# Mirror symmetry, Langlands duality, and the Hitchin system

Study group: the geometric Langlands correspondence

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This is a introduction to part of the paper

Tamás Hausel, Michael Thaddeus: Mirror symmetry, Langlands duality, and the Hitchin system, Invent. Math. **153** (2003) 197-229 (arXiv:math/0205236)

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mirror symmetry and the Hitchin system.

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  - Toric Geometry (Batirev-Borisov; Givental; Lian-Lin-Yau)

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  - Homological MS
  - Toric Geometry
  - ...
  - Strominger-Yau-Zaslow interpret MS in terms of special Lagrangian torus fibrations.

# Strominger–Yau–Zaslow (SYZ) (1997)

**Def** (1) An $n$-dimensional complex manifold $M$ is called Calabi-Yau (CY) if:

- $\exists\ \omega =$ Kähler form of a Ricci-flat metric on $M$
- $\exists\ \Omega =$ nowhere vanishing holomorphic $n$-form (holomorphic volume form)

- automatic by parallel transport if $M$ is simply connected
- required for simplicity (for calibrated geometry), but usually not part of the definition

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(2) A real submanifold $L \subset M$ of a CY is called special Lagrangian (sLag) if:

- $L \subset M$ is Lagrangian (i.e. $\omega|_L = 0$ & $\dim_{\mathbb{R}} L = n$)
- $\operatorname{Im} \Omega|_L = 0$

Let $T$ = real torus and $T^\vee$ = dual torus of $T$. Recall that

$$T \cong \left\{ \begin{array}{c} \text{flat principal} \\ U(1)\text{-bundles} \\ (E,\nabla) \text{ over } T^\vee \end{array} \right\} \quad \text{and} \quad T^\vee \cong \left\{ \begin{array}{c} \text{flat principal} \\ U(1)\text{-bundles} \\ (F,\nabla) \text{ over } T \end{array} \right\} \text{(canonically)}.$$

$$T \cong \left\{\begin{array}{c}\text{flat principal} \\ U(1)\text{-bundles} \\ (E,\nabla) \text{ over } T^\vee\end{array}\right\} \text{ and } T^\vee \cong \left\{\begin{array}{c}\text{flat principal} \\ U(1)\text{-bundles} \\ (F,\nabla) \text{ over } T\end{array}\right\}.$$

In concrete terms:

$$T = W/\Lambda \quad \text{for} \quad \Lambda \cong \mathbb{Z}^n \text{ (lattice)} \subset W \cong \mathbb{R}^n$$

$$T^\vee = W^*/\Lambda^\vee \quad \text{for} \quad \underbrace{\Lambda^\vee}_{\text{(dual lattice)}} := \{\xi : W \to \mathbb{R} \mid \langle \xi, \lambda \rangle \in 2\pi\mathbb{Z} \ \forall \lambda \in \Lambda\} \subset \underset{\mathrm{Hom}(W,\mathbb{R})}{\overset{\shortparallel}{W^*}}$$

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Each $w \in \mathbb{W}$ defines an action of $\Lambda^\vee$ on the trivial line bundle

$$\begin{matrix} \mathbb{C} \\ \downarrow \\ \mathbb{W}^* \end{matrix}$$

by

$$\begin{aligned} \Lambda^\vee \times (\mathbb{W}^* \times \mathbb{C}) &\longrightarrow \mathbb{W}^* \times \mathbb{C} \\ (\xi, (\zeta, z)) &\longmapsto (\zeta + \xi, \chi_w(\xi) z) \end{aligned}$$

in terms of the character $\chi_w : \Lambda^\vee \longrightarrow U(1) : \xi \longmapsto \exp i\langle \xi, w \rangle$.

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$T = \mathbb{V}/\Lambda$ for $\Lambda \cong \mathbb{Z}^n$ (lattice) $\subset \mathbb{V} \cong \mathbb{R}^n$

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Each $w \in \mathbb{V}$ defines an action of $\Lambda^\vee$ on the trivial line bundle $\begin{matrix}\mathbb{C}\\ \downarrow\\ \mathbb{V}^*\end{matrix}$ by

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in terms of the character $\chi_w : \Lambda^\vee \longrightarrow U(1) : \xi \longmapsto \exp i\langle \xi, w \rangle$.

Then the trivial flat connection on $\begin{matrix}\mathbb{C}\\ \downarrow\\ \mathbb{V}^*\end{matrix}$ descends to a flat connection $\nabla_w$ on the quotient line bundle $L_w$ over $T^\vee = \mathbb{V}^*/\Lambda^\vee$, and

$$(L_w, \nabla_w) = (L_{w'}, \nabla_{w'}) \iff \chi_w = \chi_{w'} \iff [w] = [w'] \text{ in } T = V/\Lambda.$$

(for $w, w' \in \mathbb{V}$)

# Original formulation of the SYZ

If $M$ and $\hat{M}$ are a mirror pair of CY $n$-folds, then $\exists$ fibrations

$$\begin{tikzcd} M \arrow[dr, "\mu"'] & & \hat{M} \arrow[dl, "\hat{\mu}"] \\ & V & \end{tikzcd}$$

with $V =$ $n$-dimensional real manifold, whose generic fibres are SLag tori dual to each other, in the sense that there are canonical identifications between the fibres, for $x \in V$,

$$L_x \cong H^1(\hat{L}_x, \underline{U(1)}), \quad \hat{L}_x \cong H^1(L_x, \underline{U(1)}),$$

sheaf of locally constant $U(1)$-valued functions

whenever $L_x := \mu^{-1}(x)$ and $\hat{L}_x := \hat{\mu}^{-1}(x)$ are non-singular tori.

## Mirror symmetry exchanges the Hodge numbers:

If $M$ and $\hat{M}$ satisfy the SYZ prescription and have other nice properties, then it is expected that $\boxed{h^{p,q}(M) = h^{p,n-q}(\hat{M})}$.

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'Proof' (see [Gross] for details).

$$\begin{tikzcd} M \arrow[dr, "\mu"'] & & \hat{M} \arrow[dl, "\hat{\mu}"] \\ & V & \end{tikzcd} \quad \text{restricts to} \quad \begin{tikzcd} M_0 \arrow[dr, "\mu_0"'] & & \hat{M}_0 \arrow[dl, "\hat{\mu}_0"] \\ & V_0 & \end{tikzcd}$$

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with $V_0 := \left\{ x \in V \;\middle|\; \begin{array}{l} L_x \text{ and } \hat{L}_x \text{ are} \\ \text{smooth SLag tori} \end{array} \right\}$.

By SYZ, the SLag fibrations admit identifications

$$\begin{tikzcd} M_0 \arrow[d, "\mu_0"'] \\ V_0 \end{tikzcd} \equiv \begin{tikzcd} R^1(\hat{\mu}_0)_*(\underline{U(1)}) \arrow[d] \\ V_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} R^1(\mu_0)_*(\underline{U(1)}) \arrow[d] \\ V_0 \end{tikzcd} \equiv \begin{tikzcd} \hat{M}_0 \arrow[d, "\hat{\mu}_0"] \\ V_0 \end{tikzcd}$$

(with $R^1(\ )_* :=$ $1^{\text{st}}$ derived direct image).

$$\boxed{h^{p,q}(M) = h^{p,n-q}(\hat{M})}$$

$$\begin{tikzcd}[column sep=small] M \arrow[dr, "\mu"'] & & \hat{M} \arrow[dl, "\hat{\mu}"] \\ & V & \end{tikzcd} \quad \text{restricts to} \quad \begin{tikzcd}[column sep=small] M_0 \arrow[dr, "\mu_0"'] & & \hat{M}_0 \arrow[dl, "\hat{\mu}_0"] \\ & V & \end{tikzcd} \quad \text{with } V_0 := \{x \in V \mid L_x \text{ and } \hat{L}_x \text{ are smooth SLag tori}\}.$$

$$\text{SYZ} \Longrightarrow \begin{tikzcd}[column sep=small] M_0 \arrow[dr, "\mu_0"'] & & \hat{M}_0 \arrow[dl, "\hat{\mu}_0"] \\ & V_0 & \end{tikzcd} \equiv \begin{tikzcd}[column sep=small] R^1(\hat{\mu}_0)_*(\underline{U}(1)) \arrow[dr] & & R^1(\mu_0)_*(\underline{U}(1)) \arrow[dl] \\ & V_0 & \end{tikzcd}$$

Fibrewise,

$$\begin{tikzcd}[column sep=small] M_x \cong T = W/\Lambda \arrow[dr] & & \hat{M}_x \cong T^\vee = W^*/\Lambda^\vee \arrow[dl] \\ & \{x\} & \end{tikzcd} \quad \text{for } x \in V$$

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$$\text{SYZ} \Rightarrow \begin{tikzcd} M_0 \arrow[dr, "\mu_0"'] & & \hat{M}_0 \arrow[dl, "\hat{\mu}_0"] \\ & V_0 & \end{tikzcd} \equiv \begin{tikzcd} R^1(\hat{\mu}_0)_*(U(1)) \arrow[dr] & & R^1(\mu_0)_*(U(1)) \arrow[dl] \\ & V_0 & \end{tikzcd}$$

Fibrewise, $L_x \cong T = W/\Lambda$ $\quad \hat{L}_x \cong T^\vee = W^*/\Lambda^\vee$ for $x \in V$,

$$\begin{tikzcd} L_x \cong T = W/\Lambda \arrow[dr] & & \hat{L}_x \cong T^\vee = W^*/\Lambda^\vee \arrow[dl] \\ & \{x\} & \end{tikzcd}$$

where $\Lambda^p W^* \otimes \Lambda^n W \longrightarrow \Lambda^{n-p} W$

$\qquad\; H^p(T,\mathbb{R}) \cong \Lambda^p W^*$, $\quad \mathbb{R} \cong \Lambda^n W$, $\quad H^{n-p}(T^\vee,\mathbb{R}) \cong \Lambda^{n-p} W$

so $H^p(T,\mathbb{R}) \xrightarrow{\simeq} H^{n-p}(T^\vee,\mathbb{R})$

$$\boxed{h^{p,q}(M) = h^{p,n-q}(\hat{M})}$$

SYZ $\Rightarrow$

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where $\Lambda^p W^* \otimes \Lambda^n W \longrightarrow \Lambda^{n-p} W$ so $H^p(T, \mathbb{R}) \xrightarrow{\simeq} H^{n-p}(T^\vee, \mathbb{R})$

$$\begin{matrix} \Lambda^p W^* & \otimes & \Lambda^n W & \longrightarrow & \Lambda^{n-p} W \\ \cong & & \cong & & \cong \\ H^p(T, \mathbb{R}) & & \mathbb{R} & & H^{n-p}(T^\vee, \mathbb{R}) \end{matrix}$$

Globally, $\Omega|_{M_x}$ is a volume form $\forall$ sLag $L_x \Rightarrow R^n(\mu_0)_* \underline{\mathbb{R}} \cong \underline{\mathbb{R}}$, so

$$R^p(\mu_0)_* \underline{\mathbb{R}} \xrightarrow{\simeq} R^{n-p}(\hat{\mu}_0)_* \underline{\mathbb{R}}$$ as well.

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$$\begin{tikzcd} R^p(\mu_0)_* \mathbb{R} \arrow[r, "\simeq"] \arrow[d, bend right] & R^{n-p}(\hat{\mu}_0)_* \mathbb{R} \\ R^p \mu_* \mathbb{R} \arrow[r, "\simeq"] & R^{n-p} \hat{\mu}_* \mathbb{R} \end{tikzcd} \quad \text{over } V_0 \subset V$$

In nice cases, $i : V_0 \hookrightarrow V$ induces isomorphisms $\left\{ \begin{array}{l} i_* R^p(\mu_0)_* \mathbb{R} \cong R^p \mu_* \mathbb{R} \\ i_* R^p(\hat{\mu}_0)_* \mathbb{R} \cong R^p \hat{\mu}_* \mathbb{R} \end{array} \right.$, so

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$$R^p(\mu_0)_*\mathbb{R} \xrightarrow{\simeq} R^{n-p}(\hat{\mu}_0)_*\mathbb{R} \text{ over } V_0 \subset V \overset{?}{\Rightarrow} R^p\mu_*\mathbb{R} \xrightarrow{\simeq} R^{n-p}\hat{\mu}_*\mathbb{R}$$

$E_2$-terms of the Leray spectral sequences for $\mu$ and $\hat{\mu}$ are

$$E_2^{p,q} = H^q(V, R^p\mu_*\mathbb{R}), \quad \hat{E}_2^{p,q} = H^q(V, R^p\hat{\mu}_*\mathbb{R}).$$

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$R^p(\mu_0)_*\mathbb{R} \xrightarrow{\simeq} R^{n-p}(\hat{\mu}_0)_*\mathbb{R}$ over $V_0 \subset V \overset{?}{\Rightarrow} R^p\mu_*\mathbb{R} \xrightarrow{\simeq} R^{n-p}\hat{\mu}_*\mathbb{R}$

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$E_2^{p,q} = H^q(V, R^p\mu_*\mathbb{R})$, $\hat{E}_2^{p,q} = H^q(V, R^p\hat{\mu}_*\mathbb{R})$.

In nice cases, the spectral sequences degenerate at $E_2$,

$h^{p,q}(M) = \dim H^p(V, R^q\mu_*\mathbb{R}) = \dim H^p(V, R^{n-q}\hat{\mu}_*\mathbb{R}) = h^{p,n-q}(\hat{M})$.

"$\square$"

# The SYZ conjecture with flat gerbes

- In the original formulation of the SYZ conjecture, the fibrations

$$\begin{tikzcd} M \arrow[dr, "\mu"] & & \hat{M} \arrow[dl, "\hat{\mu}"'] \\ & V & \end{tikzcd}$$

restrict to sLag fibrations with sections

$$\begin{tikzcd} M_0 \arrow[dr, "\mu_0"] & & \hat{M}_0 \arrow[dl, "\hat{\mu}_0"'] \\ & V_0 \arrow[ul, bend left, "s"] \arrow[ur, bend right, "\hat{s}"'] & \end{tikzcd}$$

given by

$$s : V_0 \longrightarrow M_0$$

$$x \longmapsto \begin{pmatrix} \text{trivial} \\ \text{connection} \\ \text{over } \hat{M}_x \end{pmatrix} \in L_x = H^1(\hat{M}_x, U(1))$$

and

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- Hitchin (1999) proposed to use **flat gerbes** to refine the SYZ prescription for sLag fibrations without sections.

(These gerbes are also necessary in physics to obtain a duality between the cohomology groups $H^1(M, TM)$ and $H^1(\hat{M}, T^*\hat{M})$.)

# FLAT GERBES

Let $M$ be a manifold and $\underline{U}(1) =$ locally constant sheaf of $U(1)$-valued functions over $M$.

Recall that $H^1(M, \underline{U}(1)) \cong \{$isom. classes of flat principal $U(1)$-bundles over $M\}$

where locally constant functions $\longleftrightarrow$ flatness.

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More precisely, it is a sheaf of categories (a 'stack') which is a torsor over $\underline{U}(1)$, i.e. which is locally isomorphic to the sheaf of flat principal $U(1)$-bundles with their $U(1)$-action.

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An equivalence of gerbes is an equivalence of sheaves of categories (with the $U(1)$-action).

# TRIVILIAZATIONS OF FLAT GERBES

A trivialization of a gerbe $\mathcal{B}$ is an equivalence of $\mathcal{B}$ with the trivial gerbe, i.e. the gerbe that to each open set assigns the category of flat principal $U(1)$-bundles.

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$$\mathrm{Triv}^{U(1)}(M,\mathcal{B}) := \left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{trivialisations of } \mathcal{B} \end{array} \right\} \text{ is an } H^1(M,\underline{U(1)})\text{-torsor over a point.}$$

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By the definition of gerbe, $\exists$ an open cover $\{U_\alpha\}$ s.t. $\mathcal{B}_\alpha := \mathcal{B}|_{U_\alpha}$ is trivial, and then $\mathcal{B}$ is glued by 'transition functors' $F_{\alpha\beta} : \mathcal{B}_\alpha|_{U_{\alpha\beta}} \longrightarrow \mathcal{B}_\beta|_{U_{\alpha\beta}}$ (with $U_{\alpha\beta} = U_\alpha \cap U_\beta$) given by tensoring with flat line bundles $L_{\alpha\beta}$ over $U_{\alpha\beta}$, s.t. $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ are canonically trivial over $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, so they correspond to sections $g_{\alpha\beta\gamma}$ of $\underline{U(1)}$ over $U_{\alpha\beta\gamma}$ which define a Čech class $[(g_{\alpha\beta\gamma})] \in \check{H}^2(M, \underline{U(1)})$.

# The SYZ with flat gerbes (Hitchin, 1999)

Let $(M, B)$ and $(\hat{M}, \hat{B})$ be two CY orbifolds of dim. $n$ with flat unitary gerbes.

$$\begin{tikzcd} M \arrow[r, "\mu"] & V & \hat{M} \arrow[l, "\hat{\mu}"'] \end{tikzcd}$$

They are mirror partners if $\exists$ fibrations with $V = n$-dim'l real orbifold whose generic fibres are sLag tori dual to each other, in the sense that there are canonical identifications, for $x \in V$,

$$L_x \cong \mathrm{Triv}^{U(1)}(\hat{L}_x, \hat{B}_x), \quad \hat{L}_x \cong \mathrm{Triv}^{U(1)}(L_x, B_x),$$

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whenever $L_x := \mu^{-1}(x)$ and $\hat{L}_x := \hat{\mu}^{-1}(x)$ are non-singular tori.

Recall that the RHS's are torsors for $H^1(\hat{L}_x, \underline{U}(1))$ and $H^1(L_x, \underline{U}(1))$, respectively.

# SYZ for hyperkähler manifolds

- Constructing SLags is usually very difficult, but it is much easier if $M$ is hyperkähler.

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**Def** A manifold $M$ is called **hyperkähler** if it has a metric $g$ which is simultaneously Kähler w.r.t. three complex structures $I_1, I_2, I_3$ which satisfy the commutation relations of the quaternions $i, j, k$.

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Let $\omega_1, \omega_2, \omega_3$ be the corresponding Kähler forms (ie $\omega_i = g(-, I_i -)$). Then $\omega_2 + i\omega_3$ is a complex symplectic form on $M$ which is holomorphic w.r.t. $J_1$.

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The associated volume form $\Omega = (\omega_2 + i\omega_3)^n / n!$ gives a covariantly constant trivialization of $K_M = \Lambda^{n,0} T^*_M$, so $M$ is CY w.r.t. $J_1$.

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$\Longrightarrow$ $M$ is CY w.r.t. $J_1, J_2, J_3$.

$$\boxed{\text{Then any complex submanifold } L \subset M \text{ which is complex Lagrangian w.r.t. } J_1 \text{ is sLag wrt } J_2.}$$

# SYZ for hyperkähler manifolds

$(M, J_1, J_2, J_3)$ hyperkähler $\Rightarrow$ $M$ is CY w.r.t. $J_1, J_2, J_3$.

Then any complex submanifold $L \subset M$ which is complex Lagrangian w.r.t. $J_1$ is sLag wrt $J_2$.

$\Rightarrow$ To obtain SYZ sLag fibrations by holomorphic methods, we look for holomorphic maps whose generic fibres are complex Lagrangian w.r.t. $J_1$ (for hyperkähler $M, \hat{M}$ and complex $V$)

$$\begin{tikzcd} M \arrow[dr, "\mu"'] & & \hat{M} \arrow[dl, "\hat{\mu}"] \\ & V & \end{tikzcd}$$

and then change to the cx. str. $J_2$.
(hyperkähler rotation)

# Moduli space of Higgs bundles

Problem: find examples of SYZ mirror partners.

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Def

$(E, \phi)$ is stable if $\dfrac{\deg E'}{\operatorname{rk} E'} < \dfrac{\deg E}{\operatorname{rk} E}$ $\forall\, E' \subset E$ s.t. $\phi(E') \subset E' \otimes K$.

## Definitions

$M^d(GL_r) =$ moduli space of stable Higgs bundles $(E,\phi)$ with $E$ of rank $r$ and degree $d$

$M^d(SL_r) =$ moduli space of stable Higgs bundles $(E,\phi)$ with $\mathrm{rk}\, E = r$, $\deg E = d$, $\Lambda^r E \cong L_{SL}$

(for fixed line bundle $L_{SL}$ of degree $d$)

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$M^d(GL_r)$ and $M^d(SL_r)$ are hyperkähler smooth, quasi-projective.

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$$SL_r \xleftrightarrow{\text{Langlands}} PGL_r$$

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**Notation** ("Dol" = "Dolbeault", "DR" = "de Rham"):

$M^d_{Dol}(G) := M^d(G)$ with its complex structure as a moduli of Higgs bundles

$M^d_{DR}(G) := M^d(G)$ with its complex structure as a moduli of flat connections.

## The Hitchin system: $G = GL_r, SL_r$ or $PGL_r$

$\exists$ a completely integrable Hamiltonian system, given by the Hitchin map

$$\mu : M^d_{Dol}(G) \xrightarrow{\text{proper}} V_G = \bigoplus_{i=1\,(2)}^{r} H^0(C, K^i) : (E,\phi) \longmapsto \text{char. pol. of } \phi$$

$GL_r$ ($i=1$), $SL_r, PGL_r$ ($i=2$)

with • $\dim V_G = \frac{1}{2} \dim M^d_{Dol}(G)$

• generic fibre a complex Lagrangian torsor for an abelian variety

$\Longrightarrow \begin{cases} M^d_{DR}(SL_r) \text{ and } M^d_{DR}(PGL_r) \text{ carry families of sLag} \\ \text{tori over } V = V_{SL_r} = V_{PGL_r}, \text{ as in SYZ.} \end{cases}$

To show that they are SYZ mirror partners, we need to see that these tori are dual.

## Generic fibres:

**Def** • Given $\beta = (\beta_i) \in V_G$, the spectral cover $\begin{matrix} \tilde{C}_\beta \\ \downarrow \pi \\ C \end{matrix}$ is defined by

$$\tilde{C}_\beta = \{ z \in K \mid z^r + \beta_1 z^{r-1} + \dots + \beta_r = 0 \text{ in } K^r \} \subset \text{total space of } \begin{matrix} K \\ \downarrow \\ C \end{matrix}$$

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Hitchin computes the fibre $L_\beta := \mu^{-1}(\beta)$ for $\beta \in \mathcal{U}$:

• $G = GL_r$: $L_\beta = \tilde{J}^d := \mathrm{Pic}^d \tilde{C}_\beta$ — the fibre of a $\tilde{J}^0$-torsor over $\mathcal{U}$

• $G = SL_r$: $L_\beta = P^d := Nm^{-1}(\mathcal{O}(dc))$ — the fibre of a $P^0$-torsor over $\mathcal{U}$
(the generalized Prym variety) with $c \in C$ a base point and
$Nm : \mathrm{Pic}^d \tilde{C}_\beta \to \mathrm{Pic}^d C$ the norm map.

• $G = PGL_r$: $L_\beta = \hat{P}^d := P^d / \Gamma$ — the fibre of a $\hat{P}^0$-torsor over $\mathcal{U}$.

**Lemma** $\hat{P}^0 \cong \tilde{J}^0 / J^0$

**Proof.** $P^d \times J^0 \longrightarrow \tilde{J}^d : (L, M) \mapsto L \otimes \pi^* M^{-1}$ induces $\dfrac{P^d \times J^0}{\Gamma} \cong \tilde{J}^d$

and then $Nm : \tilde{J}^d \longrightarrow \mathrm{Pic}^d C$ corresponds to the composite

$\dfrac{P^d \times J^0}{\Gamma} \longrightarrow\!\!\!\!\rightarrow \dfrac{J^0}{\Gamma} \xrightarrow{\sim} J^0 : [L, M] \mapsto M \mapsto M^{-r}$.

Dualizing now $0 \to P^0 \to \tilde{J}^0 \xrightarrow{Nm} J^0 \to 0$, we get $0 \to J^0 \xrightarrow{\pi^*} \tilde{J}^0 \to \tilde{J}^0/J^0 \to 0$

with $\tilde{J}^0/J^0 \cong \dfrac{\frac{P^0 \times J^0}{\Gamma}}{J^0} = P^0/\Gamma = \hat{P}^0$. $\square$

# Trivializations of the flat unitary gerbe

$(\mathbb{E}, \Phi)$

$\downarrow$

$M_{Dol} \times C$

- If $(r,d)$ coprime, then $\exists$ universal Higgs bundle
- If $(r,d)$ not coprime, then $\nexists$ $\mathbb{E}$, but its projectivization $\mathbb{P}\mathbb{E}$ and $\Phi : \mathbb{E} \to \mathbb{E} \otimes K$ exist anyway.

$\Rightarrow$ the restriction $\mathbb{P}\mathbb{E}|_{M^d_{Dol} \times \{c\}}$ is a projective bundle $\Psi$ on $M^d_{Dol}$.

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Using the short exact sequence $1\to \underset{\mu_r}{\overset{\shortparallel}{\mathbb{Z}_r}}\to SL_r\to PGL_r\to 1$, we define the gerbe $\mathcal{B}$ of liftings of $\Psi$ to $SL_r$:

$$U\subset M^d_{Dol} \text{ étale nghbd} \longmapsto \mathcal{B}(U)=\left\{\begin{array}{l}\text{category of pairs } (Q,\varphi) \text{ where:}\\ \bullet\ Q = SL_r\text{-bundle on } U\\ \bullet\ \varphi: (Q\times PGL_r)/SL_r \xrightarrow{\simeq} \Psi|_U \text{ isom}\end{array}\right\}$$

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<u>Lemma</u> For $G = SL_r$, the restriction of $\mathcal{B}$ to each regular fibre $P^d_\beta = \mu^{-1}(\beta)$ is trivial as a $\mathbb{Z}_r$-gerbe, hence as a $U(1)$-gerbe.

Recall that equivalence classes of $U(1)$-trivializations form a torsor for $H^1(P^d, \underline{U(1)}) \cong \mathrm{Pic}^0 P^d \cong \mathrm{Pic}^0 P^0$, so the previous lemma implies that this is $\hat{P}^0$.

**Prop.** $\forall\, d, e \in \mathbb{Z}$, $\mathrm{Triv}^{U(1)}(P^d, B^e) \cong \hat{P}^e$ as $\hat{P}^0$-torsors.

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For the converse, we need to construct a flat $U(1)$-gerbe over the orbifold $\hat{M}^d_{Dol} = M^d_{Dol}(PGL_r) = M^d_{Dol}/\Gamma$, or equivalently, a $\Gamma$-equivariant flat $U(1)$-gerbe over $M^d_{Dol}$ $= M^d_{Dol}(SL_r)$.

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equivalently, a $\Gamma$-equivariant flat $U(1)$-gerbe over $M^d_{Dol}$ $= M^d_{Dol}(SL_r)$.

One defines $\hat{B}$ as in the case of $B$, with an action of $\Gamma = \mathrm{Pic}^0 C[r]$ induced by its equivariant action on $\mathbb{PE}$ and hence on $\Psi$.

**Lemma** For $G = PGL_r$ the restriction of $\hat{\mathcal{B}}$ to each regular fibre $\hat{P}^d = \hat{\mu}^{-1}(\beta)$ is trivial.

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**Prop.** $\forall\, d, e \in \mathbb{Z}$, $\mathrm{Triv}^{U(1)}(\hat{P}^d, \hat{\mathcal{B}}^e) \cong P^e$ as $P^0$-torsors.

In conclusion:

**Thm** $\forall\, d, e \in \mathbb{Z}$, $\left\{ \begin{array}{l} \mathrm{Triv}^{U(1)}(P^d, \mathcal{B}^e) \cong \hat{P}^e \\ \mathrm{Triv}^{U(1)}(\hat{P}^e, \hat{\mathcal{B}}^d) \cong P^d \end{array} \right\}$

and hence the moduli spaces $M^d_{DR}(SL_r)$ and $M^e_{DR}(PGL_r)$, equipped with the flat unitary (orbifold) gerbes $\mathcal{B}^e$ and $\hat{\mathcal{B}}^d$, respectively, are SYZ mirror partners.

# Stringy mixed Hodge numbers

Recall that for compact CY SYZ mirror partners, without flat gerbes, one expects $h^{p,q}(M) = h^{p,n-q}(\hat{M})$.

For the Higgs moduli spaces, some generalisations are required:

# Stringy mixed Hodge numbers

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For the Higgs moduli spaces, some generalisations are required:

- moduli spaces are non-compact, possibly singular
  $\Longrightarrow$ mixed Hodge numbers
  (defined in terms of Deligne's mixed Hodge structures in cohomology)
- moduli spaces are orbifolds $\Rightarrow$ to take proper account of the orbifold singularities, one needs 'stringy mixed Hodge numbers' (using the group action).
- These numbers are encoded in the 'E-polynomial'.

- Hausel–Thaddeus' conjecture is formulated in terms of string E-polynomials for $M = M^d_{DR}(SL_r)$ and $\hat{M} = M^e_{DR}(PGL_r)$:

$$E^{B^e}_{\text{string}}\left(M^d_{DR}(SL_r)\right) \overset{?}{=} E^{\hat{B}^d}_{\text{string}}\left(M^e_{DR}(PGL_r)\right).$$

Roughly speaking, this means $h^{p,q}(M) = h^{p,q}(\hat{M})$.

- Hausel–Thaddeus proved their conjecture for $r = 2, 3$.

Better lectures by Tamás Hausel
at CRM in March 2010.

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