

Mirror symmetry, Langlands duality, and the Hitchin system

Study group: the geometric Langlands correspondence

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This is an introduction to part of the paper

Tamás Hausel, Michael Thaddeus: Mirror symmetry,
Langlands duality, and the Hitchin system,
Invent. Math. **153** (2003) 197-229 (arXiv:math/0205236)

and some related previous work on

mirror symmetry and the Hitchin system.

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Mirror Symmetry (MS)

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- * $\exists$ several (not equivalent) formulations:
 - **Physics**: 'IIA string theory' on $\mathbb{R}^{3,1} \times M$ (for $\text{volume}(M) \gg 0$) is equivalent to 'IIB string theory' on $\mathbb{R}^{3,1} \times \hat{M}$ (assuming $\dim_{\mathbb{C}}$ is 3).

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 - **Homological MS** (Kontsevich): $D\text{Goh}(M) \cong D\text{Fuk}(\hat{M})$
 - **Toric Geometry**
(Batyrev-Borisov; Givental; Lian-Lin-Yau)

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- **Homological MS**
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- ...
- **Strominger-Yau-Zaslow** interpret MS in terms of special Lagrangian torus fibrations.

Strominger - Yau - Zaslow (SYZ) (1997)

Def (1) An n -dimensional complex manifold M is called Calabi-Yau (CY) if:

- $\exists \omega =$ Kähler form of a Ricci-flat metric on M
- $\exists \Omega =$ nowhere vanishing holomorphic n -form

(holomorphic volume form)

- {
- automatic by parallel transport if M is simply connected
 - required for simplicity (for calibrated geometry), but usually not part of the definition

Strominger - Yau - Zaslow (SYZ)

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(2) A real submanifold $L \subset M$ of a CY is called special Lagrangian (sLag) if:

- $L \subset M$ is Lagrangian (i.e. $\omega|_L = 0$ & $\dim_{\mathbb{R}} L = n$)
- $\text{Im } \Omega|_L = 0$

Let T = real torus and $T^\vee$ = dual torus of T . Recall that

$$T \cong \left\{ \begin{array}{l} \text{flat principal} \\ \text{U(1)-bundles} \\ (E, \nabla) \text{ over } T^\vee \end{array} \right\} \text{ and } T^\vee \cong \left\{ \begin{array}{l} \text{flat principal} \\ \text{U(1)-bundles} \\ (F, \nabla) \text{ over } T \end{array} \right\} \text{ (canonically).}$$

$$T \cong \left\{ \begin{array}{l} \text{flat principal} \\ U(1)\text{-bundles} \\ (E, \nabla) \text{ over } T^\vee \end{array} \right\} \text{ and } T^\vee \cong \left\{ \begin{array}{l} \text{flat principal} \\ U(1)\text{-bundles} \\ (F, \nabla) \text{ over } T \end{array} \right\}.$$

In concrete terms:

$$T = W / \Lambda \quad \text{for } \Lambda \cong \mathbb{Z}^n \text{ (lattice)} \subset W \cong \mathbb{R}^n$$

$$T^\vee = W^* / \Lambda^\vee \quad \text{for } \Lambda^\vee := \{ \xi : W \rightarrow \mathbb{R} \mid \langle \xi, \lambda \rangle \in 2\pi\mathbb{Z} \forall \lambda \in \Lambda \} \subset W^* \\ \text{(dual lattice)} \quad \text{Hom}(W, \mathbb{R})$$

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Each $w \in W$ defines an action of $\Lambda^\vee$ on the trivial line bundle $\mathbb{C}$ by

$$\Lambda^\vee \times (W^* \times \mathbb{C}) \longrightarrow W^* \times \mathbb{C}$$

$$(\xi, (\zeta, z)) \longmapsto (\zeta + \xi, \chi_w(\xi)z) \quad \text{in terms of}$$

the character $\chi_w: \Lambda^\vee \rightarrow \text{U(1)}: \xi \mapsto \exp i \langle \xi, w \rangle$.

$\mathbb{C}$
 $\downarrow$
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the character $\chi_w : \Lambda^\vee \rightarrow U(1) : \xi \mapsto \exp i \langle \xi, w \rangle$.

Then the trivial flat connection on $\mathbb{C} \downarrow V^*$ descends to a flat connection ∇_w on the quotient line bundle L_w over $T^\vee = V^* / \Lambda^\vee$, and

$$(L_w, \nabla_w) = (L_{w'}, \nabla_{w'}) \Leftrightarrow \chi_w = \chi_{w'} \Leftrightarrow [w] = [w'] \text{ in } T = V / \Lambda.$$

(for $w, w' \in V$)

Original formulation of the SYZ

If M and $\hat{M}$ are a mirror pair of CY n -folds,
 then $\exists$ fibrations $M \xrightarrow{\mu} V$ and $\hat{M} \xrightarrow{\hat{\mu}} V$ with $V = n$ -dimensional real manifold, whose generic fibres are **SLag tori** dual to each other, in the sense that there are canonical identifications between the fibres, for $x \in V$,

$$L_x \cong H^1(\hat{L}_x, \underline{U(1)}) , \quad \hat{L}_x \cong H^1(L_x, \underline{U(1)}) ,$$

sheaf of locally constant $U(1)$ -valued functions

whenever $L_x := \mu^{-1}(x)$ and $\hat{L}_x := \hat{\mu}^{-1}(x)$ are non-singular tori.

Mirror symmetry exchanges the Hodge numbers:

If M and $\hat{M}$ satisfy the SYZ prescription and have other nice properties, then it is expected that

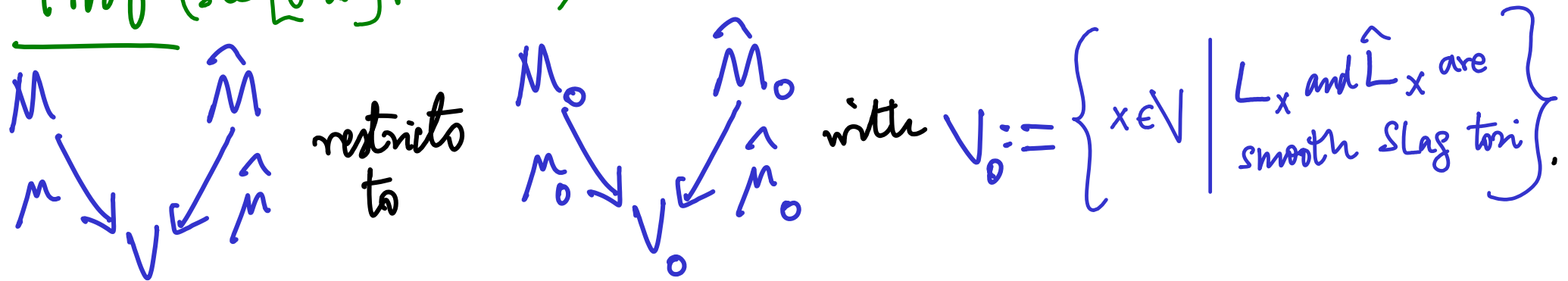
$$h^{p,q}(M) = h^{p,n-q}(\hat{M})$$

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'Proof' (see [Gross] for details).



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By SYZ, the SLAG fibrations admit identifications

$$M_0 \rightarrow V_0 \equiv R'(\hat{\mu}_0)_x(\underline{U}(1)) \rightarrow V_0 \quad \text{and} \quad R'(\mu_0)_x(\underline{U}(1)) \rightarrow V_0 \equiv \hat{M}_0 \rightarrow V_0$$

(with $R'(\)_x := 1^{\text{st}}$ derived direct image).

$$h^{p,q}(M) = h^{p,n-q}(\hat{M})$$

restricts to

$$\begin{array}{ccc} M & & \hat{M} \\ \downarrow & & \downarrow \\ M_0 & \rightarrow & \hat{M}_0 \end{array}$$

with $V_0 := \left\{ x \in V \mid L_x \text{ and } \hat{L}_x \text{ are smooth Lag tori} \right\}$.

SYZ $\Rightarrow$

$$\begin{array}{ccc} M_0 & & \hat{M}_0 \\ \downarrow & & \downarrow \\ V_0 & \rightarrow & V_0 \end{array} \equiv \begin{array}{ccc} R'(\hat{\mu}_0)_*(\underline{U}(1)) & & R'(\mu_0)_*(\underline{U}(1)) \\ \downarrow & & \downarrow \\ V_0 & & V_0 \end{array}$$

Fibrewise, $M_x \cong T = W/\Lambda$ $\hat{M}_x \cong T^V = W^*/\Lambda^V$ for $x \in V$

$\searrow \quad \swarrow$
 $\{x\}$

$$h^{p,q}(M) = h^{p,n-q}(\hat{M})$$

SYZ $\Rightarrow$

$$\begin{array}{ccc}
 M_0 & & \hat{M}_0 \\
 \downarrow \mu_0 & \searrow & \downarrow \hat{\mu}_0 \\
 & V_0 &
 \end{array}
 \equiv
 \begin{array}{ccc}
 R'(\hat{\mu}_0)_*(U(i)) & & R'(\mu_0)_*(U(i)) \\
 \downarrow & & \downarrow \\
 & V_0 &
 \end{array}$$

Fibrewise, $L_x \cong T = W/\wedge$ $\hat{L}_x \cong T^v = W^*/\wedge^v$ for $x \in V$,

$\swarrow \quad \searrow$

$\{x\}$

where

$$\begin{array}{ccc}
 \wedge^p W^* & \otimes & \wedge^n W \\
 \parallel & & \parallel \\
 H^p(T, \mathbb{R}) & & \mathbb{R}
 \end{array}
 \longrightarrow
 \begin{array}{c}
 \wedge^{n-p} W \\
 \parallel \\
 H^{n-p}(T^v, \mathbb{R})
 \end{array}$$

$\hookrightarrow H^p(T, \mathbb{R}) \cong H^{n-p}(T^v, \mathbb{R})$

$$h^{p,q}(M) = h^{p,n-q}(\hat{M})$$

SYZ $\Rightarrow$ $\begin{array}{ccc} M_0 & & \hat{M}_0 \\ \downarrow \mu_0 & \searrow & \downarrow \hat{\mu}_0 \\ V_0 & & V_0 \end{array} \equiv \begin{array}{ccc} R^1(\hat{\mu}_0)_*(U(i)) & & R^1(\mu_0)_*(U(i)) \\ \downarrow & & \downarrow \\ V_0 & & V_0 \end{array}$

Fibrewise, $L_x \cong T = W/\wedge$ $\hat{L}_x \cong T^v = W^*/\wedge^v$ for $x \in V$,

where $\wedge^p W^* \otimes \wedge^n W \xrightarrow{\quad} \wedge^{n-p} W$ $\xrightarrow{\quad} H^{n-p}(T^v, \mathbb{R})$ $\xrightarrow{\quad} H^p(T, \mathbb{R})$ $\xrightarrow{\quad} H^{n-p}(T^v, \mathbb{R})$ $\xrightarrow{\quad} H^p(T, \mathbb{R})$

so $H^p(T, \mathbb{R}) \cong H^{n-p}(T^v, \mathbb{R})$

Globally, $\Omega|_{M_x}$ is a volume form $\forall$ s.t. $L_x \Rightarrow R^1(\mu_0)_* \mathbb{R} \cong \mathbb{R}$, so

$$R^p(\mu_0)_* \mathbb{R} \xrightarrow{\cong} R^{n-p}(\hat{\mu}_0)_* \mathbb{R} \text{ as well.}$$

$$h^{p,q}(M) = h^{p,n-q}(\hat{M})$$

SYZ $\Rightarrow$

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$$R^p(\mu_0)_* R \xrightarrow{\cong} R^{n-p}(\hat{\mu}_0)_* R \text{ over } V_0 \subset V$$

In nice cases, $i: V_0 \hookrightarrow V$ induces isomorphisms

$$\left\{ \begin{array}{l} i_* R^p(\mu_0)_* R \cong R^p \mu_* R \\ i_* R^p(\hat{\mu}_0)_* R \cong R^p \hat{\mu}_* R \end{array} \right., \text{ so}$$

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$$SYZ \Rightarrow \begin{array}{c} M_0 \quad \hat{M}_0 \\ \downarrow \quad \downarrow \\ \mu_0 \quad \hat{\mu}_0 \\ \downarrow \quad \downarrow \\ V_0 \end{array} \equiv \begin{array}{c} R^1(\hat{\mu})_*(U(i)) \quad R^1(\mu)_*(U(i)) \\ \downarrow \quad \downarrow \\ V_0 \end{array}$$

$$R^p(\mu_0)_* R \xrightarrow{\sim} R^{n-p}(\hat{\mu}_0)_* R \text{ over } V_0 \subset V \stackrel{?}{\Rightarrow} R^p/\mu_* R \xrightarrow{\sim} R^{n-p}/\hat{\mu}_* R$$

E_2 -terms of the Leray spectral sequences for μ and $\hat{\mu}$ are

$$E_2^{p,q} = H^q(V, R^p/\mu_* R), \quad \hat{E}_2^{p,q} = H^q(V, R^p/\hat{\mu}_* R).$$

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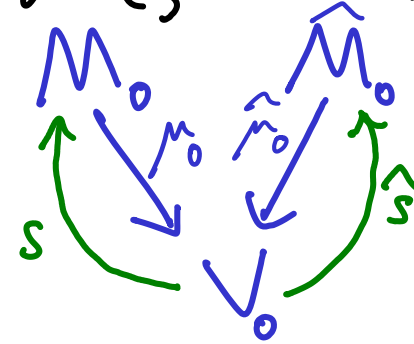
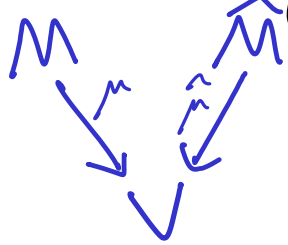
In nice cases, the spectral sequences degenerate at E_2 ,

$$h^{p,q}(M) = \dim H^p(V, R^q \mu_* R) = \dim H^p(V, R^{n-q} \hat{\mu}_* R) = h^{p,n-q}(\hat{M}).$$

"□"

The SYZ conjecture with flat gerbes

• In the original formulation of the SYZ conjecture, the fibrations restrict to lag fibrations with sections



$$s: V_0 \longrightarrow M_0$$

$$x \longmapsto \left(\begin{array}{c} \text{trivial} \\ \text{connection} \\ \text{over } \hat{M}_x \end{array} \right) \in L_x = H^1(\hat{M}_x, \mathcal{U}(1))$$

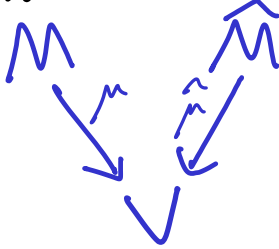
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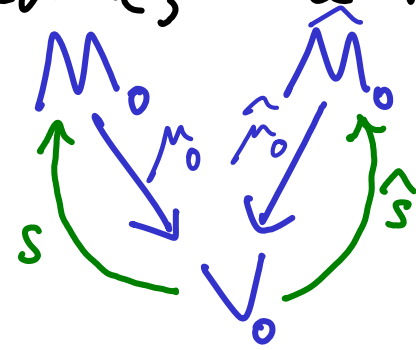
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- Hitchin (1999) proposed to use **flat gerbes** to refine the SYZ prescription for Lag fibrations without sections.

(These gerbes are also necessary in physics to obtain a duality between the cohomology groups $H^1(M, TM)$ and $H^1(\hat{M}, T^*\hat{M})$.)

FLAT GERBES

Let M be a manifold and $\underline{U(1)} =$ sheaf of $U(1)$ -valued locally constant functions over M .

Recall that $H^1(M, \underline{U(1)}) \cong \{ \text{isom. classes of } \underline{\text{flat principal } U(1)\text{-bundles over } M} \}$

where locally constant functions $\longleftrightarrow$ flatness.

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Flat unitary (or $U(1)$) gerbes are 'geometric objects over M ' s.t.
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More precisely, it is a sheaf of categories (a 'stack') which is a torsor over $\underline{U}(1)$, i.e. which is locally isomorphic to the sheaf of flat principal $U(1)$ -bundles with their $U(1)$ -action.

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sheaf $U(1)$ -valued

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An equivalence of gerbes is an equivalence of sheaves of categories (with the $U(1)$ -action).

TRIVIALIZATIONS OF FLAT GERBES

A trivialization of a gerbe $\mathcal{B}$ is an equivalence of $\mathcal{B}$ with the trivial gerbe, i.e. the gerbe that to each open set assigns the category of flat principal $U(1)$ -bundles.

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A trivialization of a gerbe B is an equivalence of B with the trivial gerbe, i.e. the gerbe that to each open set assigns the category of flat principal $U(1)$ -bundles.

$\text{Triv}^{U(1)}(M, B) := \left\{ \begin{array}{l} \text{equivalence classes of} \\ \text{trivialisations of } B \end{array} \right\}$ is an $H^1(M, U(1))$ -torsor over a point.

TRIVIALIZATIONS OF FLAT GERBES

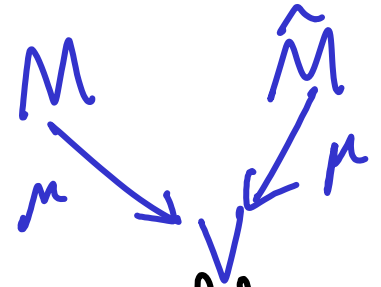
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By the definition of gerbe, $\exists$ an open cover $\{U_\alpha\}$ s.t. $B_\alpha := B|_{U_\alpha}$ is trivial, and then B is glued by 'transition functors'
 $F_{\alpha\beta}: B_\alpha|_{U_{\alpha\beta}} \rightarrow B_\beta|_{U_{\alpha\beta}}$ (with $U_{\alpha\beta} = U_\alpha \cap U_\beta$) given by tensoring with flat line bundles $L_{\alpha\beta}$ over $U_{\alpha\beta}$, s.t. $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ are canonically trivial over $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, so they correspond to sections $g_{\alpha\beta\gamma}$ of $\underline{U}(1)$ over $U_{\alpha\beta\gamma}$ which define a Čech class $[g_{\alpha\beta\gamma}] \in \check{H}^2(M, \underline{U}(1))$.

The SYZ with flat gerbes (Hitchin, 1999)

Let $(M, \mathcal{B})$ and $(\hat{M}, \hat{\mathcal{B}})$ be two CY orbifolds of dim. n with flat unitary gerbes.

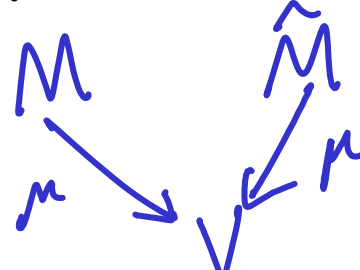
They are mirror partners if $\exists$ fibrations 
with $V = n$ -dim'd real orbifold whose generic fibres are **SLag tori** dual to each other, in the sense that there are canonical identifications, for $x \in V$,
$$L_x \cong \text{Triv}^{u(1)}(\hat{L}_x, \hat{\mathcal{B}}_x), \quad \hat{L}_x \cong \text{Triv}^{u(1)}(L_x, \mathcal{B}_x),$$

whenever $L_x := \mu^{-1}(x)$ and $\hat{L}_x := \hat{\mu}^{-1}(x)$ are non-singular tori.

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whenever $L_x := p^{-1}(x)$ and $\hat{L}_x := \hat{p}^{-1}(x)$ are non-singular tori.

Recall that the RHS's are torsors for $H^1(\hat{L}_x, \underline{U}(1))$ and $H^1(L_x, \underline{U}(1))$, respectively.

SYZ for hyperkähler manifolds

- Constructing SLags is usually very difficult, but it is much easier if M is hyperkähler.

SYZ for hyperkähler manifolds

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Permuting indices, M is CY wrt. J_1, J_2, J_3

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$\Rightarrow M$ is CY wrt. J_1, J_2, J_3 .

Then any complex submanifold $L \subset M$ which is complex Lagrangian wrt. J_1 is **isotropic** wrt. J_2 .

SYZ for hyperkähler manifolds

(M, J_1, J_2, J_3) hyperkähler $\Rightarrow M$ is CY wrt. J_1, J_2, J_3 .

Then any complex submanifold $L \subset M$ which is complex Lagrangian wrt. J_1 is slg wrt J_2 .

$\Rightarrow$ To obtain SYZ slg fibrations by holomorphic methods, we look for holomorphic maps whose generic fibres are complex Lagrangian wrt. J_1 (for hyperkähler $M, \hat{M}$ and complex V)



and then change to the cx. str. J_2 .
(hyperkähler rotation)

Moduli space of Higgs bundles

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Def

(E, ϕ) is **stable** if $\frac{\deg E'}{\operatorname{rk} E'} < \frac{\deg E}{\operatorname{rk} E} \quad \forall E' \subset E \text{ s.t. } \phi(E') \subset E' \otimes K.$

Definitions

$M^d(GL_r) =$ moduli space of stable Higgs bundles
 (E, ϕ) with E of rank r and degree d

$M^d(SL_r) =$ moduli space of stable Higgs bundles
 (E, ϕ) with $\text{rk } E = r$, $\deg E = d$, $\Lambda^r E \cong L_{SL}$
(for fixed line bundle L_{SL} of degree d)

$M^d(PGL_r) =$ geometric quotient $M^d(SL_r)/\Gamma$ by the
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$$SL_r \xleftrightarrow{\text{Langlands}} PGL_r$$

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Notation ("Dol" = "Dolbeault", "DR" = "de Rham"):

$M_{\text{Dol}}^d(G) := M^d(G)$ with its complex structure as a moduli of **Higgs bundles**

$M_{\text{DR}}^d(G) := M^d(G)$ with its complex structure as a moduli of **flat connections**.

The Hitchin system: $G = GL_r, SL_r$ or PGL_r

$\exists$ a completely integrable Hamiltonian system, given by the Hitchin map

$$\mu: M_{\text{Dol}}^d(G) \xrightarrow{\text{proper}} V_G = \bigoplus_{i=1}^r H^0(C, K^i) : (E, \phi) \mapsto \text{char. pol. of } \phi$$

$GL_r \nearrow \quad \nwarrow SL_r, PGL_r$

with • $\dim V_G = \frac{1}{2} \dim M_{\text{Dol}}^d(G)$

• generic fibre a complex Lagrangian torus for an abelian variety

$\Rightarrow \begin{cases} M_{\text{DR}}^d(SL_r) \text{ and } M_{\text{DR}}^d(PGL_r) \text{ carry families of } \text{slag} \\ \text{tori over } V = V_{SL_r} = V_{PGL_r}, \text{ as in SYZ.} \end{cases}$

To show that they are SYZ mirror partners, we need to see that these tori are dual.

Generic fibres:

Def • Given $\beta = (\beta_i) \in V_G$, the spectral cover $\begin{matrix} \tilde{C}_\beta \\ \downarrow \pi \\ C \end{matrix}$ is defined by

$$\tilde{C}_\beta = \{z \in K \mid z^r + \beta_1 z^{r-1} + \dots + \beta_r = 0 \text{ in } K^r\} \subset \text{total space of } \begin{matrix} K \\ \downarrow \\ C \end{matrix}$$

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Hitchin computes the fibre $L_\beta := \mu^{-1}(\beta)$ for $\beta \in U$:

- $G = GL_r$: $L_\beta = \tilde{J}^d := \text{Pic}^d \tilde{C}_\beta$ — the fibre of a $\tilde{J}^0$ -torsor over U
- $G = SL_r$: $L_\beta = P^d := \text{Nm}^{-1}(\mathcal{O}(dc))$ — the fibre of a P^0 -torsor over U
 (the generalized Prym variety) with $c \in C$ a base point and
 $\text{Nm}: \text{Pic}^d \tilde{C}_\beta \rightarrow \text{Pic}^d C$ the norm map.
- $G = PGL_r$: $L_\beta = \hat{P}^d := P^d / T$ — the fibre of a $\hat{P}^0$ -torsor over U

Lemma

$$\hat{P}^0 \cong \tilde{J}^0 / J^0$$

Proof. $P^d \times J^0 \rightarrow \tilde{J}^d : (L, M) \mapsto L \otimes \pi^* M^{-1}$ induces $\frac{P^d \times J^0}{\Gamma} \cong \tilde{J}^d$

and then $Nm : \tilde{J}^d \rightarrow \text{Pic}^d C$ corresponds to the composite

$$\frac{P^d \times J^0}{\Gamma} \rightarrow \frac{J^0}{\Gamma} \xrightarrow{\sim} J^0 : [L, M] \mapsto M \mapsto M^{-r}.$$

Dualizing now $0 \rightarrow P^0 \rightarrow \tilde{J}^0 \xrightarrow{Nm} J^0 \rightarrow 0$, we get $0 \rightarrow J^0 \xrightarrow{\pi^*} \tilde{J}^0 \rightarrow \tilde{J}^0 / J^0 \rightarrow 0$

$$\text{with } \tilde{J}^0 / J^0 \cong \frac{\frac{P^0 \times J^0}{\Gamma}}{J^0} = P^0 / \Gamma = \hat{P}^0. \quad \square$$

Trivializations of the flat unitary gerbe $(\mathbb{E}, \Phi)$

- If (r, d) coprime, then $\exists$ universal Higgs bundle $M_{\text{Dol}} \times C$
- If (r, d) not coprime, then $\nexists \mathbb{E}$, but its projectivization $\mathbb{P}\mathbb{E}$ and $\Phi: \mathbb{E} \rightarrow \mathbb{E} \otimes K$ exist anyway -

$\Rightarrow$ the restriction $\mathbb{P}\mathbb{E}|_{M_{\text{Dol}}^d \times \{c\}}$ is a projective bundle $\mathbb{P}$ on M_{Dol}^d .
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Using the short exact sequence $1 \rightarrow \overset{\mu_r}{\mathbb{Z}_r} \rightarrow SL_r \rightarrow PGL_r \rightarrow 1$, we define the gerbe $\mathcal{B}$ of liftings of Ψ to SL_r :

$U \subset M_{\text{Dol}}^d$ étale nghbd $\mapsto \mathcal{B}(U) = \left\{ \begin{array}{l} \text{category of pairs } (Q, \psi) \text{ where:} \\ \bullet Q = SL_r\text{-bundle on } U \\ \bullet \psi: (Q \times PGL_r) /_{SL_r} \xrightarrow{\cong} \Psi|_U \text{ isom} \end{array} \right\}$

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Lemma For $G = SL_r$, the restriction of $\mathcal{B}$ to each regular fibre $P = \pi^{-1}(\beta)$ is trivial as a $\mathbb{Z}_r$ -gerbe, hence as a $U(1)$ -gerbe.

Recall that equivalence classes of $\mathcal{U}(1)$ -trivializations form a torsor for $H^1(P^d, \underline{\mathcal{U}(1)}) \cong \text{Pic}^0 P^d \cong \text{Pic}^0 P^0$, so the previous lemma implies that this is $\hat{P}^0$.

Prop. $\forall d, e \in \mathbb{Z}, \text{Triv}^{\mathcal{U}(1)}(P^d, \mathcal{B}^e) \cong \hat{P}^e$ as $\hat{P}^0$ -torsors.

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For the converse, we need to construct a flat $U(1)$ -gerbe over the orbifold $\hat{M}_{\text{Dol}}^d = M_{\text{Dol}}^d(\text{PGL}_r) = M_{\text{Dol}}^d / \Gamma$, or

equivalently, a Γ -equivariant flat $U(1)$ -gerbe over $M_{\text{Dol}}^d = M_{\text{Dol}}^d(\text{SL}_r)$.

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One defines $\hat{\mathcal{B}}$ as in the case of $\mathcal{B}$, with an action of $\Gamma = \text{Pic}^0 C[r]$ induced by its equivariant action on $\mathcal{P}\mathcal{E}$ and hence on $\bar{\Psi}$.

Lemma For $G = \mathrm{PGL}_r$ the restriction of $\hat{\mathcal{B}}$ to each regular fibre $\hat{P}^d = \hat{\pi}^{-1}(\beta)$ is trivial.

Prop. $\forall d, e \in \mathbb{Z}$, $\mathrm{Triv}^{u(1)}(\hat{P}^d, \hat{\mathcal{B}}^e) \cong P^e$ as P^0 -torsors.

Lemma For $G = \mathrm{PGL}_r$, the restriction of $\hat{B}$ to each regular fibre $\hat{P}^d = \hat{\pi}^{-1}(\beta)$ is trivial.

Prop. $\forall d, e \in \mathbb{Z}$, $\mathrm{Triv}^{u(1)}(\hat{P}^d, \hat{B}^e) \cong P^e$ as P^0 -torsors.

In conclusion:

Thm $\forall d, e \in \mathbb{Z}$, $\left\{ \begin{array}{l} \mathrm{Triv}^{u(1)}(P^d, B^e) \cong \hat{P}^e \\ \mathrm{Triv}^{u(1)}(\hat{P}^e, \hat{B}^d) \cong P^d \end{array} \right\}$

and hence the moduli spaces $M_{\mathrm{DR}}^d(\mathrm{SL}_r)$ and $M_{\mathrm{DR}}^e(\mathrm{PGL}_r)$, equipped with the flat unitary (orbifold) gerbes B^e and $\hat{B}^d$, respectively, are SYZ mirror partners.

Stringy mixed Hodge numbers

Recall that for compact CY SYZ mirror partners, without flat gerbes, one expects $h^{p,q}(M) = h^{p,n-q}(\hat{M})$. For the Higgs moduli spaces, some generalisations are required:

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For the Higgs moduli spaces, some generalisations are required:

- moduli spaces are non-compact, possibly singular
 $\Rightarrow$ mixed Hodge numbers

(defined in terms of Deligne's mixed Hodge structures in cohomology)

- moduli spaces are orbifolds $\Rightarrow$ to take proper account of the orbifold singularities, one needs 'stringy mixed Hodge numbers' (using the group action).
- These numbers are encoded in the 'E-polynomial'.

- Hausel-Thaddeus' conjecture is formulated in terms of string E-polynomials for $M = M_{\text{DR}}^d(\text{SL}_r)$ and $\hat{M} = M_{\text{DR}}^e(\text{PGL}_r)$:

$$E_{\text{string}}^{\text{B}^e}(M_{\text{DR}}^d(\text{SL}_r)) \stackrel{?}{=} E_{\text{string}}^{\hat{\text{B}}^d}(M_{\text{DR}}^e(\text{PGL}_r)).$$

Roughly speaking, this means $h^{p,q}(M) = h^{p,q}(\hat{M})$.

- Hausel-Thaddeus proved their conjecture for $r = 2, 3$.

Better lectures by Tamás Hausel
at CRM in March 2010.

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