S-duality and Geometric Langlands

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Slides available at www.icmat.es/seminarios/langlands/

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Langlands dual of a group

Let G = complex reductive Lie group

$H \subset G$ maximal torus

$\mathfrak{g} = \mathrm{Lie}\, G$ Lie algebra of G

$\mathfrak{h} = \mathrm{Lie}\, H$ Lie algebra of H.

Root data = quadruple $(\Lambda, \Delta, \Lambda^\vee, \Delta^\vee)$ = (weight lattice, root lattice, coweight lattice, coroot lattice)

$\Lambda = \mathrm{Hom}(H, \mathbb{C}^*) =$ character group of H = weight lattice $\subset \mathfrak{h}^*$ by differentiation

$\Delta = \{$roots (defined by H-action on $\mathfrak{g}$)$\}$ = root lattice $\subset \Lambda$

$\Lambda^\vee = \mathrm{Hom}(\mathbb{C}^*, H) =$ cocharacter grp of H = coweight lattice $\subset \mathfrak{h}$

$\Delta^\vee = \{$coroots$\}$ = coroot lattice $\subset \Lambda^\vee$

in terms of Killing form κ on $\mathfrak{g}^*$, $\Delta^\vee = \{h_{\alpha^\vee} \mid \alpha \in \Delta\}$ with $\langle h_{\alpha^\vee}, \phi\rangle = \kappa(\alpha^\vee, \phi)\ \forall \phi$, $\alpha^\vee = \dfrac{2\alpha}{K(\alpha,\alpha)}$.

Langlands dual ${}^LG$ of G — Dualize root data:

$\tilde{G}$ = another complex reductive Lie group with maximal torus $\tilde{H}$ and root data $(\tilde{\Lambda}, \tilde{\Delta}, \tilde{\Lambda}^\vee, \tilde{\Delta}^\vee)$

$\tilde{G}$ = Langlands dual if $\exists$ isomorphism $H \cong \tilde{H}^\vee$ dualizing root data, i.e. identifying

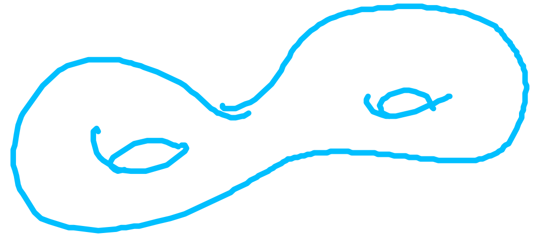
$$(\Lambda, \Delta) = (\tilde{\Lambda}^\vee, \tilde{\Delta}^\vee), \quad (\Lambda^\vee, \Delta^\vee) = (\tilde{\Lambda}, \tilde{\Delta}).$$

G	${}^LG$	
GL_n	GL_n	type A
SL_n	PSL_n	type A
Sp_{2n}	SO_{2n+1}	types $B \longleftrightarrow C$
$Spin_{2n+1}$	$Sp_{2n}/\mathbb{Z}_2$	types $B \longleftrightarrow C$
$Spin_{2n}$	$SO_{2n}/\mathbb{Z}_2$	type D
simply connected form	adjoint form	

Notes: (1) The root data determines G up to isomorphism.

(2) There is a functorial definition of ${}^LG$.

C = compact Riemann surface



= smooth connected complex projective curve of genus $g > 1$

$\mathrm{Bun}_G C$ = moduli space (stack) of (holomorphic) G-bundles over C

i.e. equipped with a flat connection

$\mathrm{Conn}_{{}^L G} C$ = moduli space (stack) of flat ${}^L G$-bundles over C

$D(\mathrm{Bun}_G C, \mathcal{D})$ = derived category of $\mathcal{D}$-modules over $\mathrm{Bun}_G C$

$D(\mathrm{Conn}_{{}^L G} C, \mathcal{O})$ = derived category of $\mathcal{O}$-modules over $\mathrm{Conn}_{{}^L G} C$

Conjecture (Geometric Langlands, rough form) [Beilinson & Drinfeld]

There is an equivalence of categories

$$D(\mathrm{Bun}_G C, \mathcal{D}) \cong D(\mathcal{C}onn_{{}^LG} C, \mathcal{O})$$

- intertwining the 'Hecke actions'
- taking the skyscraper sheaves $\mathcal{O}_L \in D(\mathcal{C}onn_{{}^LG} C, \mathcal{O})$ at flat ${}^LG$-bundles $L \in \mathcal{C}onn_{{}^LG}$ to the 'characters' or 'Hecke eigensheaves' $\mathrm{Aut}_L \in D(\mathrm{Bun}_G C, \mathcal{D})$, i.e. the $\mathcal{D}$-module Aut_L satisfies

Hecke operators

$$H_{x,V} \mathrm{Aut}_L = L_V|_x \otimes \mathrm{Aut}_L$$

eigenvector for all Hecke operators with eigenvalues determined by L

The Langlands dual group shows up in physics (S-duality).

Maxwell's equations on $\mathbb{R}^{1,3}$ (in the presence of electric charges)

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad\qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times B - \frac{\partial\vec{E}}{\partial t} = \vec{j} \qquad\qquad \vec{\nabla}\times\vec{E} + \frac{\partial\vec{B}}{\partial t} = 0 \qquad (c=1)$$

where

$\vec{E}(x)$ = electric field $\in \mathbb{R}^3$, $\vec{B}(x)$ = magnetic field $\in \mathbb{R}^3$ $(x \in \mathbb{R}^{1,3})$

$\rho(x)$ = electric density $\in \mathbb{R}$, $\vec{j}(x) = (j^1, j^2, j^3)$ = electric current $\in \mathbb{R}^3$

The Langlands dual group also shows up in physics (S-duality).

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$$\vec{\nabla}\cdot\vec{E}=\rho \qquad \vec{\nabla}\cdot\vec{B}=0$$
$$\vec{\nabla}\times\vec{B}-\frac{\partial\vec{E}}{\partial t}=\vec{J} \qquad \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0$$

These equations are invariant under Lorentz transformations.

Minkowski space $\mathbb{R}^{1,3}=\mathbb{R}^4$ with metric $\begin{pmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{pmatrix}$.

Points $x=(x^0,x^1,x^2,x^3)\in\mathbb{R}^{1,3}$; $\mu=0,1,2,3$ space-time indices

$\mu=0$ time index

$i=1,2,3$ space indices

Define 2-form $F = \frac{1}{2} F_{\mu\nu} \, dx^{\mu} \wedge dx^{\nu} =$ electromagnetic field-strength by $F_{0i} = -F_{i0} = E^{i}$
$F_{ij} = -\epsilon_{ijk} B^{k}$

and 1-form $j = j_{\mu} \, dx^{\mu} =$ electric current 1-form by $j_0 = \rho$
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and 1-form $j = j_\mu\, dx^\mu$ = electric current 1-form by $j_0 = \rho$, $j_i = -j^i$.

Maxwell's equations $\longleftrightarrow$ $\boxed{\begin{aligned} d * F &= j \\ dF &= 0 \end{aligned}}$

Now the invariance of Maxwell's eqns under Lorentz transformations is 'manifest'

where $* : \Omega^2 \to \Omega^2$ is Hodge star operator.

When there are no electric sources, $j=0$, so Maxwell's eqns are $\boxed{\begin{aligned} d*F &= 0 \\ dF &= 0 \end{aligned}}$

They are invariant under electromagnetic duality:

$$\boxed{\begin{aligned} F &\longmapsto *F \\ *F &\longmapsto -F \end{aligned}} \quad \longleftrightarrow \quad \text{exchange electric and magnetic fields:} \quad \boxed{\begin{aligned} \vec{E} &\longmapsto \vec{B} \\ \vec{B} &\longmapsto -\vec{E} \end{aligned}}$$

this sign comes from $*^2 = -1$ in $\mathbb{R}^{1,3}$

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It would be nice to have the same symmetry when $j \neq 0$.

In this case we need to assume that there are both electric and magnetic sources (but magnetic sources are not detected in Nature).

Assume that there are electric and magnetic sources:

Plug 1-form $k = k_\mu dx^\mu$ = magnetic current 1-form given by $k_0 = \sigma$, $k_i = -k^i$

with $\sigma(x)$ = magnetic density $\in \mathbb{R}$,

$\vec{k}(x) = (k^1, k^2, k^3)$ = magnetic current $\in \mathbb{R}^3$.

Assume that there are electric and magnetic sources:

Plug 1-form $k = k_\mu \, dx^\mu =$ magnetic current 1-form

Modified Maxwell's equations

$$\boxed{\begin{aligned} d * F &= j \\ dF &= k \end{aligned}}$$

$\longleftarrow$ invariant under electromagnetic duality:

$$\boxed{\begin{aligned} F &\longmapsto *F, \quad & j &\longmapsto k, \\ *F &\longmapsto -F, \quad & k &\longmapsto -j. \end{aligned}}$$

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Then the electric charge $q = \int \rho \, d^3x$ and magnetic charge $g = \int \sigma \, d^3x$ are also exchanged by the em duality:

$$\boxed{\begin{aligned} q &\longmapsto g \\ g &\longmapsto -q \end{aligned}}$$

$M = \mathbb{R}^{1,3} = \mathbb{R} \times X$ – Minkowski space

L – hermitian line bundle

$\downarrow$

$X = \mathbb{R}^3$ – space

Dirac observed that the modified Maxwell's equations have highly nontrivial consequences in the quantum theory.

Here is a modern treatment [after Wu-Yang]:

Maxwell's equns (with no magnetic sources)

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

admit solution

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

with $\vec{A}$ = magnetic potential vector

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Equ of motion of a charged particle of charge q : $\dfrac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v}\times\vec{B}\right)$

$$\therefore \quad \frac{d\vec{p}}{dt} + q\frac{\partial\vec{A}}{\partial t} = \vec{F} \quad \text{with} \quad \vec{F} := e\,\vec{v}\times\vec{B}$$

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$\therefore \dfrac{d\vec{p}}{dt} + q\dfrac{\partial \vec{A}}{\partial t} = \vec{F}$ with $\vec{F} := e\,\vec{v}\times\vec{B}$ — Lorentz force of field $\vec{B}$ on the particle

$\therefore$ the Schrödinger eqn is $H\psi = i\hbar\dfrac{\partial \psi}{\partial t}$, where

$\hat{P} = \dfrac{\hbar}{i}\vec{\nabla} + q\vec{A}$ and $H = -\dfrac{1}{2m}\hat{P}^2$ = Hamiltonian operator for a particle moving in a magnetic field.

mass

$\longrightarrow$ the vector potential is essential in the quantum theory, but it is only unique up to transformations $\vec{A} \mapsto \vec{A} - \vec{\nabla} f$, for functions f.

This ambiguity in the choice of $\vec{A}$ and hence ψ is related to gauge symmetry:

A solution of this problem is obtained by viewing the wave functions ψ as the (L^2-) sections of a Hermitian line bundle:

$$\begin{array}{l} L \quad \text{Hermitian line bundle} \\ \downarrow \\ \mathbb{R}^3 = \text{Space } X \end{array}$$

Minkowski space-time $M = \mathbb{R}^{1,3} = \mathbb{R} \times X$

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Identifying the 1-forms and the vectors in $\mathbb{R}^3$ in the canonical way, we see that the moment operator $\hat{p} = \frac{\hbar}{i}\vec{\nabla} + q\vec{A}$ can be viewed as a differential operator

$$\hat{p} = \frac{\hbar}{i} d_A : \Gamma(X, L) \to \Omega^1(X, L),$$ with $\tilde{A} = i e \vec{A}$, $e := \frac{q}{\hbar}$.

$$\hat{P} = \frac{\hbar}{i} d_A \implies \vec{B} = \vec{\nabla} \times \vec{A} = \epsilon^{ijk} \partial_j A_k \, \partial/\partial x^i$$ corresponds to the 1-form

$B = d\vec{A} = \frac{1}{ie} d\tilde{A} = \frac{1}{ie} F_A$, where $F_A = (d_A)^2 = d\tilde{A} =$ curvature of A.

In conclusion, $\boxed{F_A = ieB}$ with $e := \frac{q}{\hbar}$.

Suppose now that there is a magnetic charged particle at the origin $0 \in \mathbb{R}^3$.

The modified Maxwell's equation $dF = k$ contradicts the Bianchi id.

$\Rightarrow$ we cannot look for connections A s.t. $\vec{B} = \vec{\nabla} \times \vec{A}$ (i.e. $F =$ curvature of A)

because $k \neq 0$ is a distribution supported at the origin.

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However we can do so on $\mathbb{R}^3 \setminus \{0\}$. One solution to the modified Maxwell's equations in this case is Dirac's magnetic monopole

$$\vec{B} = \frac{1}{4\pi} g \frac{\hat{r}}{|r|^2} \quad (r \in \mathbb{R}^3, \ \hat{r} = \frac{r}{|r|} \text{ unit vector})$$

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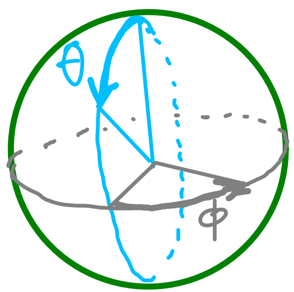
we need two 'trivialisations': Northern and Southern half-space.

Northern half $0 \le \theta \le \pi/2$: $\vec{B} = \vec{\nabla} \times \vec{A}_+$ with $\vec{A}_+ = \frac{g}{4\pi} \frac{1-\cos\theta}{\sin\theta} \vec{e}_\phi$

Southern half $\pi/2 \le \theta \le \pi$: $\vec{B} = \vec{\nabla} \times \vec{A}_-$ with $\vec{A}_- = \frac{-g}{4\pi} \frac{1+\cos\theta}{\sin\theta} \vec{e}_\phi$



spherical coordinates

On each half space we have a Schrödinger equation

$$-\frac{\hbar^2}{2m}\left(\vec{\nabla}+ie\vec{A}_\pm\right)^2\psi_\pm(x)=i\hbar\frac{\partial}{\partial t}\psi_\pm(x).$$

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$\Rightarrow$ to have a single-valued wave function we require

$A_+ - A_- = \vec{\nabla}\chi$ s.t. $e^{-ie\chi}$ is single-valued at $\theta=\pi/2$.

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Now, $A_+ - A_- = \vec{\nabla}\chi$ with $\chi=\frac{g}{2\pi}\phi$ so $e^{-ie\chi}$ is single-valued for $0\leq\phi\leq 2\pi$ $\Longleftrightarrow$ $e^{-ie\chi}\Big|_{\phi=0}=e^{-ie\chi}\Big|_{\phi=2\pi}$ $\Longleftrightarrow$

$$e^{ieg}=1 \Longleftrightarrow \boxed{eg\in 2\pi\mathbb{Z}}.$$

This is Dirac's quantisation law:

$$\boxed{eg = 2\pi n \text{ for some integer } n}$$

(recall that $e := q/\hbar$).

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Another derivation of this law is as follows:

Away from $0 \in \mathbb{R}^3$, $\vec{\nabla}.\vec{B}=0$, so $B = \frac{1}{i}\frac{\hbar}{q}F_A$ for a connection A over $\mathbb{R}^3 \setminus \{0\}$.

Now, the magnetic charge at $0 \in \mathbb{R}^3$ is the flux of $\vec{B}$ across a sphere centred at $0 \in \mathbb{R}^3$:

$$g = \oiint_{S^2} \vec{B} \times d\vec{S} = \frac{1}{ie}\int_{S^2} F_A \in \frac{1}{e}2\pi\mathbb{Z} \text{ because } \int_{S^2}\frac{i}{2\pi}F_A \in \mathbb{Z}.$$

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CONSEQUENCES: 1) The existence of a magnetic monopole would imply the quantisation of the electric charge (as observed experimentally).

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2) The electromagnetic duality maps $\begin{matrix} e \longmapsto g \\ g \longmapsto -e \end{matrix}$, so it relates abelian gauge theories with small e to another ones with large e (b/c $eg = 2\pi n$).
This is the first instance of S-duality, relating strong/weak couplings.

Dirac's quantisation law for nonabelian gauge theories

Goddard, Nuyts & Olive generalized Dirac's law to nonabelian gauge theories.

G = compact connected real Lie group

$\underline{g}$ = its Lie algebra

$F_A = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu \in \Omega^2 \otimes \underline{g}$ = $\underline{g}$-valued 2-form over $\mathbb{R}^4$

(or over $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$ as for Dirac's magnetic monopole)

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Yang-Mills equations with 'electric' and 'magnetic' sources

$d_A * F = j$ have the same duality symmetry $F \mapsto *F$, $j \mapsto k$

$d_A F = k$ $*F \mapsto -F$, $k \mapsto -j$

and the quantum theory leads to a charge quantisation.

Suppose that there is a magnetic charged particle at the origin $0 \in \mathbb{R}^3$.

This nonabelian magnetic monopole in this case is

$$\vec{B} = \frac{g(r)}{4\pi} \frac{\hat{r}}{|r|^2} \longleftrightarrow \text{2-form } F_{ij} = \epsilon_{ijk} B^k = \epsilon_{ijk} \frac{g(r)}{4\pi} \frac{\hat{r}^k}{|r|^2} = *_3\, d\left(\frac{1}{4\pi} \frac{g}{r}\right)$$

with $g(r) \in \underline{\mathfrak{g}}$ covariantly constant magnetic charge $Dg(r) = 0$.

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with $g(r) \in \underline{g}$ covariantly constant magnetic charge.

To obtain a consistent quantum theory for a particle with 'electric' charge q in an irred. represent. ρ of G, we have as before a covariant derivative $d_A = d + \tilde{A}$ with $\tilde{A} = ie\rho(\vec{A})$, $e = q/\hbar$.

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To obtain a consistent quantum theory for a particle with 'electric' charge e in an irred. represent. ρ of G, we have as before a covariant derivative $d_A = d + \tilde{A}$ with $\tilde{A} = ie\rho(\vec{A})$, $e = q/\hbar$

Dirac's argument + some homotopy theory $\Longrightarrow$ the analogue of Dirac's condition $e^{ieg} = 1$ is now

$$\boxed{\exp(eg) = 1 \in G}.$$

The magnetic charge $g_0 := g(r_0)$ is not physical.
fixed pt
What is physical is its orbit under the adjoint action of G on $\underline{g}$.

$\Longrightarrow$ up to this action, the condition $\exp(e\, g_0) = 1$ means that the magnetic charge g_0 (suitibly normalised by e) is a dominant weight of the Langlands dual group ${}^L G$.

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Note also that 'electric' charged particles are classified by irreducible representations of G, i.e. dominant weights of G (as above).

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Note: Goddard, Nuyts & Olive (1977) did not know Langlands dual already existed, so called $G^\vee = {}^L G$ the 'magnetic dual'. Very soon after that, Atiyah suggested that this could be related to (classical) Langlands.

This is the basis of Montonen-Olive's conjecture (S-duality):

$$\boxed{\text{Yang-Mills theories with structure groups } G \text{ and } {}^LG \text{ are isomorphic } \textbf{on the quantum level} \text{ and this isomorphism exchanges electric and magnetic sources.}}$$

This is the basis of Montonen-Olive's conjecture (S-duality):

Yang-Mills theories with structure groups G and ${}^L G$ are isomorphic **on the quantum level** and this isomorphism exchanges electric and magnetic sources.

On the quantum level one should consider a path integral

$$Z = \sum_{[E]} \int \mathcal{D}A \; e^{-S(A)}$$

with Yang-Mills action

sum over all topological classes

$$S(A) = \int_X \Big(-\frac{1}{e^2} \underbrace{\operatorname{tr} F \wedge * F}_{\text{Yang-Mills}} + \frac{i\theta}{8\pi^2} \underbrace{\operatorname{tr} F \wedge F}_{\text{Chern class}} \Big).$$

space-time

This theory has 2 coupling constants $e^2>0$, θ which we combine in a complex number $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \in \mathbb{H} :=$ upper-half plane.

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Using Dirac's quantisation rule one finds that for $\theta=0$, the dual gauge coupling is $\check{e}^2 = \frac{16\pi^2}{e^2} n_{\underline{g}}$ with

$$n_{\underline{g}} = \frac{\|\text{long root}\|^2}{\|\text{short root}\|^2} = 1 \ (A,D,E),\ 2\ (B,C,F_4),\ 3\ (G_2).$$

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Another obvious symmetry of Z (which does not transform the group): $\mathbb{H} \longrightarrow \mathbb{H} : \tau \longmapsto \tau + 1$

$\therefore$ Both symmetries generate a subgroup of $PSL(2,\mathbb{R}) \circlearrowright \mathbb{H}$.

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Strong evidence (Sen, Vafa-Witten), but not a theorem yet.

... Geometric Langlands might be more evidence!

$\mathcal{N}=4$ super Yang-Mills theory:

Riemannian 4-mfd

Bosons:

- Connection 1-form A on principal G-bundle $E \to X$
- 6 scalar fields $\phi_i \in \Gamma(X, \operatorname{ad} E)$ $\quad i=1,\dots,6$

$\mathcal{N}=4$ susy

Fermions:

- 4 spinor fields $\bar{\lambda}_a \in \Gamma(X, (\operatorname{ad} E)\otimes S_-)$ $\quad \underbrace{a=1,\dots,4}$
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Action: $$S_{\mathcal{N}=4} = S_{YM} + \frac{1}{e^2}\int_X \sum_i \operatorname{tr} d_A\phi_i \wedge * d\phi_i + \mathrm{vol}_X \sum_{i<j} \operatorname{tr}[\phi^i, \phi^j] + \text{fermionic terms}$$

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S-duality:
- bosonic symmetry generators are mapped trivially
- supersymmetry generators:

$$\bar{Q}_a \mapsto e^{i\phi/2}\bar{Q}_a\,, \quad Q^a \mapsto e^{-i\phi/2} Q^a\,, \quad e^{i\phi} := \frac{|\tau|}{\tau}$$

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R-symmetry: $\mathrm{Spin}(6)$

Twisting $\mathcal{N}=4$ super Yang-Mills theory:

This is the process of constructing a 'topological field theory' out of $\mathcal{N}=4$ SYM.

Twisting $\mathcal{N}=4$ super Yang-Mills theory:

We obtain a family $Q = u\,Q_\ell + v\,Q_r$ of BRST operators parametrized by $\boxed{t = v/u \in \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}}$, and hence a family of topological field theories with action $S = \{Q, V\} + \frac{i\Psi}{4\pi}\int_X \mathrm{tr}\, F \wedge F$, for certain V, where

$$\Psi = \frac{\theta}{2\pi} + \frac{t^2-1}{t^2+1}\,\frac{4\pi i}{e^2}$$

combines τ and t.

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Bosons: Connection 1-form $\boxed{A}$

ad E-valued 1-form $\boxed{\phi \in \Omega^1(X, \mathrm{ad}\,E)}$

$\sigma \in \Gamma(X, \mathbb{C}\otimes \mathrm{ad}\,E)$

Fermions: 0-forms $\eta, \tilde{\eta} \in \Gamma(X, \mathbb{C}\otimes \mathrm{ad}\,E)$

1-forms $\psi, \tilde{\psi} \in \Omega^1(X, \mathbb{C}\otimes \mathrm{ad}\,E)$

2-form $\chi \in \Omega^2(X, \mathbb{C}\otimes \mathrm{ad}\,E)$

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(similar to Hitchin's eqn in 4d)

$t=i$: $F_B = 0, \quad d_B * \phi = 0,$ where $B = \underbrace{A + i\phi}_{\text{complex connection}}$

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$$\begin{matrix} t=i & & t=1 \\ \theta=0 & \longmapsto & \theta=0 \\ G & & {}^L G \end{matrix}$$

Dimensional reduction $d=4 \longrightarrow d=2$:

Study the TFT on $X = \Sigma \times C$ for Riemann surfaces Σ, C, in the limit $\mathrm{volume}(C) \to 0$.

Then A and ϕ are pulled-back from C and satisfy Hitchin equs

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For each t, the effective theory is a topological σ-model with worldsheet Σ and target $\mathcal{M}_H(G, C) =$ moduli space solutions of Hitchin's eqns / gauge symmetry.

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$t=i$: topological σ-model is B-model

i.e. its correlators depend on the complex str. of the target but not on its sympl. str.

for complex str. given by identification

$$\mathcal{M}_H(G,C) = \mathcal{M}_{flat}(G^c, C) :=$$ moduli space of flat G^c-bundles

$\left(G^c := \text{complexification of } G \right)$

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t = 1 : topological σ-model is A-model

i.e. its correlators depend on the sympl. str. of the target but not on its complex str.

for sympl. str. $\omega_K = \frac{2}{e^2}\int_C \mathrm{tr}\, \delta\phi \wedge \delta A$ ← this sympl. str does not depend on complex str of C

∴ Prediction of S-duality for $\theta = 0$:

$t = i$, $\theta = 0$		$t = 1$, $\theta = 0$
twisted $\mathcal{N}=4$ SYM with gauge group ${}^L G$	$\xleftrightarrow{\text{equivalence}}$	twisted $\mathcal{N}=4$ SYM with gauge group G
$\parallel$		$\parallel$
B-model with target $\mathcal{M}_{flat}({}^L G^{c}, C)$		A-model with target $(\mathcal{M}_H(G, C), \omega_K)$

∴ Prediction of S-duality for $\theta = 0$:

$t = i \;, \theta = 0$		$t = 1 \;, \theta = 0$
twisted $\mathcal{N}=4$ SYM with gauge group ${}^LG$	equivalence $\longleftrightarrow$	twisted $\mathcal{N}=4$ SYM with gauge group G
$\parallel$		$\parallel$
B-model with target $\mathcal{M}_{flat}({}^LG^{\mathbb{C}}, C)$		A-model with target $(\mathcal{M}_H(G, C), \omega_K)$

By mirror symmetry, this implies:

CONCLUSIONS

1) Strominger–Yau–Zaslow $\Rightarrow$ $\mathcal{M}_{flat}({}^LG^{\mathbb{C}}, C)$ and $(\mathcal{M}_H(G, C), \omega_K)$ are a mirror pair, with SYZ fibrations = the Hitchin fibrations

further work: Hausel–Thaddeus

CONCLUSIONS

1) $\mathcal{M}_{flat}(G^c, C)$ and $(\mathcal{M}_H({}^LG^c, C), \omega_K)$ are a mirror pair, with SYZ fibrations = the Hitchin fibrations

2) Homological mirror symmetry $\Longrightarrow$ Derived category of coherent sheaves on $\mathcal{M}_{flat}({}^LG^c, C)$ $\cong$ Category of A-branes on $(\mathcal{M}_H(G, C), \omega_K)$

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$\updownarrow$ further work

twisted $\mathcal{D}$-modules over

$M(G^c, C) :=$ moduli space of stable G^c-bundles $\subset \mathrm{Bun}_{G^c} C$

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Hecke eigensheaf

$\mathrm{Aut}_L \in D(\mathrm{Bun}_{G^c} C, \mathcal{D})$

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$\updownarrow$ Hecke eigensheaf $\mathrm{Aut}_L \in D(\mathrm{Bun}_{G^c} C, \mathcal{D})$

This 'implies' Geometric Langlands

$$D(\mathrm{Conn}_{{}^LG^c} C, \mathcal{O}) \cong D(\mathrm{Bun}_{G^c} C, \mathcal{D})$$

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