Resumen:

Let $C(G)$ denote the continuous functions on a compact group $G$. By the Gelfand-Naimark Theorem, this $C^*$-algebra captures the topology on $G$. If we now define

$$\Gamma : C(G) \rightarrow C(G) \otimes_{\text{min}} C(G) \simeq C(G \times G) \quad \text{by} \quad \Gamma(f)(s,t) = f(st),$$

then $\Gamma$ captures the group structure and, as $f((st)u) = f(s(tu))$, the $*$-homomorphism $\Gamma$ will satisfy $(id \otimes \Gamma) \circ \Gamma = (\Gamma \otimes id) \circ \Gamma$, that is, $\Gamma$ is a comultiplication. A compact quantum group is thus defined to be a pair $(A, \Gamma)$ consisting of a unital $C^*$-algebra $A$ and a comultiplication $\Gamma$ satisfying certain conditions. The category of compact quantum groups also includes the reduced group $C^*$-algebras of discrete groups. However it also contains new and interesting examples of deformations of classical groups (e.g., Woronowicz’s quantum $SU(2)$).

Another motivation for the analytic approach to quantum groups was the desire to generalize the Pontryagin duality theorem for locally compact abelian groups: If $G$ is a locally compact abelian group, then $\hat{G} := \{ \phi : G \rightarrow \mathbb{C} \mid \phi(g_1g_2) = \phi(g_1)\phi(g_2) \}$ is again a locally compact abelian group and $\hat{\hat{G}}$ is canonically isomorphic to $G$. However, this fails for nonabelian groups. Vaes and Kustermans’ locally compact quantum groups include the locally compact groups, their dual objects, and satisfy a Pontryagin duality theorem.

Compact and locally compact quantum groups will be defined, and the basics of their duality/representation theory will be described (concentrating mainly on the compact quantum group case), before briefly discussing some examples.