig)

$Gr_\lambda := G(\mathcal{O}) \cdot \lambda(t) G(\mathcal{O}) \subset G(F)/G(\mathcal{O})$

$G(\mathcal{O})$-orbit

$$\overline{Gr_\lambda} = \bigcup_{\substack{\mu \le \lambda \\ \mu \in \check P_+}} Gr_\mu$$

$IC_\lambda \in D(\overline{Gr_\lambda})$ intersection cohomology ~~sheaves~~ sheaf

Theorem $\quad a_{\lambda\mu} = \sum_i a^{(i)}_{\lambda\mu} q^{i/2}$,

$a^{(i)}_{\lambda\mu}$ = dim of i-th stalk cohomology of IC_λ on $Gr_\mu \subset \overline{Gr_\lambda}$.

$a_{\lambda\mu}(1)$ = coef's of simple factors in Jordan-Hölder decomposition of Verma modules.

4 cont'd) Geometric version

$F = \mathbb{C}((t))$

$\mathcal{O} = \mathbb{C}[[t]]$

$Gr_G = G(F)/G(\mathcal{O})$ affine Grassmannian

Orbits of $G(\mathcal{O}) \curvearrowright Gr_G$ are finite dim'l, of shape

$$\begin{array}{l} Orb_\lambda = G(\mathcal{O}) \cdot \check{\lambda}\, G(\mathcal{O}) \\ \parallel \\ Gr_\lambda \end{array}$$

$\mathcal{P}_{G(\mathcal{O})}$ = category of $G(\mathcal{O})$ equivariant perverse sheaves on Gr_G.

Bernstein-Drinfeld: $\mathcal{P}_{G(\mathcal{O})}$ generated by IC_λ, $\lambda \in \check{P}_+$.

$$K(\mathcal{P}_{G(\mathcal{O})}) \simeq Rep[{}^LG].$$

Mirković - Vilonen promote previous isomorphism of algebras to equivalence of Tannakian categories:

$$(\mathcal{P}_{G(\mathcal{O})}, \star, H) \xrightarrow{\simeq} (Rep({}^LG), \otimes, \omega)$$

by defining

$\star$ = convolution product

H = weight (fiber) functor (hypercohomology)

Corollary Canonical construction of ${}^LG$ (Tannaka duality) (over $\mathbb{Z}$).

Observation $IC_\lambda \longleftrightarrow V_\lambda$

5) ~~Eigenvalues of Hecke~~

Hecke eigensheaves

X curve

Bun_G = stack of G-bundles $/X$

$$\mathcal{H}ecke = \left\{ (\mathcal{M}, \mathcal{M}', x, \beta) \;\middle|\; \begin{array}{l} \mathcal{M}, \mathcal{M}' \ G\text{-bdls}/X, \quad x \in X, \\ \beta : \mathcal{M}|_{X \setminus x} \xrightarrow{\simeq} \mathcal{M}'|_{X \setminus x} \end{array} \right\}$$

$$\begin{tikzcd} & \mathcal{H}ecke \arrow[dl, "h^{\leftarrow}"'] \arrow[dr, "h^{\rightarrow}"] & \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{tikzcd}$$

$$(h^{\rightarrow})^{-1}(x, \mathcal{M}') = \mathcal{M}'(\mathcal{O}_x) \underset{G(\mathcal{O}_x)}{\times} \overbrace{\mathrm{Gr}_x}^{G(F_x)/G(\mathcal{O}_x)}$$

Hecke functor

$\mathcal{F}$ = perverse sheaf $/ \mathrm{Bun}_G$

$\lambda \in \check{P}_+$

$$\boxed{H_\lambda(\mathcal{F}) := h^{\rightarrow}_*\left(h^{\leftarrow *}(\mathcal{F}) \otimes IC_\lambda\right)}$$

Let E be a ${}^LG$ local system ~~on~~ on X

$\mathcal{F}$ is a Hecke eigensheaf with eigenvalue E if

$$\forall \lambda \ \exists \quad \iota_\lambda : H_\lambda(\mathcal{F}) \xrightarrow{\simeq} (E \underset{{}^LG}{\times} V_\lambda) \boxtimes \mathcal{F}$$

and $\{\iota_\lambda\}$ compatible with Tannakian structure on both sides.

5 cont'd)

<u>Example</u> $G = GL(n, \mathbb{C})$.

$\mathcal{P}(G)$ generated as tensor category by

$\{ IC_\lambda \mid \lambda$ fundamental weights$\}$

$\updownarrow$

$\{ \Lambda^i V \}$ $V =$ fund. representation

induce usual Hecke transforms