The Hitchin system

(G - Higgs bundles)

Oscar García-Prada
CSIC

CRM, 3 September 2009

Higgs bundles (Hitchin)

E — holomorphic vector bundle
$\downarrow$
X — compact Riemann surface

$\Phi : E \longrightarrow E \otimes K$ Higgs field

K: canonical line bundle

(E, Φ) stable $\Leftrightarrow$ $\mu(E') < \mu(E)$ $\forall E' \subset E$, Φ-invariant

$\left(\mu(E') = \dfrac{\deg E'}{\operatorname{rk} E'} \right)$

(E, Φ) polystable : $(E, \Phi) = \bigoplus (E_i, \Phi_i)$, stable, $\mu(E_i) = \mu(E)$

$\mathcal{M}(n, d)$ — moduli space of polystable Higgs bundles $\operatorname{rk} E = n$, $\deg E = d$

$\mathcal{M}(n,d)$: irreducible complex quasi-projective variety

$$\dim \mathcal{M}(n,d) = 2 + 2n^2(g-1)$$

(twice $\dim M(n,d)$)

$\mathcal{M}^s(n,d) \subset \mathcal{M}(n,d)$: open smooth subvariety

Again, $GCD(n,d) = 1 \Rightarrow \mathcal{M}(n,d) = \mathcal{M}^s(n,d)$

<u>Fact</u>

moduli space of stable bundles

$$T^* M^s(n,d) \subset \mathcal{M}^s(n,d)$$

open

- $E \in M^s(n,d)$: $T_E M^s(n,d) \cong H^1(\mathrm{End}\, E) \cong H^0(\mathrm{End}\, E \otimes K)^*$

Serre duality

- $\forall\, \Phi \in H^0(\mathrm{End}\, E \otimes K) = T_E^* M^s(n,d)$

(E, Φ) stable !

$SL(n,\mathbb{C})$ – Higgs bundles

- $\det E \cong \mathcal{O}$
- $Tr(\Phi) = 0$

trace

$\mathcal{M}(SL(n,\mathbb{C}))$ – Moduli space of polystable Higgs bundles with $rk\, E = n$, $\det E \cong \mathcal{O}$, $Tr(\Phi) = 0$

$$\dim \mathcal{M}(SL(n,\mathbb{C})) = 2n^2(g-1)$$

We want to consider the theory of Higgs bundles for arbitrary semisimple Lie groups.

G - Higgs bundles

G — connected complex semisimple Lie group

E — holomorphic principal G-bundle

$$E \downarrow X$$

$\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$

adjoint bundle

$Ad : G \longrightarrow GL(\mathfrak{g})$

Stability of (E, φ)

- $P \subset G$ — maximal parabolic subgroup
- $\sigma : E_P \longrightarrow E$ — reduction of structure group
- $\chi : P \to \mathbb{C}^*$ — antidominant character

- (E, φ) <u>stable</u>:

For every P, σ, χ such that $\varphi \in H^0(X, E_P(\rho) \otimes K)$ then

$\deg \underbrace{E(\sigma, \chi)}_{\text{line bundle associated to } E_P \text{ via the character } \chi} > 0$ $\left(\frac{\text{semistable}}{\geq 0} \right)$

- <u>Polystability</u>

(E, φ) semistable

For every P, σ, χ as above such that — <u>strictly</u> antidominant
$\varphi \in H^0(X, E_P(\rho) \otimes K)$ and $\deg E(\sigma, \chi) = 0$
there is a reduction $E_L \subset E_P$ — Levi of P
such that $\varphi \in H^0(X, E_L(\iota) \otimes K)$.
((E_L, φ_L) stable L-Higgs bundle)

<u>Moduli space of G-Higgs bundles</u>

$d \in \pi_1(G)$ — Topological type of G-bundle

$\mathcal{M}_d(G)$: Moduli space of isomorphism classes of polystable G-Higgs bundles of topological type $d \in \pi_1(G)$

- $\mathcal{M}_d(G)$ is a quasi-projective complex algebraic variety of dimension $= 2 \dim \mathfrak{g} \, (g-1)$
- $\mathcal{M}_d(G)$ is <u>connected</u>
- $\mathcal{M}_d(G) \supset T^* \underbrace{M_d^s(G)}$ — smooth (stable + simple)

Moduli space of G-bundles.

Hitchin equations

(E, φ) G-Higgs bundle

- $H \subset G$ – maximal compact subgroup

$$(H^{\mathbb{C}} = G)$$

- $h \in \Gamma(E(G/H))$ – reduction of structure group to H

$$E_H \longrightarrow E$$

(For vector bundles $H = U(n)$ and h is a Hermitian metric)

- $F_h \in \Omega^2(X, E_H(\mathfrak{h}))$ Lie algebra of H

curvature of Chern connection

- Conjugation

$$\tau : \Omega^1(X, E(\mathfrak{g})) \to \Omega^1(X, E(\mathfrak{g}))$$

combination of conj. defined by h and complex conj. on 1-forms

- For vector bundles

$$\tau(\varphi) = -\varphi^*$$

- $[\varphi, \tau(\varphi)] \in \Omega^2(X, E_H(\mathfrak{h}))$

<u>Theorem</u> (Hitchin '87, Simpson '92)

A reduction h of structure group of E from G to H satisfies the <u>Hitchin equation</u>

$$\boxed{F_h - [\varphi, \tau(\varphi)] = 0}$$

if and only if (E, φ) is polystable

(stable $\Longleftrightarrow$ irreducible solutions)
+ simple

Moduli space of solutions

$\mathbb{E}_H$ — C^∞ H-bundle with fixed topological class $d \in \pi_1(H)$

<u>Equations for a pair</u> (A, φ):

A — H-connection on $\mathbb{E}_H$

$\varphi \in \Omega^{1,0}(X, \mathbb{E}_G(\mathfrak{g}))$

extension of structure group

$$\boxed{\begin{aligned} &F_A - [\varphi, \tau(\varphi)] = 0 \\ &\bar{\partial}_A \varphi = 0 \end{aligned}}$$

<u>Hitchin equs</u>

$\bar{\partial}_A$: — $(0,1)$-part of the covariant derivative d_A

defines a holomorphic structure on $\mathbb{E}_G$

__Gauge group__

$$\mathcal{H} = \mathrm{Aut}(\mathbb{E}_H)$$

acts on the space of solutions

- $$\boxed{\left\{ \begin{array}{l} (A, \varphi) \text{ satisfying} \\ \text{Hitchin equs} \end{array} \right\} \Big/ \mathcal{H} \cong \mathcal{M}_d(G)}$$

  homeomorphism

- Hitchin equations originate locally as __dimensional reduction__ of __instantons__ $(F_A^+ = 0)$ in $\mathbb{R}^4$

  Self-duality equation

  (__invariance__: Translation in two directions)

Representation of surface groups

S closed oriented surface of genus $g \geq 2$

G connected semisimple / algebraic Lie group

Moduli space of representations

$$\mathcal{R}(G) := \mathrm{Hom}^{\text{reductive}}(\pi_1(S), G) \,/\, G$$

$$\underset{\text{reductive}}{\rho : \pi_1(S) \to G} \overset{\text{Def.}}{\iff} \underset{\text{reductive group}}{\overline{\rho(\pi_1(S))}}$$

Zariski closure

$\mathcal{R}(G)$ — algebraic variety

→ Interest: geometry and topology of $\mathcal{R}(G)$.

Geometrically:

$$\mathrm{Hom}(\pi_1(S), G)/G \cong H^1(S, G)$$

equivalence classes of flat G-bundles

Universal covering of G:

$$1 \to \pi_1(G) \to \tilde{G} \to G \to 1$$

Long sequence:

$$H^1(S, \tilde{G}) \to H^1(S, G) \xrightarrow{c} H^2(S, \pi_1(G)) \cong \pi_1(G)$$

Topological invariant:

$$R(G) \xrightarrow{c} \pi_1(G)$$

Consider:

$$R_d(G) := c^{-1}(d), \quad d \in \pi_1(G)$$

Question: Is R_d non-empty and connected for every $d \in \pi_1(G)$?

Harmonic bundles

$\mathbb{E}_G$ — C^∞ G-bundle with fixed topological class $d \in \pi_1(G)$

• D — G-connection on $\mathbb{E}_G$

Unique decomposition

$$D = d_A + \psi \quad \psi \in \Omega^1(X, \mathbb{E}_H(\mathfrak{h}))$$

d_A : H-connection on $\mathbb{E}_H \hookrightarrow \mathbb{E}_G$

$$\boxed{\begin{aligned} &F_A + \tfrac{1}{2}[\psi,\psi] = 0 \\ &d_A \psi = 0 \\ &d_A^* \psi = 0 \end{aligned}}$$

$F_D = 0$

Harmonicity equations

<u>Theorem</u> (Donaldson '87, Corlette '88)

$$\{D : F_D = 0 \text{ reductive}\} / \mathcal{G} \cong \left\{\begin{array}{l}\text{Solutions} \\ (A,\psi) \text{ to} \\ \text{Harmonicity} \\ \text{equations}\end{array}\right\} / \mathcal{G}$$

$\mathcal{G}$ — $\mathrm{Aut}(\mathbb{E}_G)$

Theorem: ($X = (S, J)$) complex structure

$$\mathcal{M}_d(G) \cong \left\{ (\bar\partial_E, \varphi) : \bar\partial_E \varphi = 0 \ \text{polystable} \right\} \Big/ \mathcal{H}^{\mathbb{C}} \cong \left\{ \begin{array}{l} (A, \varphi): \\ \bar\partial_A \varphi = 0 \\ F_A - [\varphi, \tau(\varphi)] = 0 \end{array} \right\} \Big/ \mathcal{H}$$

$$(\bar\partial_E, \varphi) \longmapsto (d_A = \bar\partial_A + \partial, \varphi)$$

$$(A, \varphi) \downarrow (A, \psi = \varphi - \tau(\varphi))$$

Corlette

$$\wr\| \qquad\qquad \wr\|$$

$$R_d(G) \cong \left\{ D : F_D = 0 \ \text{reductive} \right\} \Big/ \mathcal{G} \cong \left\{ \begin{array}{l} (A, \psi): \\ F_A + \frac{1}{2}[\psi, \psi] = 0 \\ d_A \psi = 0 \\ d_A^* \psi = 0 \end{array} \right\} \Big/ \mathcal{H}$$

$$D \longmapsto (d_A, \psi) \qquad D = d_A + \psi$$

$$\tilde{X} \longrightarrow G/H$$

Symplectic and Kähler quotients

(M, ω) — symplectic manifold $\left(\begin{array}{c}\omega \in \Omega^2_M \\ d\omega = 0 \\ \text{non-deg.}\end{array}\right)$

G — Lie group

$G \times M \longrightarrow M$ — symplectic action

$$g^* \omega = \omega \quad \forall g \in G.$$

<u>Moment map:</u>

$\mu : M \longrightarrow \mathfrak{g}^*$ <u>G-equivariant</u>

$$(*) \quad \boxed{d\mu_a|_m(v) = \omega(\tilde{a}_m, v)} \qquad m \in M,\ v \in T_m M,\ a \in \mathfrak{g}$$

$\mu_a : M \to \mathbb{R}$, $\mu_a(m) := \mu(m)(a)$

$\tilde{a}$ — vector field defined by $a \in \mathfrak{g}$.

$(*) \iff \mu_a$ Hamiltonian function for $\tilde{a}$

- μ unique up to constant element in centre of $\mathfrak{g}^*$.

Symplectic quotient (Marsden - Weinstein)

$\mu^{-1}(0)/G$ - symplectic manifold

- If $\dim M = n$, $\dim \mu^{-1}(0)/G = n - 2\dim G$

(M, g, I, ω) — Kähler manifold

g — Riemannian metric

$I : TM \to TM$ — complex structure $(I^2 = -id)$

$\omega(u,v) := g(u, Iv)$ symplectic form

Action : $G \times M \to M$

- preserving the symplectic form
- by isometries.

$\mu^{-1}(0)/G$ — Kähler manifold

(algebraic set up: $\mu^{-1}(0)/G \cong M/\!\!/G^c$)

GIT quotient

<u>Moduli space as Kähler quotient</u>

$\mathbb{E}_G$ — smooth G-bundle over X

E_H — reduction of $\mathbb{E}_G$ to H.

$\mathcal{A}$ — <u>space of H-connections on $\mathbb{E}_H$</u>

$$T_A \mathcal{A} = \Omega^1(E_H(\mathfrak{h}))$$

Killing form

$$\omega_{\mathcal{A}}(\psi, \eta) = \int_X B(\psi \wedge \eta) \quad ; \psi, \eta \in T_A \mathcal{A}$$

Symplectic form

$\mathcal{A} \cong \mathcal{C}$ — holomorphic structures on $\mathbb{E}_G$

$$A \mapsto \bar{\partial}_A$$

$$T_{\bar{\partial}_A} \mathcal{C} = \Omega^{0,1}(\mathbb{E}_G(\mathfrak{g}))$$

$$I_{\mathcal{A}}(\alpha) = i\alpha, \quad \alpha \in \Omega^{0,1}(\mathbb{E}_G(\mathfrak{g}))$$

- $(\mathcal{A}, \omega_{\mathcal{A}}, I_{\mathcal{A}})$ — Kähler manifold

- $\Omega := \Omega^{1,0}(X, \mathbb{E}_G(\mathfrak{g}))$

$$\omega_\Omega(\psi, \eta) = -i \int_X B(\psi \wedge \tau(\eta)), \quad \psi, \eta \in T_\varphi \Omega = \Omega$$

conj.

$$I_\Omega(\alpha) = i\alpha \quad , \alpha \in T_\varphi \Omega = \Omega$$

$(\Omega, \omega_\Omega, I_\Omega)$ — Kähler manifold

- <u>Consider</u>

$$\mathcal{X} = \mathcal{A} \times \Omega$$

$$\omega_{\mathcal{X}} = \omega_{\mathcal{A}} + \omega_\Omega$$

$$I_{\mathcal{X}} = I_{\mathcal{A}} + I_\Omega$$

The action of $\mathcal{H} = Aut(\mathbb{E}_H)$ preserves $\omega_{\mathcal{X}}$ and $I_{\mathcal{X}}$ and there is a moment map:

Hyperkähler quotients (Hitchin et. al.)

(M, g, I_1, I_2, I_3) — hyperkähler manifold

$\omega_1 \quad \omega_2 \quad \omega_3$

(M, g, I_i, ω_i) — Kähler $\quad I_i^2 = -\mathrm{Id}, \ I_1 I_2 = I_3$, etc.

Let G act preserving $g, \omega_1, \omega_2, \omega_3$:

$$\mu_i : M \longrightarrow \mathfrak{g}^*$$ — moment map

$$\bar{\mu} : M \longrightarrow \mathfrak{g}^* \otimes \mathbb{R}^3$$

$$\bar{\mu} := (\mu_1, \mu_2, \mu_3)$$

$$\bar{\mu}^{-1}(0) / G$$ — hyperkähler manifold

- Moment maps

$$\mu_1(A,\varphi) = F_A - [\varphi, \tau(\varphi)]$$

$$\mu_2(A,\varphi) = \mathrm{Re}(\bar{\partial}_A \varphi)$$

$$\mu_3(A,\varphi) = \mathrm{Im}(\bar{\partial}_A \varphi)$$

$$\bar{\mu} = (\mu_1, \mu_2, \mu_3) : \mathcal{X} \to \mathrm{Lie}\,\mathcal{H}^*$$

$$\bar{\mu}^{-1}_{\mathrm{irr.}}(0) / \mathcal{H}$$ — hyperkähler manifold

- Holomorphic symplectic form

$$\omega = \omega_2 + i\,\omega_3$$

Holomorphic with respect to I_1

Hyperkähler structure on $\mathcal{X} = \mathcal{A} \times \Omega$

$$\alpha \in \Omega^{0,1}(X, E_G(\mathfrak{g})) = T_{\bar\partial_A}\mathcal{C} \cong T_A \mathcal{A}$$

$$\psi \in \Omega^{1,0}(X, E_G(\mathfrak{g})) = \Omega = T_\psi \Omega$$

$$I_1(\alpha, \psi) = (i\alpha, i\psi) \qquad (I_1 = I_{\mathcal{X}})$$

$$I_2(\alpha, \psi) = (-i\tau(\psi), i\tau(\alpha))$$

$$I_3(\alpha, \psi) = (\tau(\psi), -\tau(\alpha))$$

I_1, I_2, I_3 define a hyperkähler structure on $\mathcal{X}$ with symplectic structures $\omega_1 = \omega_{\mathcal{X}}$, ω_2 and ω_3.

Action of $\mathcal{H}$ on $\mathcal{X}$ preserves this hyperkähler structure.

$$\mu_{\mathcal{X}} : \mathcal{X} \longrightarrow \Omega^2(X, E_H(\mathfrak{h})) \cong \operatorname{Lie} \mathcal{H}^*$$

$$(A, \varphi) \longmapsto F_A - [\varphi, \tau(\varphi)]$$

$$\mathcal{N} = \{ (A, \varphi) \in \mathcal{X} : \bar{\partial}_A \varphi = 0 \}$$
$$\cup$$
$$\mathcal{N}^*$$ — smooth points

$$\mu = \mu_{\mathcal{X}}|_{\mathcal{N}^*} : \mathcal{N}^* \longrightarrow \Omega^2(X, E_H(\mathfrak{h}))$$

$\mu_{irr}^{-1}(0)/\mathcal{H}$ — moduli space of (irreducible) solutions to Hitchin equations

$$\boxed{\mathcal{M}_d^s(G) \cong \mu_{irr}^{-1}(0)/\mathcal{H}}$$

Integrable systems

(M, ω) — symplectic manifold

$\underset{\text{function}}{f} \in C^\infty(M) \rightsquigarrow X_f$ — vector field:

$$df = i(X_f)\omega$$

$$\omega: \quad T^*M \xrightarrow{\cong} TM$$
$$df \longmapsto X_f$$

Poisson bracket:

f, g — functions on M

$$\{f, g\} = X_f \cdot g = -X_g f = -\{g, f\}$$

f, g Poisson commute $\overset{\text{def}}{\iff} \{f, g\} = 0$

Completely integrable Hamiltonian systems:

- (M, ω) — $2n$-dimensional symplectic manifold
- $f_1, \dots, f_n$ — n functions on M

<u>such that:</u>

- $df_1 \wedge \dots \wedge df_n$ generically nonzero
- $\{f_i, f_j\} = 0$ for every $i, j = 1, \dots, n$

<u>Consider the map:</u>

$$f = (f_1, \dots, f_n) : M \longrightarrow \mathbb{C}^n$$

- $f^{-1}(c)$ – n-dimensional submanifold for generic c
- $X_{f_1}, \dots, X_{f_n}$ — n – linearly independent commuting vector fields on $f^{-1}(c)$

Spectral curves

$\tilde{X} \longrightarrow X$ covering

Fibre of Hitchin map is a Prym variety.

Classical groups (Hitchin)

• $SL(m,\mathbb{C})$	$\tilde{X} \xrightarrow{m:1} X$	$\mathrm{Prym}(\tilde{X}/X)$
• $Sp(2m,\mathbb{C})$	$\tilde{X} \xrightarrow{2m:1} X$ $\iota : \tilde{X} \to \tilde{X}$ involution $\bar{X} = \tilde{X}/\iota$ $\downarrow$ X	$\mathrm{Prym}(\tilde{X}/\bar{X})$
• $SO(m,\mathbb{C})$	$\tilde{X} \xrightarrow{m:1} X$ $\iota : \tilde{X} \to \tilde{X}$ involution $\bar{X} = \tilde{X}/\iota$	n odd : $\mathrm{Prym}(\tilde{X}/\bar{X})$ n even : $\mathrm{Prym}(\tilde{X}'/\bar{X}')$ desingularization

Any two of these function Poisson commute in $\mathcal{A} \times \Omega^{0,1}(\mathbb{E}(\mathfrak{g}))$ since they are function of $\Omega^{0,1}(\mathbb{E}, \mathfrak{g})$ alone

$$\Longrightarrow \{f_i, f_j\} = 0 \text{ in } \mathcal{M}^s_d(G).$$

<u>Need to show:</u>

- generic fibre of Hitchin map is n-dimensional $\Rightarrow df_1 \wedge \cdots \wedge df_n \neq 0$

- generic fibre is an open set of an abelian variety with the vector fields X_{f_i} linear.

- <u>Now</u>

$$\underset{\mu_2 + i\mu_3}{\overset{}{\mu_c}} : \mathcal{A} \times \Omega^{0,1}(E(\mathfrak{g})) \to \operatorname{Lie} \mathcal{G}^*$$

$$(A, \varphi) \longmapsto \bar{\partial}_A \varphi$$

$$\mathcal{M}^s_{id}(G) \underset{\text{open}}{\subset} \mu_c^{-1}(0) \big/ \mathcal{G}$$

- <u>Have r $\mathcal{G}$-invariant holomorphic functions</u> defined by

$$\mathcal{A} \times \Omega^{0,1}(E(\mathfrak{g})) \longrightarrow \bigoplus_{i=1}^{k} \Omega^0(X, K^{d_i})$$

$$(A, \varphi) \longmapsto (P_1(\varphi), \dots, P_k(\varphi))$$

On $\mu_c^{-1}(0)$ these go to $\bigoplus_{i=1}^{k} H^0(X, K^{d_i})$

since $\bar{\partial}_A \varphi = 0$.

Recall:

$$\left(\mathcal{M}^s_d(G), \omega_c \right)$$ holomorphic symplectic manifold of dim = $2n$

Claim: $\{f_i, f_j\} = 0 \qquad i,j = 1,\dots,n$

Sketch of proof:

- $\mu : M \longrightarrow \mathfrak{g}^*$ moment map for action of G
- f : G-invariant function on M define a G-invariant function on $\mu^{-1}(0)$ and hence a function $\tilde{f}$ on $\mu^{-1}(0)/G$
- f, g G-invariant functions on M then

$$\{f, g\} = 0 \Longrightarrow \{\tilde{f}, \tilde{g}\} = 0$$

By Riemann-Roch

$$\dim \bigoplus_{i=1}^{k} H^0(X, K^{d_i}) = \sum_{i=1}^{k} (2d_i - 1)(g-1)$$

But

$$\dim G = \sum_{i=1}^{k} (2d_i - 1)$$

Kostant : principal TDS breaks $\mathfrak{g}$ into rep. spaces of dim. $2d_i - 1$

Hence:

$$\dim \bigoplus_{i=1}^{k} H^0(X, K^{d_i}) = \dim G\,(g-1)$$

$$= \frac{1}{2} \dim \mathcal{M}_d(G)$$

$\Rightarrow$ The map h gives $n = \dim G(g-1)$ holomorphic functions on $\mathcal{M}_d(G)$.

$f_1, \dots, f_n$

<u>Hitchin map</u>

p – G-invariant homogeneous polynomial on $\mathfrak{g}$ of degree d

<u>p defines a map:</u>

$$p : H^0(X, \underbrace{E}_{G\text{-bundle on } X}(\mathfrak{g}) \otimes K) \longrightarrow H^0(X, K^d)$$

Let $p_1, \dots, p_k$ be a basis for the ring of invariant polynomials on $\mathfrak{g}$.

We have a map:

$$h : \mathcal{M}_d(G) \longrightarrow \bigoplus_{i=1}^{k} H^0(X, K^{d_i})$$

$$(E, \varphi) \longmapsto (p_1(\varphi), \dots, p_k(\varphi))$$

$d_i = \deg p_i$

The system is said to be <u>algebraically completely integrable</u> if

- $f^{-1}(c)$ is an open set in an abelian variety for generic $c \in \mathbb{C}^n$

- The Hamiltonian vector fields $X_{f_1}, \dots, X_{f_n}$ are linear on $f^{-1}(c)$.

<u>Theorem</u> (Hitchin '87)

smooth

$\mathcal{M}_d^s(G)$ is an algebraically completely integrable Hamiltonian system.

- Hitchin did it for

$$T^* M_d^s(G) \subset \mathcal{M}_d(G)$$