LECTURES ON b-REEB DYNAMICS

CÉDRIC OMS

ABSTRACT. These are expanded notes from the author's minicourse at the International ICMAT Summer School on Geometry, Dynamics and Field Theory that was held at Miraflores de la Sierra in Madrid, Spain, June 20-25, 2024. These notes are intended to be accessible to undergraduate students.

CONTENTS

Over the last 40 years, the study of Hamiltonian and Reeb dynamics has significantly developed. A relatively recent addition to this field is the exploration of b-symplectic structures, which can be interpreted as symplectic structures away from a codimension one submanifold. On this hypersurface, the symplectic form admits a singularity. The investigation of these structures is motivated by problems arising in celestial mechanics, raising the question of the extent to which classical results in Hamiltonian dynamics apply to this realm.

In this mini-course, we will delve into the dynamical study of these manifolds. On the one hand, we will develop the theory of contact forms with singularities and explore the dynamical properties of the Reeb vector field. On the other hand, we will investigate the classical Arnold conjecture concerning the fixed points of Hamiltonian diffeomorphisms on b-symplectic manifolds.

Disclaimer: None of the material presented in these notes is new. These notes are a self-contained and completed recompilation of the results contained in [39, 40, 41, 17, 16] and my thesis. Also, there may be typos in these notes. Please let me know if you spot any!

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1. The three body problem: going to infinity

As all of you know, Hamiltonian dynamics originates in classical mechanics, where the manifold is the phase space of the Euclidean 3-dimensional space $\mathbb{E}^3$. One would like to describe the masses' motion under the influences of the forces. Guiding examples have been (and still are) planets moving under the law of gravitation in $\mathbb{E}^3$. Newton, Kepler, and Poincaré understood the case of two planets well, as they consist of an integrable system. The 3-body problem, however, is not an integrable system anymore, and a lot of research is still to be done to help understand this puzzling problem. We will quickly review some basic facts on the 3-body problem, details can be found in [36].

1.1. The planar restricted 3-body problem.

The n-body problem is the dynamical system obtained when n point masses in the Euclidean 3 space are left to interact according to Newton's second law of gravitation. Given initial positions and velocities, the problem consists of predicting the future positions and velocities of the bodies. Mathematically speaking, let us denote by $q_i(t) \in \mathbb{R}^3$ the position of the body i (for $i \in \{1, \dots, n\}$) at time t and by m_i its mass. The Hamiltonian is given by

$$H = K + U,$$

where K is the kinetic energy given by

$$K = \frac{1}{2}\sum_{i=1}^{n} \frac{|p_i(t)|^2}{m_i},$$

where $q_i(t) = \dot{p}_i(t)$ is the momenta, and U is the potential energy, given by

$$U = -\sum_{1 \leq i < j \leq n} \frac{G m_i m_j}{|q_i - q_j|}.$$

Here, G denotes the universal gravitational constant. The motion of the n-problem is described by Hamilton's equations given by

$$\begin{cases} \frac{dp}{dt} = \frac{\partial H}{\partial q} \\ \frac{dq}{dt} = -\frac{\partial H}{\partial p}. \end{cases}$$

In what follows, we will assume that $n = 3$, and therefore the equations simplify considerably. The model to have in mind is given by the earth $q_E(t)$, moon $q_M(t)$ and a satellite $q(t)$, moving under the gravitational law in $\mathbb{R}^3$. the respective masses are denoted by m_E, m_M and m. The *restricted planar circular three body problem*, short RPC3BP, is obtained assuming the following assumptions:

- Restricted: $m = 0$. Physically speaking, the satellite has negligible mass compared to the masses of the Earth and the moon;
- Circular: Under the first assumption, the earth and the moon are governed by the 2-body problem. It is classically known that the possible trajectories for this integrable system are either circular, elliptic, parabolic, or hyperbolic. We assume that the motion of the earth and the moon is of the first type, meaning circular;
- Planar: The satellite's motion is subjected to the potential from the Earth and the moon. Planar means that the satellite's motion is assumed to be in the plane spanned by the Earth and the moon.

The restricted planar three-body problem involves understanding the satellite's motion under the above assumptions. To do so, the above Hamiltonian can be written as follows (normalizing $G = 1$ and $m_E + m_M = 1$) :

$$H_t(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{|q - q_M(t)|} - \frac{1-\mu}{|q - q_E(t)|},$$

where $\mu = \frac{m_M}{m_E + m_M}$ is called the relative mass. Note that the Hamiltonian is not defined when $q = q_M$ or $q = q_E$, that is when the satellite collides with one of the two massive bodies. In the following, we will not consider collisions, and therefore the phase space is given by $T^*(\mathbb{R}^2 \setminus \{q_M, q_E\})$. We continue by choosing a suitable reference frame to simplify the equations further. Under the circular symmetry in the circular three-body problem, we choose the origin at the center of mass. In this reference frame, the motion of the earth and the moon are given respectively by $q_E(t) = (\mu \cos(t), \mu \sin(t))$ and $q_M(t) = ((1-\mu)\cos(t), (1-\mu)\sin(t))$. Note that we used the three conditions of the PRC3BP to deduce this equation.

The Hamiltonian is still time-dependent. This is unfortunate, as time-dependent Hamiltonians are not preserved quantities. However, in a rotating coordinate system, we can assume that the earth and the moon are fixed at $q_E = (\mu, 0)$ and $q_M = (1-\mu, 0)$. The Hamiltonian thus becomes *time-independent*, with the Hamiltonian given by

$$H(q,p) = \frac{1}{2}|p|^2 - \frac{\mu}{q - M} - \frac{\mu}{q - E} + p_1 q_2 - p_2 q_1. \tag{1}$$

This is now a time-independent Hamiltonian - the price to pay. However, it is no longer a *mechanical* Hamiltonian, meaning that it is no longer the sum of kinetic and potential energy. This is because the mixed term $L(q,p) = p_1 q_2 - p_2 q_1$, which can be interpreted as the Coriolis force due to the choice of inertial frame.

There are five critical points of H, which are called the Lagrangian points L_i, $i = 1, \dots 5$, $H(L_1) < H(L_2) < H(L_3) < H(L_4) = H(L_5)$.

Given an energy value c, we denote by $\Sigma_c = H^{-1}(c) \subset T^*\mathbb{R}^2$. As mentioned earlier, the level-sets Σ_c are preserved, as H is time-independent. To analyse the position sets of Σ_c, consider the natural projection $\pi : T^*\mathbb{R}^2 \to \mathbb{R}^2$. Using this projection, we find the sets $\mathcal{E}_c = \pi(\Sigma_c)$, which are called the *Hill's region*. These represent the positions that can be obtained by satellites having energy c. For c small enough, namely when $c < H(L_1)$, one can prove that $\mathcal{E}$ consists of three connected components: one close to the moon, denoted by $\mathcal{E}_M$, one close to the earth, $\mathcal{E}_E$, (both of them are compact sets in $\mathbb{R}^2$), and one connected component far away from the earth and the moon. **Exercise**. The connected component of the hypersurface in $T^*\mathbb{R}^2$ lying above $\mathcal{E}_M$ (respectively Σ_E is denoted by Σ_M (respectively Σ_E): more precisely $\pi^{-1}(\mathcal{E}_M) =: \Sigma_M$. When the energy crosses the critical value $H(L_1)$, the two bounded connected components Σ_M and Σ_E "become" one connected component, in fact the topological connected sum between Σ_M and Σ_E!

It is important to note that, even though $\mathcal{E}_M$ and $\mathcal{E}_E$ are compact, Σ_M and Σ_E are *not* compact. This is due to the collision that the satellite can have with the Earth and the moon.

To resume: in the RC3BP, we obtained two energy level-sets of dimension 3 (non-compact) for low energy values contained in $T^*\mathbb{R}^2$, provided with the canonical symplectic structure. This raises the natural question of whether methods from symplectic topology can be applied to find periodic orbits of this system. As we can see, non-compactness is a troublemaker. First, let us review some basic facts about symplectic geometry.

1.2. **The very basics on symplectic geometry.** An excellent reference to get started in symplectic geometry is [13, 34].

Definition 1.1. *A manifold M with a non-degenerate closed 2-form $\omega \in \Omega^2(M)$ is called a symplectic manifold.*

A first trivial example of a symplectic manifold is orientable surfaces:

Example 1.2. *Any orientable surface is a symplectic manifold.*

The most 'natural' example, however, is the cotangent bundle of a manifold, as it comes equipped with a canonical symplectic structure.

Example 1.3. *If λ is the Liouville 1-form on T^*M, then $\omega = d\lambda$ is symplectic. Given coordinates $(p_1, \dots, p_n)$ on M, the Liouville 1-form is given by $\lambda = \sum_{i=1}^n p_i dq_i$.*

If ω is a symplectic form on a manifold M^{2n}, then ω^n is a volume form, and thus M is always orientable. Nevertheless, not all orientable manifolds admit a symplectic form. For instance, S^4 is an orientable manifold that does not admit any symplectic form: in fact, $H^2(M)$ obstructs the existence of a symplectic form. Assume that there is a symplectic form ω on M. As ω is closed, $[\omega] \in H^2(S^4)$, $[\omega] \neq 0$. But $H^2(S^4) = 0$ contradicts this. More generally, this proves

Proposition 1.4. *If M^{2n} is a compact smooth manifold and admits a symplectic structure, then $H^2(M) \neq 0$.*

This implies that S^{2n} is not a symplectic manifold for $n > 1$. Symplectic manifolds have only one single local invariant: the dimension. This is the content of Darboux's theorem.

Theorem 1.5 (Darboux theorem)**.** *Let (M^{2n}, ω) be a symplectic manifold. Then, for all $p \in M$, there exists a local coordinate system $(x_1, y_1, \dots, x_n, y_n)$ such that $\omega = \sum_{i=1}^n dx_i \wedge dy_i$ in a neighbourhood U of p.*

Given a smooth function H (as the Hamiltonian H in the symplectic manifold $(T^*\mathbb{R}^2, d\lambda_{st})$ in the previous section), the Hamiltonian vector field is given by the equation

$$\iota_{X_H}\omega = -dH.$$

The function H is called Hamiltonian function.

We can now define the Poisson bracket associated with the symplectic structure.

Definition 1.6. *Let $f, g \in C^\infty(M)$ two smooth function on the symplectic manifold (M, ω). The Poisson bracket is defined by $\{f, g\} = \omega(X_f, X_g)$.*

Exercise: Check that the Poisson bracket is a Lie bracket on the space of smooth function (that is, it is skew-symmetric and defines a Leibniz rule) and $[X_f, X_g] = -\{f, g\}$. Furthermore, check that it satisfies Jacobi identity. Check that in Darboux coordinates, it is given by

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} \right). \tag{2}$$

By the motivating example of the previous section, we are particularly interested in level sets of the Hamiltonian functions - and, more precisely, in the dynamics that those Hamiltonian functions generate. Therefore:

Exercise: Given a Hamiltonian function H, check that X_H preserves the level-set $H^{-1}(c)$, where c is a regular value. As in the previous section, one would like to know whether there are periodic orbits of X_H. The previous remark shows that the vector field X_H can be seen as a vector field on the hypersurface $\Sigma_c := H^{-1}(c)$. In some cases, where some additional geometric structure is given, the existence of periodic orbits of X_H can be proved when there is a transverse Liouville vector field to the hypersurface.

Definition 1.7. *A vector field X that satisfies $\mathcal{L}_X\omega = \omega$ is called a Liouville vector field.*

This means that the vector field *expands* the symplectic form exponentially. In the cotangent bundle (Example 1.3), a Liouville vector field is given by $X = \sum_{i=1}^n p_i \frac{\partial}{\partial p_i}$. Given a hypersurface Σ in a symplectic manifold (M, ω) that has a transverse Liouville vector field, it is easy to check that $(\Sigma, \iota_X\omega)$ is a contact manifold, namely that satisfies the following definition. **Exercise**.

Definition 1.8. *A manifold M^{2n+1} with a 1-form $\alpha \in \Omega^1(M)$ is called contact if $\alpha \wedge d\alpha \neq 0$.*

The reference for contact geometry is [21].

A first example of a contact structure is the 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$, given by $\alpha = dx + ydz$. This contact form is called the *standard* contact form because any contact form locally is given by the standard one. This is the content of the Darboux theorem for contact forms.

Theorem 1.9. *Darboux theorem for contact manifolds Let α be a contact form on M^{2n+1}. Then locally $\alpha = dz + \sum_{i=1}^n x_i dy_i$.*

A contact manifold has an intrinsic vector field called the *Reeb vector field*.

Definition 1.10. *The Reeb vector field R_α is the uniquely defined vector field by*

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0, \\ \iota_{R_\alpha}\alpha = 1. \end{cases}$$

In the case of the standard contact form, the associated Reeb vector field is given by the linear vector field ∂_z.

In the case of a hypersurface Σ given by the regular level-set of a Hamiltonian function in a symplectic manifold (M, ω), and a transverse Liouville vector field X, there are thus two vector fields that are of interest to us: the Hamiltonian vector field on Σ and the Reeb vector field of the contact form $\alpha = \iota_X \omega$. **Exercise:** It is easy to check that both agree, up to reparametrisation. The integral curves of both thus agree, and looking for periodic orbits of X_H means that we are looking for periodic orbits of the Reeb vector field.

A good example to keep in mind is the following:

Example 1.11. *Consider the symplectic manifold $(\mathbb{R}^4, dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$. The unit sphere is the regular level-set given by $H^{-1}(1)$, where $H = r^2$ in radial coordinates. One way to compute the Hamiltonian vector field X_H is as follows: The radial vector field $r\partial_r$ is a Liouville vector field and thus transverse to any star-shaped hypersurface. In particular, it is transverse to the unit sphere S^3. This provides S^3 thus with a contact form. The Reeb vector field's integral curves associated with this contact form are just the fibers of the Hopf fibration of S^3. In particular, every Reeb vector field is periodic.* ***Exercise:*** *Do the explicit computations of this.*

Another star-shaped hypersurface is given by the ellipsoid defined by the Hamiltonian $H^{a,b} = \frac{\pi r_1^2}{a} + \frac{\pi r_2^2}{b}$, where $r_1^2 = x_1^2 + y_1^2$, and $r_2^2 = x_2^2 + y_2^2$. ***Exercise:*** *Check that if a and b are rationally dependent, the associated Reeb (or Hamiltonian) only has 2 periodic orbits.*

In the previous examples, under the condition that the contact manifold is *compact*, the associated Reeb vector field always has a periodic orbit (2 to be more precise). In the non-compact case, however, as in the case of the standard contact form, such dynamical invariants do not need to exist: the flow in of $R_{\alpha_{st}}$ is just the linear flow. This is the content of the Weinstein conjecture.

Conjecture 1.12 (Weinstein conjecture)**.** *Let (M, α) be a compact contact manifold. Then, a periodic Reeb orbit always exists.*

The quest for periodic orbits of the Reeb vector field has a rich history and is, and still is, a driving force in developing symplectic topology. In particular, the above conjecture has been the source of an immense amount of research in the last 30 years. In what follows, we briefly overview some of the most notable results obtained to understand the Reeb dynamics on contact manifolds better.

Most notably, the conjecture was proved in the following cases (chronologically ordered):

- '86 Viterbo: compact contact hypersurfaces in $(\mathbb{R}^{2n}, \omega_{\text{st}})$;
- '93 Hofer: so-called overtwisted contact manifolds;
- '05: Taubes: general compact M of dimension 3.

The conjecture is still open in general, but much has been done. For instance it is known that in dimension 3, there are always 2 or inifinitely many periodic Reeb orbits.

1.3. **Back to the RPC3BP.** Returning to the restricted planar circular planar three-body problem, we would like to apply the symplectic and contact topology results to predict the existence of periodic orbits on the energy level-sets Σ_c. To apply the results from contact topology, we, therefore, need to show that a Liouville vector field is transverse to these level sets. If this is the case, then the contraction of this Liouville vector field with canonical symplectic form gives rise to a contact form. Let us assume that the energy c is below the first critical value below $H(L_1)$, where L_1 is the first Lagrange point. In this case, we saw three connected components: the projections under π of these connected components give two bounded ones (around the earth, respectively, the moon) and one non-compact one. Let us focus on the preimage of the compact connected component around the earth. In other words: the connected component Σ_c^E such that the Hill region is $\pi(\Sigma_c^E) = \mathcal{E}_c$.

There is a natural candidate for the Liouville vector field to be transverse to this connected component: we introduce the vector field

$$X = (q - M)\frac{\partial}{\partial q}.$$

This is a Liouville vector field for the canonical symplectic structure on $T^*\mathbb{R}^2$. However, what is not so clear is that this vector field is indeed transverse to Σ_c^E.

Proposition 1.13 (Proposition 5.1, [1])**.** *For $c < H(L_1)$, X intersects Σ_c^M transversally.*

The proof of this is that rather tedious computations are omitted here. This means that we are on an excellent basis to apply the previously mentioned results coming from contact topology. Indeed, we know that $(\Sigma_c^E, \iota_X\omega_{st})$ is a contact manifold, and that the Hamiltonian vector field is a reparametrization of the Reeb vector field associated to this contact form. However, as mentioned in Section 1.1, this is *not* a compact hypersurface due to the collision of the satellite with the moon.

However, the collision of the satellite with the moon is a 2-body collision. It has been known since Kepler that 2-body collision can be regularized. Intuitively, upon collision, this reguliarization makes the satellite bounces back from where it came from. Topologically, this regularization is a compactification of Σ_c^M (or Σ_c^E). This regularization is known as Moser's regularization. For more information on this, we refer the interested reader to the original paper [1] and to the book of Augustin Moreno [44].

Theorem 1.14 (Theorem A in [1])**.** *For $c < H(L_1)$ both regularized connected components $\widetilde{\Sigma}_c^M$ and $\widetilde{\Sigma}_c^E$ admit a compatible contact form λ.*

As it is known that for contact compact 3-dimensional manifolds, there is always a periodic Reeb vector field, it follows thus that there is always at least one periodic orbit of the satellite in the *regularized* level-set $\widetilde{\Sigma}_c^E$. For physical applications, one would like to know whether or not this periodic orbit is an orbit that goes through the collision set or is, in fact, an actual 'physical' periodic orbit.

Question 1.15. *Does this periodic orbit intersect the collision set?*

In these notes, we will propose a strategy to answer this question: namely, we will define symplectic structure with *singularities*. These singularities capture the collision set, or more generally, the sets one would like to avoid for physical applications. The price to pay is then to develop the necessary tools, similar to the tools in symplectic and contact topology, to this set-up and analyze the dynamics on these manifolds with symplectic structures that admit singularities.

As a disclaimer: this strategy is still relatively new, and a lot must be done to adapt the techniques. This disclaimer should serve as a motivation to join the research on b-symplectic manifolds.

To give an example of a b-symplectic structure (in fact, it will be b^3-symplectic), we will analyze another connected component of the level-set, namely the connected component where the satellite can escape to infinity.

1.4. **...and going to infinity.** To analyze the connected component of Σ_c where the satellite can escape to infinity, we use another regularization known as the *McGehee* regularization. This regularization is used to study the dynamical behaviour close to infinity. To describe it, we first introduce some further changes in coordinates.

As we are in the planar restricted three-body problem, the configuration space is given by $T^*\mathbb{R}^2$. Furthermore, in the *circular* RP3BP, we have a circular symmetry, and thus, we introduce polar coordinates given by

$$\begin{cases} q_1 = r\cos\alpha \\ q_2 = r\sin\alpha. \end{cases}$$

We change the momenta accordingly as follows to have a symplectic change of coordinate, that is, a diffeomorphism that preserves the symplectic form.

$$\begin{cases} p_1 = P_r\cos\alpha - P_\alpha\sin\alpha \\ p_1 = P_r\sin\alpha + P_\alpha\cos\alpha. \end{cases}$$

In this way, we obtain that $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ (**Exercise**). The McGehee change of coordinates consists of a change of coordinates that is *not* symplectic. It is given by the following change of coordinates

$$r := \frac{2}{x^2}, \quad x \in \mathbb{R}^+. \tag{3}$$

Geometrically, this change of coordinates exchanges the infinity with the origin. This change of coordinates is thus useful to study the dynamical properties when the satellite escapes to infinity.

However, this is not a symplectic change of coordinate, meaning that the symplectic form gets changed. A naive computation of what the canonical 'symplectic form' looks like in these new coordinates gives the following: first, observe that $dr = -\frac{4dx}{x^3}$, and thus, the obtained geometric structure is given by

$$\omega = -4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha, \tag{4}$$

and the domain is given by $\mathbb{R}^+ \times S^1 \times \mathbb{R}^2$. This form is still a symplectic form away from $\{x = 0\}$ - however, on the hypersurface $\{x = 0\}$, the form has a singularity.

To overcome this singularity, we do the following exercise:

Exercise: Compute that the 'associated'[1] Poisson structure by computing $\omega(df, dg)$ in the above coordinates. Check that this is indeed a Poisson structure (see Definition 2.1 for the definition of Poisson structure).

Continuing, we would like to develop a formal set-up to study the symplectic geometry of those forms with singularities. The strategy is, therefore, the following:

- Study manifolds with boundary
- study forms with singularities on hypersurfaces
- define what it means to be symplectic for these structures
- study the Hamiltonian dynamics of these structures.

2. b-SYMPLECTIC STRUCTURES

Before diving into symplectic structures with singularities, we dive into a more general setting of symplectic structures, namely the one of Poisson structures.

2.1. **Poisson structures.** In this subsection, we give the necessary information about Poisson structures. We recommend the references [12, 50] for more details on this.

Recall that given a symplectic structure, there is an associated bracket as in Definition 2.1 that equips the space of smooth function with a Lie bracket that satisfies Jacobi identity. A Poisson structure imitates this. We, therefore, obtain the following definition.

[1] We still don't t know that this is a Poisson structure - but this will be proved in the next pages.

Definition 2.1. *A Poisson structure on a manifold M is a bracket*

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$
$$(f,g) \mapsto \{f,g\}$$

that satisfies the following

(1) *it s skew-symmetric, i.e. $\{f,g\} = -\{g,f\}$,*
(2) *it is a derivation, i.e. $\{fg,h\} = f\{g,h\} + \{f,h\}g$,*
(3) *and it satisfies Jacobi identity, i.e. $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$.*

As already mentioned, given a symplectic manifold (M,ω), we can associated *a* Poisson structure defining the Poisson bracket by $\{f,g\} = \omega(X_f, X_g)$. However, Poisson structures are much more general, as the zero bracket trivially satisfies the definition.

An alternative definition of Poisson structures can be given using bi-vector fields.

Definition 2.2. *A multivector field is a section of the bundle $\Lambda^k TM$.*

Given a Poisson bracket, we can define a bi-vector field as follows. Let $f,g \in C^\infty(M)$ be two smooth functions, and the Poisson bi-vector field is determined by

$$\Pi(df,dg) := \{f,g\}.$$

For instance, the bi-vector field coming from the symplectic Poisson bracket (as in Equation 2) is given by

$$\Pi = \sum_{i=1}^n \left(\frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}\right).$$

Note that this looks very much like the standard Darboux symplectic form - just *dual* in some sense (this can be made formal!). Vice-versa, given a bi-vector field Π, we can define a bracket on the space of smooth functions as above. By the definition of the space of multi-vector fields, this bracket is skew-symmetric and is a derivation; the integrability condition, however, is not necessarily satisfied. We, therefore, need a way to express the integrability condition in the space of multivector fields. To do so, we need to define how to *derive* a multivector field concerning another one. This is a generalization of the Lie bracket for vector fields to the set-up of multivector fields.

Given vector fields $X_1,\dots,X_n, Y \in \mathfrak{X}(M)$, it is natural to define the following operation:

$$[X_1 \wedge \cdots \wedge X_n, Y] := \sum_{i=1}^n (-1)^{i+1} X_1 \wedge \cdots \wedge X_{i-1} \wedge \hat{X}_i \wedge X_i \wedge \cdots X_n \wedge [X_i, Y], \tag{5}$$

where $[X_i,Y]$ is just the usual Lie bracket, and the hat symbol denotes the absence of this term.

Theorem 2.3 (Schouten bracket)**.** *There exists a unique $\mathbb{R}$-linear extension of the Lie derivative L_X to the operation*

$$[\cdot,\cdot] : \mathfrak{X}^p(M) \times \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q}(M)$$

such that (5) holds.

Exercise: Prove this. Furthermore, one can check that the following properties hold.

Proposition 2.4. *Given X,Y,Z multivector fields of respective degree p,q,r, the following holds:*

(1) $[X,Y] = (-1)^{pq}[Y,X]$
(2) $[X, Y\wedge Z] = [X,Y]\wedge Z + (-1)^{pq+q} Y \wedge [X,Z]$
(3) $(-1)^{pr}[X,[Y,Z]] + (-1)^{pq}[Y,[Z,X]] + (-1)^{qr}[Z,[X,Y]] = 0$, *for $X \in \mathfrak{X}^p(M)$, $Y \in \mathfrak{X}^q(M)$, $Z \in \mathfrak{X}^r(M)$.*

Exercise: Prove this. The reader may read more about the Schouten-Bracket in [12]. Having to our disposal this bracket, one can now prove that the following integrability condition describes the Jacobi identity:

Proposition 2.5. *A bi-vector field* Π *defines a Poisson bracket if* $[\Pi, \Pi] = 0$*.*

Exercise: Prove this. A Poisson structure can thus be described equivalently by a bi-vector field that satisfies $[\Pi, \Pi] = 0$. Such a bi-vector field is called a *Poisson bi-vector field*. Surfaces equipped with a bi-vector field are always Poisson manifolds - for dimensional reasons.

Example 2.6. *Consider a surface* Σ *with a bi-vector field* Π*. As* $[\Pi, \Pi]$ *is a* 3*-vector field on a* 2*-dimensional manifold, it is automatically zero. Thus,* Π *is a Poisson bi-vector field.*

In this mini-course, a leading example is given by the following example:

Example 2.7. *Consider* $\mathbb{R}^{2n}$ *equipped with the bi-vector field* $\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$*.*
Exercise: *Check that this is a Poisson bi-vector field, i.e., that* $[\Pi, \Pi] = 0$*.*

Naively speaking, the *dual* of this Poisson vector field is given by $\frac{1}{x_1} dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i$. This is, of course, not well-defined on x_1. But this strongly resembles the geometric structure we encountered in Equation (4)! But before giving more information about these geometric structures, we turn attention to a central theorem theorem in Poisson geometry, that will be useful to describe an associated foliation to a Poisson structure.

It turns out that Poisson structures are suitable geometric structures for studying classical mechanics. This is mainly because Hamiltonian vector fields can easily be defined on Poisson manifolds.

Definition 2.8. *The Hamiltonian vector field associated with a smooth function* f *is given by* $X_f = \Pi(df, \cdot)$*.*

Alternatively, associated with a Poisson structure, we obtain a map $\Pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$, defined by $\beta(\Pi^\sharp(\alpha)) := \Pi(\alpha, \beta)$, where $\alpha, \beta \in \Omega^1(M)$. The Hamiltonian vector field is thus just the image of df under $\Pi^\sharp$.

The distribution given by all the Hamiltonian vector fields, that is, $\mathcal{D} := \Pi^\sharp\left(\Omega^1(M)\right)$, is called the *symplectic foliation*. This comes from proving that the distribution $\mathcal{D}$ is integrable. However, this has to be understood in a more general sense than in the sense of Frobenius theorem because the rank of this distribution is not constant. The rank of the distribution at a point p is the dimension of the image of Π at p and is always even-dimensional **Exercise: Check this** (but once more, to be clear: the rank can change). Furthermore, one can prove that the Poisson bi-vector field induces a symplectic structure on the leaves. This thus motivates the name of *symplectic* foliation, for more information see [12, Chapter 4]. Loosely speaking, one can thus think of a Poisson structure as a foliation of symplectic leaves of different ranks being glued together. A more precise description is given by the Weinstein splitting theorem.

Theorem 2.9 (Weinstein splitting theorem)**.** *Let* (M, Π) *be a Poisson manifold. Around a point* $x \in M$*, where* $\operatorname{rank} \Pi_x = 2k$*, there exists a coordinate chart* $(x_1, y_1, \cdots, x_k, y_k, z_1, \cdots, z_l)$*, where* $2k + l = \dim M$*, such that*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum \varphi_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

which $\varphi(0) = 0$*.*

We omit the proof of this fundamental theorem in Poisson geometry. Note that this theorem says that locally, the manifold M can be split into a product of a manifold N^{2n} and Z^r, with coordinates $(x_1, y_1, \cdots, x_k, y_k)$ on N and coordinates $(z_1, \cdots, z_l)$ where the manifold N is equipped with a symplectic structure. This is thus the formal statement that Π induces a symplectic foliation.

Two trivial examples on which we can 'see' Weinstein splitting theorem is

- any manifold with the zero Poisson structure: the leaves are 0-dimensional manifolds
- a symplectic manifold: there is only one left, that is the whole manifold. Note that Weinstein's splitting theorem is thus a generalization of the classical Darboux theorem!

The above examples are not very interesting (from a Poisson point of view, at least). Here, we present the first non-trivial examples and describe their associated symplectic foliation.

Example 2.10. *Consider the Poisson bi-vector field* $\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$ *on* $\mathbb{R}^{2n}$*. Away from the hypersurface* $Z = \{x_1 = 0\}$*, the bi-vector field satisfies that* $\Pi^n \neq 0$*, meaning that it defines a symplectic structure away from* $Z = \{x_1 = 0\}$*. Alternatively, the hypersurface* Z *can be defined by* $Z = \{\Pi^n = 0\}$*. The hypersurface* Z *is foliated by leaves* $\mathcal{L}_c = \{y_1 = c|\}$*. The restriction of the Poisson bi-vector field yields to those leaves yields* $\sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$*, which yields a symplectic structure on these yields. Resuming, the symplectic foliation given by the Poisson bi-vector field* Π *admits two leaves of maximal rank* $Z^{\pm} = \{\pm z > 0\}$ *and a family of codimension* 2 *symplectic leaves given by* $\{y_1 = c, x_1 = 0\}$*.*

The following example consists of a compact example that the reader can visualize easily.

Example 2.11. *Consider the* 2*-dimensional sphere* S^2*, equipped with cylindrical coordinates* (θ, h) *with the bi-vector field* $\Pi = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$*. As we have observed in Example 2.6, this is a Poisson bivector field for dimensional reasons. Away from the equator* $Z = \{h = 0\}$*, the bi-vector field defines a symplectic structure because here* $\Pi \neq 0$*. The equator* Z *is foliated by symplectic leaves given by* $\{\theta = c\}$*, which are just* 0*-dimensional symplectic manifolds. The symplectic foliation is thus given by two leaves of dimension* 2*,* $S^2_{\pm} = \{\pm h > 0\}$*, and by a family of codimension* 2 *'symplectic' leaves, which are just* 0*-dimensional symplectic points.*

The two last examples satisfy the following:

Definition 2.12. *A b-Poisson manifold is a Poisson manifold* (M, Π) *of dimension* $2n$ *that satisfies* $\Pi^n \pitchfork 0$*.*

This means that Π^n defines a section of $\bigwedge^{2n} TM$. The latter is a 1-dimensional bundle (because the dimension of the manifold M is $2n$). Thus, here, $\Pi^n \pitchfork 0$ means that the 1-dimensional bundle is transverse to the zero-section of the bundle $\bigwedge^{2n} TM$. By transversality, it thus follows that $Z := \{x \in M | \Pi^n_x = 0\}$ is a codimenison 1-submanifold. This hypersurface is called *critical set*, or *critical hypersurface*. The nomenclature will be clarified in the next subsection, but here is already a spoiler: b stands for *boundary*, because the critical set can be viewed as the boundary of a connected component of $M \setminus Z$.

One can apply the Weinstein splitting theorem (Theorem 2.9) for Poisson manifolds to the case of b-Poisson structures and show that any b-Poisson structure is given around a point $p \in Z$ as in Example 2.10. We thus obtain local normal forms for b-Poisson manifolds. This has been done in [25, Proposition 20].

As we saw, Poisson structures are very flexible: any manifold is Poisson and equipped with the zero Poisson structure. Symplectic structures (which form a particular class of

Poisson structures) are very far away and form a much more rigid behavior. For instance, only an orientable, even dimensional manifold admits a symplectic structure. Or even stronger topological conditions obstruct the existence of symplectic structures: a 2-form ω that is closed on a *closed* manifold defines a cohomology class. If this cohomology class is zero, that ω cannot be symplectic for topological reasons. **Exercise**: Prove this. In this sense, b-symplectic manifolds represent the next nicest family of Poisson manifold - after symplectic manifolds of course. Many natural questions arise from the very definition:

(1) How flexible are b-Poisson manifolds, i.e. when does an even dimensional manifold admit a b-Poisson structure?
(2) b-Poisson manifolds are not too far from symplectic structures: they are symplectic away from the critical hypersurface. Can symplectic techniques employed in this set-up?
(3) Related to the previous question: What about the known results for chasing periodic orbits of Hamiltonian X_H? Can those questions be handled in this setup?
(4) More precisely, given a hypersurface in a b-Poisson manifold that admits a transverse "Liouville" vector field, can one prove the existence of periodic Reeb orbits?
(5) As the first motivating example, as in Section 1, comes from celestial mechanics, can those techniques be used to prove the existence of periodic orbits in the 3BP, and additionally, 'localize' them, in the sense that they do not collide with the earth/moon?

To deal with these questions, one would like to view b-Poisson manifolds as a symplectic structure. A naive way to do this is to take the dual of these structures: for instance, the dual to the b-Poisson structure given in Example 2.10 yields

$$\frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i,$$

whereas the one dual to the b-Poisson of Example 2.11 yields

$$\frac{dh}{h} \wedge d\theta.$$

The 'problematic' terms in these expressions are $\frac{dx_1}{x_1}$, respectively $\frac{dh}{h}$. We will see in the next subsection that the singularities can be hidden in a vector bundle - not the tangent bundle, but the b-tangent bundle. We will then proceed to do symplectic geometry over this b-tangent bundle. But one thing after the other: we will define the b-tangent bundle.

2.2. **The b-tangent bundle.** In this section, we will detail the construction of the b-tangent bundle. The b-tangent bundle was initially defined by Melrose in his proof of the Atiyah-Index-Singer theorem for manifolds with boundary [35]. The b's that we will encounter thus stand all for **b**oundary.

Definition 2.13. *A b-manifold is a closed manifold M with a hypersurface Z, called critical hypersurface*

The 'problematic' terms mentioned at the end of Subsection 2.1 are the duals of vector fields that are tangent to the critical hypersurface. We would thus like to define a vector bundle whose sections are the vector fields tangent to Z. To construct this vector bundle, we will use the following theorem by Serre-Swan:

Theorem 2.14 (Serre-Swann theorem)**.** *Given a finitely generated $C^\infty(M)$-module $\mathcal{M}$, there exists a vector bundle whose sections equal $\mathcal{M}$.*

In our case, the module we consider is the set of vector fields that are tangent to Z.

Definition 2.15. *A b-vector field is a vector field tangent to Z. The set of vector fields that is tangent to Z is denoted by ${}^b\mathfrak{X}(M)$.*

Locally, the hypersurface Z is given by the zero-level set of a *locally* defined function $z : M \to \mathbb{R}$, i.e. $Z = \{z = 0\}$. Around a point $p \in Z$, we can complete this to a coordinates chart $(z, x_2, \ldots, x_n)$. In these coordinates, the b-vector fields, which are the vector fields that are tangent to Z, are spanned by

$$\langle z\frac{\partial}{\partial z}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\rangle.$$

Any vector field X tangent to Z can be written as

$$X = f_1 z\frac{\partial}{\partial z} + f_2\frac{\partial}{\partial x_2}\cdots + f_n\frac{\partial}{\partial x_n},$$

where $f_1, \ldots, f_n \in C^\infty(M)$. It follows that the set of b-vector fields is a $C^\infty(M)$-finite module, as every b-vector field can be written as a combination of the above generators. Serre-Swann theorem (Theorem 2.14) thus yields the existence of a vector bundle E whose sections are the b-vector fields. In what follows, we will denote this vector bundle ${}^bTM$ and call it the b-tangent bundle.

It follows by the definition that away from Z, the b-tangent bundle and the tangent bundle agree, i.e.

$$ {}^bT_pM = T_pM \quad \text{for } p \in M \setminus Z. \tag{6}$$

This ceases to remain true on the critical hypersurface Z. At a point $p \in Z$, this does not remain true anymore: the b-tangent bundle on Z can be described as

$$ {}^bT_pM = T_pZ \oplus \langle \left(z\frac{\partial}{\partial z}\right)_p \rangle. \tag{7}$$

Important observation: The rank of the b-vector bundle is of rank $\dim M = m$. At the point p, the vector field $z\frac{\partial}{\partial z}$ is a **non-vanishing** b-vector field, even though, after the inclusion of b-vector fields in the set of smooth vector fields, i.e. $i : {}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$, it is vanishing: $i(z\frac{\partial}{\partial z})|_p = 0$.

The above equations explain the local picture of the b-tangent bundle. Thus, understanding this vector bundle locally is relatively easy. However, as is often the case, the global picture makes these vector bundles interesting. Before giving a more intuitive picture, that allows to understand this vector bundle globally, we briefly mention more general way of looking at b-tangent bundle.

The b-tangent bundle is an example of a *Lie algebroid* over the tangent bundle. A Lie algebroid is a triple $(\mathcal{A}, [\cdot,\cdot], \rho)$, where $\mathcal{A}$ is a vector bundle, $[\cdot,\cdot]$ is a Lie bracket on the space of sections over $\mathcal{A}$ and $\rho : \mathcal{A} \to TM$ is a morphism, called *anchor map* such that the bracket and anchor map satisfy the Leibniz rule, meaning that $[X, fY] = f[X,Y] + \rho(X)(f)\cdot Y$, for $X, Y \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$. As the inclusion map $i : {}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$ is $C^\infty(M)$-linear, it comes from a vector bundle map

$$\rho : {}^bTM \to TM. \tag{8}$$

Furthermore, the Lie bracket of two tangent vector fields to Z is still a vector field tangent to Z. Thus, the Lie bracket on $\Gamma({}^bTM)$ is just the usual Lie bracket. **Exercise:** Check the

Leibniz formula. A deeper look into Lie algebroids can be found in [20]. From Z, ρ is the identity, as seen in Equation (7). On Z, however, the anchor map is no longer the identity:

$$\rho|_Z : {}^bTM|_Z \to TZ$$

admits a 1-dimension kernel that is spanned by $\langle z\frac{\partial}{\partial z}\rangle$. **Exercise:** Check that this does not depend on the choice of defining function. We have thus a 1-dimensional line bundle over Z that we denote by $\mathbb{L}_Z$. A non-vanishing section of $\mathbb{L}_Z$ is called *normal b-vector field*. The b-vector field $z\frac{\partial}{\partial z}$ is an example of such a vector field.

Another way of looking at the b-tangent bundle can be obtained as follows: let us assume that the hypersurface is globally defined, that is that there exists a *globally* defined function $z : M \to \mathbb{R}$ such that $z^{-1}(0) = Z$. This means that the manifold M can be split as $M = M^+ \cup M^- \cup Z$, where $M^\pm = \{\pm z > 0\}$. The vector field $\frac{\partial}{\partial z}$ is a transverse normal vector field, pointing from M^- towards M^+. The case of the b-tangent bundle is very much different: the vector field $z\frac{\partial}{\partial z}$ is transverse to the hypersurfaces $\{z = \pm\epsilon\}$ for $\epsilon > 0$ small enough, but pointing *away* from Z. This means that to construct the b-tangent bundle out of TM, we can glue the smooth tangent bundles around a connected component of Z in the following way. For $\{z > \epsilon\}$ we consider the vector field $\frac{\partial}{\partial z}$ and we glue it to the vector field $-\frac{\partial}{\partial z}$. Mathematically speaking, this can be described as follows:

The b-tangent bundle of M is the vector bundle obtained by gluing

$$TM|_{M\setminus M-} \to TM|_{M\setminus M_+}$$

by the constant diagonal map

$$\mathrm{Id} \oplus (-1) : Z \to \mathrm{GL}(TZ \oplus \langle\frac{\partial}{\partial z}\rangle).$$

Some examples are in order.

Example 2.16. *Consider S^1 with one marked point p, thus (S^1, p) is a b-manifold. The b-tangent bundle ${}^bTS^1$ is isomorphic to the Moebius strip.*
However, if we consider S^1 with two marked points p, q, the b-tangent bundle associated to the b-manifold $(S^1, p \cup q)$ is isomorphic to $TS^1 \cong S^1 \times \mathbb{R}$.

The above examples are easy to understand. For higher dimensional examples, the global understanding of the b-tangent bundle becomes trickier.

Proposition 2.17 (Brugués, [43])**.** *The b-tangent bundle associated to (S^2, S^1) is parallelizable.*

Thus, in the above case, the b-tangent bundle is not isomorphic to TS^2. To conclude this subsection, we mention the following:

Proposition 2.18 ([9])**.** *Let M^3 be a three-dimensional manifold with a separating hypersurface Z defined by the zero-level set of a global defining function f. Then the b-tangent bundle associated with (M, Z) is isomorphic to TM.*

2.3. ***b*-geometry.** Equipped with the tool of b-tangent bundle, we would like to do geometry over this vector bundle and, ultimately, talk about the dynamics of this vector bundle. To do so, the first step is to define differential forms; the second is to define an extension of the exterior derivative to this bundle. The first one will be straightforward: this follows from the usual definitions of vector bundles:

Definition 2.19. *The b-cotangent bundle is the dual of the b-tangent bundle and is denoted by ${}^bT^*M$. A b-differential form of degree k is a section of $\bigwedge^k {}^bT^*M$. The set of b-forms of degree k is denoted by ${}^b\Omega^k(M)$.*

To grasp what b-forms look like, we make use of our understanding of the b-tangent bundle. In fact, away from Z, we know by Equation (7) that the b-cotangent bundle is isomorphic to the b-tangent bundle, and therefore

$$^b\Omega^k_p(M) = \Omega^k_p(M), \quad p \in M \setminus Z.$$

On a point $p \in Z$, the map $\rho_p : {}^bT_pM \to T_pZ$ as in Equation (8) is surjective, meaning that the dual $\rho_p^* : T_p^*Z \to {}^bT_p^*M$ is injective and the image of ρ_p^* is given by $\{\alpha \in {}^bT_p^*M | \alpha(z\frac{\partial}{\partial z}) = 0\}$.

Away from Z, the one-form $\frac{dz}{z}$ defines a well-defined 1-form. The evaluation of this one-form on b-vector fields can still be *smoothly* extended over Z, and the 1-form can be interpreted as a *smooth* section of ${}^bT^*M$. Moreover, $\frac{dz}{z}(z\frac{\partial}{\partial z}) = 1$, and thus $\frac{dz}{z} \notin \mathrm{Im}\rho_p^*$. We get thus a splitting at $p \in Z$

$$^bT_p^*M = T_p^*Z \oplus \langle \left(\frac{dz}{z}\right)_p \rangle \tag{9}$$

A b-form of degree one, that is a section over ${}^bT^*M$ can thus be written as $\alpha = f\frac{dz}{z} + \beta$, where f is a smooth function over M and β is a smooth 1-form.

In the above arguments, we chose a preferred section of the line bundle $\mathbb{L}_Z$. However, one can choose that the above constructions are independent of the choice of a non-vanishing section of $\mathbb{L}_Z$.

A b-form of degree $\omega \in {}^b\Omega^1(M)$ can thus be decomposed as $\omega = f\frac{dz}{z} + \alpha$, where $\alpha \in \Omega^1(M)$ and $f \in C^\infty(M)$. These arguments generalize to higher wedges of ${}^bT^*M$, and we thus obtain that a b-form of degree k can be written as

$$\omega = \alpha \wedge \frac{dz}{z} + \beta, \quad \alpha \in \Omega^{k-1}(M), \beta \in \Omega^k(M). \tag{10}$$

This decomposition lets us define an exterior derivative on the space ${}^b\Omega^k(M)$.

Definition 2.20. *The exterior derivative is defined on* ${}^b\Omega^k(M)$ *by*

$$d\omega = d(\alpha \wedge \frac{dz}{z} + \beta) := d\alpha \wedge \frac{dz}{z} + d\beta$$

This turns ${}^b\Omega^k(M)$ in a differential graded algebra. It follows from the definition that the exterior derivative on ${}^b\Omega^k(M)$ is an extension of the usual exterior derivative. Note that a Lie algebroid comes already equipped with an exterior derivative, and one can check that the exterior derivative we define here is, the one, coming from the general Lie algebroid construction.

Equipped with the b-differential forms and the exterior derivative, we are ready to start to symplectic geometry over ${}^bTM$.

2.4. b-symplectic geometry.

Definition 2.21. *An even dimension b-manifold* (M^{2n}, Z) *with a b-form of degree* 2, $\omega \in {}^b\Omega^2(M)$ *is a b-symplectic manifold if*

- ω *is non-degenerate as a form over* ${}^bTM$,
- ω *is closed for the differential defined in Definition 2.20.*

We continue with two examples:

Example 2.22. *Consider the b-form of degree 2 given by $\omega = \frac{dx_1}{x_1} \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. From the definition of the exterior derivative, $d\omega = 0$. Away from the hypersurface $Z = \{x_1 = 0\}$, the b-form satisfies that $\omega^n \neq 0$, meaning that it defines a symplectic structure away from $Z = \{x_1 = 0\}$. A direction computation yields that*

$$\omega^n = \frac{dx_1}{x_1} \wedge dy_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots dx_n \wedge dy_n,$$

and therefore ω is non-degenerate as a b-form.

The following example consists of a compact example.

Example 2.23. *Consider the 2-dimensional sphere S^2, equipped with cylindrical coordinates (θ, h) with the b-form of degree 2 given by $\omega = \frac{dh}{h} \wedge d\theta$. For dimensional reasons, this is a closed b-form. Furthermore, this b-form is non-vanishing as a b-form, and therefore it defines a b-symplectic structure on S^2.*

The last two examples are very close, almost identical to Example 2.10 and Example 2.11, examples of b-Poisson structures. This is no coincidence.

Proposition 2.24 (Guillemin–Miranda–Pires, Proposition 20 in [25])**.** *A b-form is b-symplectic if its dual bi-vector field is b-Poisson.*

The proof is mainly using Weinstein's splitting theorem.

In contrast to the two examples presented from the b-symplectic point of view, Example 2.10 and Example 2.11 present a codimension 1 foliation on the critical set. This codimension 1-foliation can be seen from the point of b-symplectic geometry, which we will see in a second.

Proposition 2.25 (Proposition 10 in [25])**.** *Let (M^{2n}, Z) be a b-manifold with b-symplectic form ω. By Equation 10, we can decompose $\omega = \alpha \wedge \frac{dz}{z} + \beta$. Denote by $i : Z \hookrightarrow M$ the inclusion of the critical hypersurface to M. Then the forms $\tilde{\alpha} = i^*\alpha \in \Omega^1(Z)$ and $\tilde{\beta} := i^*\beta \in \Omega^2(Z)$ are closed forms. Furthermore*

1. *(1) $\tilde{\alpha}$ does not depend on the choice of defining function for Z and is nowhere vanishing. It thus defines a codimension 1 foliation on Z.*
2. *(2) Let L be a leaf of the foliation defined by $\ker \tilde{\alpha}$. Then the form $\tilde{\beta}|_L$ defines a symplectic form on the leaf L.*
3. *(3) In the decomposition of Equation (10), $\alpha \wedge \beta \wedge \frac{dz}{z}$ is nowhere vanishing.*

This proposition gives a neat description of the critical set Z: the b-symplectic structure induces a regular codimension 1 symplectic foliation. An $(2n-1)$ dimensional manifold equipped with a closed 1-form and a closed 2-form β such that $\alpha \wedge \beta^{n-1}$ is a volume form is called a cosymplectic structure. Thus, the induced structure on the critical set is a *cosymplectic* structure.

Codimension 1-symplectic foliations are rigid: in particular, Proposition 2.25 shows that there are some topological conditions on a given manifold for it to admit a b-symplectic structure.

For instance, there is no b-symplectic structure on S^4. **Exercise**: Show this.

As mentioned earlier, using the Weinstein splitting theorem, one can show that locally around a point $p \in Z$, a b-Poisson manifold is locally given as in Example 2.10. The correspondence between b-Poisson and b-symplectic manifolds proves a local standard form theorem for b-symplectic manifolds as well. However, using Moser's path method, one can prove the same using the same approach as for symplectic manifolds.

Proposition 2.26 (b-Darboux theorem, Theorem 37 in [25]). *Let (M, Z) be a b-symplectic manifold and $p \in Z$. Then there exists local coordinates around p such that*

$$\omega = \frac{dz}{z} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i. \tag{11}$$

This means that b-symplectic manifolds are locally well understood. But even the semi-local understanding of Z is well-understood. Recall that it follows from Proposition 2.25 that a codimension 1 symplectic foliation exists on Z. If one of the symplectic leaves is compact, all of them are symplectomorphic, and Z is topologically a mapping torus.

Proposition 2.27. *Let (M, ω) be a b-symplectic manifold. Assume that Z is compact and a compact leave of the symplectic foliation L exists. Then $Z \cong L \times [0, T]/\{(x, 0) = (\phi(x), T)\}$, for a symplectomorphism ϕ and $T > 0$.*

The above isomorphism is called a *mapping torus*. This proposition follows from the existence of a transverse vector field to the symplectic foliation on Z. In the above statement, ϕ is the flow of this vector field and T the *return time*. This vector field is the *modular vector field* in Poisson geometry. First, let us give a general definition of an oriented Poisson manifold.

Definition 2.28 (Proposition 49 in [25]). *Let (M^m, Π) a Poisson manifold with a volume form $\Omega \in \Omega^m(M)$. The modular vector field is defined as the derivation*

$$v^{\Omega}_{\mathrm{mod}} : f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

In local Darboux coordinates as in Proposition 2.26, the modular vector field is given by $v^{\Omega}_{\mathrm{mod}} = \frac{\partial}{\partial y_1}$, and therefore it is tangent to Z, but transverse to the leaves. To be precise, we need to specify the volume form Ω, which is given here by $\Omega = dz \wedge dy_1 \wedge dx_2 \wedge dy_2 \wedge \cdots dx_n \wedge dy_n$. However, the choice of volume form is irrelevant: **Exercise** two different volume forms lead to the same modular vector field, up to addition of a Hamiltonian vector field. The Hamiltonian vector field is by definition tangent to the foliation, and therefore, any modular vector field is tangent to Z but transverse to the symplectic foliation. Furthermore, the flow of this vector field preserves the symplectic structure **Exercise** and is, therefore, a symplectic flow. This shows that all the leaves are symplectomorphic as there is always a fixed point of the symplectomorphism after a finite time $T > 0$, Proposition 2.27 follows.

We will continue investigating a further generalization of b-symplectic structures.

2.5. **b^m-symplectic structures.** The structure of b-symplectic structures is based on studying the symplectic geometry of the b-tangent bundle. This bundle is constructed so that its sections are tangent to the given hypersurface Z. From the Poisson point of view, this thus yields a transversality condition in the definition of the maximum wedge of the Poisson bi-vector field. The tangency of the vector fields can, of course, be easily generalized by asking that the vector fields are tangent *to order k*. This has first been studied in [46].

Definition 2.29. *A* b^m-vector field *is a vector field v on a b-manifold (M, Z)* tangent *of order m at Z.*

One would, therefore, like to define the b^m-tangent bundle similarly as one defines the b-tangent bundle—however, the order of vanishing *depends* on the choice of defining function.

Example 2.30. *Consider the b-manifold* $(\mathbb{R}^2, Z = \{y = 0\})$*. The critical set* Z *is defined by the defining function* $f_1 = y$ *and by the defining function* $f_2 = e^x y$*. The vector field* $X = \frac{\partial}{\partial x}$ *satisfies that* $\mathcal{L}_X f_1 = 0$ *and* $\mathcal{L}_X f_2 = e^x y$*. This means that the order of tangency is not well-defined.*

Definition 2.29 is therefore not well-defined and has to be replaced by a slightly more subtle definition, as one has to fix the data of a $(k-1)$-jet of Z. For the definition to be well-defined, we must ask that the defining function be an element of this $(k-1)$-jet bundle. We will not detail this; the reader is referred to [46, Definition 2.11].

Given this definition, we obtain the b^m-tangent bundle. One can prove that the results of the Subsection 2.4 generalize to this set-up: out of the b^m-tangent bundle, one can produce b^m-symplectic forms, and loosely speaking, Subsection 2.4 can be adapted *mutatis mutandis* by replacing b, by b^m (up to the additional data of $(k-1)$-jet bundle, of course).

2.6. **Relation to smooth symplectic structures.** To what extent are b^m-symplectic structures and smooth symplectic structures related? Of course, away from the critical set, a b^m-symplectic structure is symplectic. The question does, therefore, ask for global considerations of the manifold. An approach to tackle this question is to change the singular term $\frac{dz}{z}$ to a smooth term and thus 'desingularize' the b^m-symplectic structure. This is the content of [27] and [10]. However, this strongly depends on the parity of the b^m-symplectic structure: b^{2k}-symplectic structures can be desingularized to symplectic structures, while b^{2k+1}-symplectic structures can be desingularized to so called *folded* symplectic structures.

Theorem 2.31 ([**?**, Theorem 3.1 and 5.1])**.** *Let* (M, Z, ω) *be a* b^m*-symplectic manifold.*

(1) If $m = 2k$*, then there exists a family of* ***symplectic*** *forms* ω_ε *which coincide with the* b^{2k}*-symplectic form* ω *outside an* ε*-neighborhood of* Z*.*
(2) If $m = 2k+1$*, then there exists a family of folded symplectic forms* ω_ε *which coincides with the* b^{2k+1}*-symplectic form* ω *outside an* ε*-neighbourhood of* Z*.*

Here *folded* symplectic means that there is a non-degenerate 2-form such that $\omega^n \pitchfork 0$. A direct corollary of this theorem is that if a manifold admits a b^{2m}-symplectic structure, then it admits a symplectic structure.

2.7. **A guide through b-symplectic literature.** The first instance of these geometric structures can be found in Melrose's book [35], where he studied the Atiyah–Patodi–Singer index theorem on manifolds with *b*oundaries, followed by the work of Tsygan–Nest [47] on formal deformations.

The study of these geometric structures was picked up again by Radko [45], the first instance of b-symplectic structures - even though the name b-symplectic does not appear there. The reason for this is that in this paper, these structures were dealt with more from the point of view of Poisson geometry. The first ones to study these structures using Melrose's b-tangent bundle are Eva Miranda, Victor Guillemin, and Ana Rita Pires in [25]. Many of their results are based on their previous paper [24]. The present lecture notes are based on their description, and many of the results can be found here.

Almost simultaneously, the here described object was studied under the name of *log*-symplectic structures in [23].

A practical guide through the literature of b-symplectic geometry can be found in [4] - even though the literature has expanded since then!

Their description gave rise to a series of new developments. Here is an overview of what has been studied; however, this list is far from exhaustive and will probably (and hopefully!) continue to grow in the future.

- The obstructions to the existence were studied in [32], and later further developed in [10] and [31]. More on the topology, particular applications of the h-principle can be found in [19].
- Hamiltonian actions were considered in [26], where a Delzant-type theorem for toric b-symplectic manifolds was proved.
- KAM and action-angle coordinates have been studied in [29].
- Higher order singularities were initiated in [46], and continued in [3].
- Marsden–Weinstein reduction theory on b^m-symplectic manifolds is investigated in [33].
- The relation to smooth symplectic structures was studied in [27].
- The geometric quantization was studied in [28, 5], see also [37] for lecture notes on this.
- The symplectic geometry of more general Lie algebroids has been initiated in [42].
- For more on the deformation of Lagrangian submanifolds, see [22], and for Lagrangian Floer theory [30].
- The relation with dissipative systems is described in [11].
- Floer theory in b-symplectic manifolds was initiated in [7].
- For Gauge theories in b-symplectic manifolds and their generalizations, see [38].

Having a good understanding of b-symplectic manifolds, we will now verge into the study of its odd-dimensional sibling.

3. Reeb dynamics on b-contact manifolds

3.1. b-contact manifolds.

Having gone through the definition of b-symplectic, the following definition should not be a surprise:

Definition 3.1. *Given an odd dimensional manifold (M^{2n+1}, Z), a b-form of degree 1, $\alpha \in {}^b\Omega^1(M)$ is a b-contact form if $\alpha \wedge (d\alpha)^n \neq 0$.*

We start with a couple of examples.

Example 3.2. *Consider the b-manifold $(\mathbb{R}^3, Z = \{z = 0\})$.*

- *The b-form $\alpha_1 = \frac{dz}{z} + x dy$ is a b-contact form.*
- *Another example of a b-contact form is given by $\alpha_2 = dx + y\frac{dz}{z}$.*

The codimension 1 distribution $\xi \subset {}^bTM$ given by $\xi = \ker \alpha$ is called the b-contact structure.

Recall that when viewed as a b-vector field, the vector field $z\frac{\partial}{\partial z}$ is a non-vanishing vector field. Therefore:

> **Important observation:** ξ is a codimension 1 regular distribution in ${}^bTM$. At the point p, we can look at the inclusion of ξ under the map ρ as defined in Equation (8), and $\rho(\xi)$ is a distribution of TM.

The b-contact distribution ξ, viewed as a smooth distribution, i.e., $\rho(\xi)$, is a distribution that consists of a distribution that is tangent to Z. Away from Z, we have an honest codimension 1-plane distribution (given by a contact structure). On Z, however, the rank of the smooth distribution $\rho(\xi)$ may drop. **Exercise:** Check this.

We include here one more example. displace?

Example 3.3. *Consider the torus $\mathbb{T}^2$ as a b-manifold where the boundary component if given by two disjoint copies of S^1. The unit cotangent bundle $S^*\mathbb{T}^2$, diffeomorphic to the 3-torus $\mathbb{T}^3$ is a b-contact manifold with b-contact form given by $\alpha = \sin\phi \frac{dx}{\sin(x)} + \cos\phi dy$, where ϕ is the coordinate on the fiber and (x, y) the coordinates on $\mathbb{T}^2$.*

Associated to a b-contact form α, we can define a unique b-vector field, the *b-Reeb vector field*. We omit the 'b' whenever it is clear that we talk about a b-contact form and not a smooth contact form.

Definition 3.4. *The Reeb vector field associated to a b-contact form α is a b-vector field, defined by*

$$\begin{cases} \iota_{R_\alpha} d\alpha = 0 \\ \iota_{R_\alpha} \alpha = 1. \end{cases}$$

By definition, the b-Reeb vector field is a b-vector field. The computation of the b-Reeb vector field of the b-contact forms in Example 3.2 yields

- $R_{\alpha_1} = z\frac{\partial}{\partial z}$
- $R_{\alpha_2} = \frac{\partial}{\partial x}$.

Under the inclusion of b-vector fields in the set of smooth vector field $i : {}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$, the above example thus shows that the b-vector field looked at as a smooth vector field, or more precisely, the image under the inclusion map i as above, is vanishing when restricted to Z.

Using the flow of the vector field R_α, we can prove a local normal form, similar to the Darboux theorem for contact forms (Theorem 1.9). Loosely speaking, the Reeb vector field together with the rank of the smooth distribution $\rho(\ker\alpha)$ determines the local normal form of the b-contact form. For simplicity, we include the statement only in dimension 3:

Theorem 3.5 (b-Darboux theorem, Theorem 5.4 in [40])**.** *Let α be a b-contact form inducing a b-contact structure ξ on a b-manifold (M^3, Z) and $p \in Z$. We can find a local chart $(\mathcal{U}, z, x, y)$ centered at p such that on $\mathcal{U}$ the hypersurface Z is locally defined by $\{z = 0\}$ and*

(1) if $i(R_p) \neq 0$, where $i : {}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$,

(a) and ξ_p is singular, then

$$\alpha|_{\mathcal{U}} = dx + y\frac{dz}{z},$$

(b) and ξ_p is regular, then

$$\alpha|_{\mathcal{U}} = dx + y\frac{dz}{z} + \frac{dz}{z}$$

(2) if $i(R_p) = 0$, then $\tilde{\alpha} = f\alpha$ for $f(p) \neq 0$, where

$$\tilde{\alpha}_p = \frac{dz}{z} + xdy.$$

For higher dimensional b-contact manifold, say of dimension $(2n + 1)$, the same holds by adding 'regular terms' given by $\sum_{i=2}^n x_i dy_i$ to the above expressions. This theorem can be helpful when doing local computations.

We end this subsection with two more constructions that explain the relation between b-contact and b-symplectic manifolds.

Given a b-contact manifold (M, α), we obtain a b-symplectic structure by going to the symplectization: it follows from the definitions that $(M \times \mathbb{R}, Z \times \mathbb{R})$ is a b-manifold and the b-form of degree 2 given by $\omega = d\left(e^t\alpha\right)$ is a b-symplectic form.

Vice-versa, having a b-symplectic manifold (M, ω) with a given hypersurface Σ transverse to Z and a b-vector field X that satisfies

- $d\iota_X\omega = \omega$,
- X is transverse to Σ,

then $(\Sigma, \iota_X\omega)$ is a b-contact manifold, with critical set given by $Z \cap \Sigma$. By analogy to the smooth symplectic set-up, a vector field satisfying the above conditions is called *Liouville vector field*. **Exercise:** Proof this.

An example is given as follows:

Example 3.6. *Consider the unit sphere S^3 in the standard b-symplectic manifold $(\mathbb{R}^4, \frac{dz}{z} \wedge dy_1 + dx_2 \wedge dy_2)$. The vector field $X = \frac{1}{2}z\frac{\partial}{\partial z} + y_1\frac{\partial}{\partial y_1} + r\frac{\partial}{\partial r}$, where $r^2 = x_2^2 + y_2^2$ is a Liouville vector field.* ***Exercise:*** *This vector field is transverse to the S^3 and therefore $(S^3, \iota_X\omega =: \alpha)$ is a b-contact manifold. The b-contact form is explicitly given by*

$$\alpha = \frac{1}{2}\left(dy_1 - 2y_1\frac{dz}{z} + x_2dy_2 - y_2dx_2\right).$$

The above example will be important when studying the global behavior of the b-Reeb vector field. Before doing so, we will look deeper at the critical set of a b-contact manifold.

3.2. **Jacobi structures.** As we observed in Subsection 2.1, the symplectic manifolds can be seen as a particular case of Poisson manifolds, and the associated Poisson bi-vector field is non-degenerate. In between lie b-symplectic manifolds, where the maximum wedge of the bi-vector field is asked to be transverse to the zero-section. This description allowed us to use Poisson geometry tools to describe the critical set's symplectic foliation. When wanting to do a similar construction for b-contact, one runs into the following problem.

Contact manifolds do not fit into this picture: they are *not* an example of Poisson manifolds. However, contact manifolds are a particular case of the following geometry.

Definition 3.7. *A Jacobi structure on a manifold M is a couple (Λ, R) of a bi-vector field $\Lambda \in \mathfrak{X}^2(M)$ and a vector field $R \in \mathfrak{X}(M)$ that satisfy the following equations:*

(1) $[\Lambda, \Lambda] = 2\Lambda \wedge R$,
(2) $\mathcal{L}_R\Lambda = 0$. *The couple (M, Λ, R) is called a Jacobi structure.*

A good reference for more information about Jacobi structure is [49]. Here, we will only deal with the necessary basics to describe the geometry of the critical set. This definition has a few equivalent descriptions. Similar to the description of Poisson manifolds, a Jacobi structure can be described in terms of a bracket on the space of smooth function $C^\infty(M)$, that is skew-symmetric, satisfies Jacobi identity, and is a *local derivation*, meaning that it is a bi-linear, bi-differential operator. This bracket can be described as

$$\{f, g\} := \Lambda(df, dg) + f(R(g)) - g(R(f)).$$

From this definition, it should be clear that Poisson manifolds are a particular type of Jacobi manifolds, namely when the bracket is not only a local derivation but satisfies Leibniz rule, or equivalently speaking, that R in Definition 3.7 is zero.

Another example of Jacobi structure are - as announced just above - contact manifolds.

Example 3.8. *Let (M^{2n+1}, α) be a contact manifold. Given a smooth function $f \in C^\infty(M)$, the* contact Hamiltonian vector field *X_f is defined by the equations*

$$\begin{cases} \iota_{X_f}d\alpha = -df + R_\alpha(f)\alpha \\ \iota_{X_f}\alpha = f. \end{cases}$$

We can define the following bi-vector field $\Lambda(df, dg) = d\alpha(X_f, X_g)$ and $R = R_\alpha$. ***Exercise:*** *Then (M, Λ, R) is a Jacobi structure. Furthermore, $\Lambda^n \wedge R \neq \{0\}$.*

As in the case of Poisson manifolds, associated with a Jacobi manifold, a singular foliation is generated by the Hamiltonian vector fields. The Hamiltonian vector field associated with f is defined by $X_f = \Lambda(df, \cdot) + fR$. The distribution $\mathcal{D}(M) = \{X_f | f \in C^\infty(M)\}$ can be shown to be integrable (**Exercise, see** [49]). In contrast to the foliation in Poisson geometry, the leaves of this foliation are even dimensional if $R \in \mathrm{Im}\Lambda^\sharp$, but odd dimensional when $R \notin \mathrm{Im}\Lambda^\sharp$. The Jacobi structure induces the following structure on the leaf:

(1) If the leaf is odd-dimensional, then the induced structure is a contact structure
(2) If the leaf is even-dimensional, then the induced structure is a *locally conformally symplectic* structure.

Here *locally conformally symplectic*, short lcs, means that locally, it is a symplectic structure up to the multiplication of a locally defined function.

Contact manifolds are a particular case of Jacobi manifolds and the *non-degenerate* odd-dimensional Jacobi manifolds. In this sense, the following definition and proposition should not be a surprise.

Definition 3.9. *An odd dimensional Jacobi manifold (M^{2n+1}, Λ, R) is b-Jacobi if $\Lambda^n \wedge R \pitchfork \{0\}$.*

Proposition 3.10. *There is a one-to-one correspondence between b-Jacobi and b-contact manifolds.*

As a result of this, we understand the associated foliation on the critical set. By the transversality condition of b-Jacobi structures, the leaves on the critical set can either be of dimension $2n$ (and thus have an induced locally conformally symplectic structure) or of dimension $(2n-1)$ and have an induced contact structure on this leaf.

Having a good understanding of the geometry of b-contact manifolds, we dive now into the central theme of this mini-course: studying the dynamics of the b-Reeb vector field.

3.3. **The Singular Weinstein conjecture.** In this section, we study the b-Reeb vector field dynamics. More precisely, given the Reeb vector field associated with a b-contact form α as in Equation 3.4, we regard R_α as a smooth vector field, meaning we take into consideration $i(R_\alpha)$ where $i : {}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$. In particular, we would like to know if $i(R_\alpha)$ admits periodic orbits. However, we will omit i in the notation whenever we discuss the dynamics of the b-Reeb vector field.

We start by comping the Reeb vector field in a couple of examples.

Example 3.11. *Consider $(\mathbb{T}^3, \sin x \frac{d\phi}{\sin\phi} + \cos x dy)$. The Reeb vector field is given by $R_\alpha = \sin x \sin\phi \frac{\partial}{\partial\phi} + \cos x \frac{\partial}{\partial y}$. The critical set is given by two disjoint copies of the 2-torus $\mathbb{T}^2$, and the Reeb flow restricted to it is given by $\cos x \frac{\partial}{\partial y}$. As in the last example, the critical set Z is given by periodic orbits (except when $\cos x = 0$, where the Reeb vector field is singular).*

Example 3.12. *Consider the b-contact 3-sphere (S^3, α) as in Example 3.6. The associated Reeb b-vector field can be computed to be* ***Exercise***

$$R_\alpha = \frac{2}{1+y_1^2}(-y_1 z\partial_z + z^2\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2}). \tag{12}$$

Up to the conformal factor, the vector field is given by

$$-y_1 z\partial_z + z^2\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2} \tag{13}$$

and therefore defines the same orbits as the vector field R_α. In cylindrical coordinates (h,θ) on $Z = S^2$, this vector field is the Hamiltonian vector field for the Hamiltonian function $H = h$ given by the symplectic form $dh \wedge d\theta$.

Remark: We do not know if there exists a b-contact form on $\mathbb{S}^3$ which produces the vector field of Equation (13) as a Reeb vector field (i.e. R_α, but without the conformal factor).

Observations: In the above 3-dimensional examples, we observe that
- there a points on Z where the Reeb vector field vanishes;
- The critical set has infinitely many periodic orbits.

We will see that we can turn this observation, in fact, into a proposition.

More precisely, the following will allow us to thoroughly understand the dynamics of the critical set in dimension 3.

Proposition 3.13 (Theorem 5.7 in [40])**.** *Let $(M, \alpha = u\frac{dz}{z} + \beta)$ be a b-contact manifold of dimension 3. Then the restriction to Z of the 2-form*

$$\Theta = ud\beta + \beta \wedge du \tag{14}$$

is symplectic and the Reeb vector field is Hamiltonian with respect to Θ with Hamiltonian function u, i.e. $\iota_{R_\alpha}\Theta = du$.

Proof. In the decomposition, α is given by $\alpha = u\frac{dz}{z} + \beta$. We compute

$$\begin{aligned}\alpha \wedge d\alpha &= \left(u\frac{dz}{z} + \beta\right) \wedge \left(du \wedge \frac{dz}{z} + d\beta + \frac{\partial \beta}{\partial z} \wedge dz\right) \\ &= \left(ud\beta + \beta \wedge du + z\beta \wedge \frac{\partial\beta}{\partial z}\right) \wedge \frac{dz}{z} \neq 0.\end{aligned}$$

When restricting to Z, we obtain that the 2-form $\Theta := ud\beta + \beta \wedge du$ on Z has to be non-vanishing, meaning that it is a symplectic form. In the same decomposition, let us write the Reeb vector field as $R_\alpha = g \cdot z\frac{\partial}{\partial z} + X$, where $g \in C^\infty(M)$ and $X \in \mathfrak{X}(Z)$. The Reeb vector field has to satisfy the equations as in Definition 3.4, and we therefore obtain the following equations:

$$\begin{aligned} g \cdot u + \beta(X) &= 1, \\ -gdu + \iota_X d\beta &= 0, \\ \iota_X du &= 0. \end{aligned}$$

A straightforward computation using those equations yields that $\iota_X\Theta = du$, hence the restriction of R_α to Z is the Hamiltonian vector field for the function $-u$. $\square$

Note that this proposition breaks down in higher dimensions: one obtains a volume form on the even dimensional hypersurface - and of course in dimension 2, volume form and symplectic form coincides.

In the compact case, we obtain a first corollary, proving that the second observation we remarked higher up is true.

Corollary 3.14. *Let (M, α) be a 3-dimensional compact b-contact manifold. Then, there are at least two points where the Reeb vector field vanishes.*

The 2-form Θ of Equation 14 is closed in dimension 3, and by Stokes theorem, if Z is closed, then Θ cannot be exact. Thus, the function $u|_Z$ cannot be constant, as otherwise, the form Θ would be written as $ud\beta$, which is exact if $u|_Z$ is constant. We will prove that infinitely many periodic orbits exist around the function's critical points u.

Proposition 3.15 (Proposition 2.6 in [39]). *Let $(M, \alpha = u\frac{dz}{z} + \beta)$ be a 3-dimensional b-contact manifold with a closed critical hypersurface Z, where $u \in C^\infty(M)$ and $\beta \in \Omega^1(M)$ as before. Then the function $u|_Z$ is non-constant. Furthermore, there exists infinitely many periodic Reeb orbits on Z.*

Note that the critical hypersurface Z is closed if a global function defines Z and the ambient manifold M is compact.

Proof. We have already checked and proved that the function u is non-constant. By Proposition 3.13, we know that the Reeb vector field is the Hamiltonian vector field on Z for the function $-u$. Let $p \in Z$ be a point such that $du_p \neq 0$ (which exists because u is non-constant). As the preimage of a closed topological set is closed and a closed set of a compact manifold is compact, circles give the level sets, and the Reeb vector field, contained in the level-set, is non-vanishing in view of

$$\iota_{R_\alpha}(ud\beta + \beta \wedge du) = du.$$

Hence, the Reeb vector field is periodic on $u^{-1}(p)$. $\square$

Short: in three dimensions and when Z is compact, there always exist zeros of the vector field and infinitely many periodic Reeb orbits on the critical set around the zeros of the vector field. But what about the periodic orbits away from Z? We will return to the previous examples and analyze the dynamics away from Z to answer this question.

Example 3.16. *The Reeb vector field on the 3-torus equipped with the b-contact form as in Example 3.11 is given by $R_\alpha = \sin x \sin\phi \frac{\partial}{\partial\phi} + \cos x \frac{\partial}{\partial y}$. Applying Proposition 3.13, we find that the Reeb vector field on Z is described by the Hamiltonian vector field given by the Hamiltonian function $H(x, y) = H(x) = -\sin x$ associated to the area form $dx \wedge dy$. To analyze the flow of R_α away from Z, notice that R_α preserves H. Hence the integral curve of R_α through a point (x_0, y_0, ϕ_0) satisfies*

$$\dot{\phi}(t) = H(x, y)\sin\phi(t) = H(x_0, y_0)\sin\phi(t).$$

We can integrate this differential equation: the explicit solutions are given by

$$\phi(t) = 2\cot^{-1}\left(\exp\left(c - H(x_0, y_0)t\right)\right),$$

where c is the constant such that $\phi(0) = \phi_0$. Thus, no periodic orbits away from Z exist for this vector field. However, we can observe another dynamical phenomenon.

Taking as initial condition $\{x_0 = \pm\frac{\pi}{2}\}$, the differential of H is zero at (x_0, y_0), and therefore this integral curve satisfies thus that $\lim_{t\to\pm\infty}\phi(t)$ is at a zero of the restriction of R_α in Z.

In the above example, we have thus an orbit that comes 'out' of the zero of R_α on Z and limits to one of those zeros of R_α on Z. We call such an orbit *singular periodic orbit*.

Definition 3.17. *Let (M, Z, α) be a b-contact manifold. A singular periodic orbit is an integral curve $\gamma : \mathbb{R} \to M \setminus Z$ of the Reeb b-vector field such that $\lim_{t\to\pm\infty}\gamma(t) = p_\pm \in Z$, where $p\pm$ is a zero of the Reeb vector field on Z, i.e.$R(p_\pm) = 0$.*

We continue to analyze the dynamics of the Reeb vector field on S^3.

Example 3.18. *The Reeb vector field on S^3 equipped with the b-contact form as in Example 3.12 is given by $R_\alpha = \frac{2}{1+y_1^2}(-y_1 z\partial_z + z^2\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2})$. Once again, this vector field cannot have periodic orbits away from Z, since the flow satisfies the equation $y_1(t) = z^2(t)$, which is strictly positive away from Z. The flow can thus not be periodic. However, in what follows we will understand the flow away from Z.*

We can use the stereographic projection onto $\mathbb{R}^3$ from the north pole $(1,0,0,0)$ of $\mathbb{S}^3$ to visualize the flow. The stereographic projection is given by

$$\Psi : \mathbb{S}^3 \setminus \{(1,0,0,0)\} \to \mathbb{R}^3$$
$$(x_1,y_1,x_2,y_2) \mapsto \left(\frac{x_2}{1-x_1}, \frac{y_2}{1-x_1}, \frac{y_1}{1-x_1}\right).$$

The stereographic projection of the usual Hopf vector field $-y_1\partial_{x_1} + x_1\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2}$ can be written in cylindrical coordinates (ρ,ϕ,z) as

$$-z\rho\partial_\rho - \frac{1-\rho^2+z^2}{2}\partial_z + \partial_\phi. \tag{15}$$

*The stereographic projection of the b-vector field $-zy_1\partial_{x_1} + z^2\partial_{y_1} - y_2\partial_{x_2} + x_2\partial_{y_2}$ is in cylindrical coordinates, and letting $r^2 = x^2+y^2+z^2$ given by (**Exercise**)*

$$\frac{r^2-1}{r^2+1}\left(-z\rho\partial_\rho - \frac{1-\rho^2+z^2}{2}\partial_z\right) + \partial_\phi. \tag{16}$$

We will call this vector field the b-Hopf field, because of the analogy to the usual Hopf field. Notice that the only difference between Equations (15) *and* (16) *is the $\frac{r^2-1}{r^2+1}$ factor, which multiplies the ∂_ρ and ∂_z components, but crucially not the ∂_ϕ component, of the Hopf b-vector field. Figure 1 shows the flow of the Hopf b-vector field on the y = 0 plane. Notice the white circle representing stationary points at the critical surface $r = 1$. The points on this circle rotate. according to ∂_ϕ along the parallels of the critical $\mathbb{S}^2$.*

<u>Observation:</u> In the above 3-dimensional examples, we observe that

- there can be no periodic orbit away from Z on a compact b-contact manifold. From the point of contact geometry, $M\setminus Z$ is a non-compact contact manifold, and therefore this makes sense. However, from a 'b-contact geometry' point of view, it would be nice to have the existence of periodic orbits on b-contact manifolds;
- there exists an orbit that comes from the zero of R_α on Z and returns to another zero on Z, which we called in Definition 3.17 a *singular periodic orbit*.

It follows that all hope for the Weinstein conjecture to hold for the existence of periodic orbits away from Z is gone: some examples do not satisfy this. However, a new dynamical invariant appears in the above examples: singular periodic Reeb orbits. They seem to exist - at least in all of the examples we have encountered. We thus conjecture

Conjecture 3.19 (Singular Weinstein Conjecture)**.** *On a compact b-contact manifold, singular periodic Reeb orbits exist or a periodic Reeb orbit outside the critical set.*

In what follows, we will analyze this conjecture. This conjecture is of global nature: it asks for a globally defined integral curve coming and going to a singular point of the Reeb vector field on Z. We will start by analyzing this conjecture from a semi-local point of view. Namely, we will analyze the behavior of the Reeb vector field in a tubular neighborhood around the critical set. To do so, we need to take a detour to the world of hydrodynamics, more precisely to the one of Euler flows.

3.4. **Euler flows and the contact/Beltrami mirror.** In [15], a correspondence between contact forms and Beltrami vector fields was shown. This gives an exciting reinterpretation of both fields that we will analyze in this subsection in the set-up of b-geometry. Here, we first define what Beltrami vector fields are.

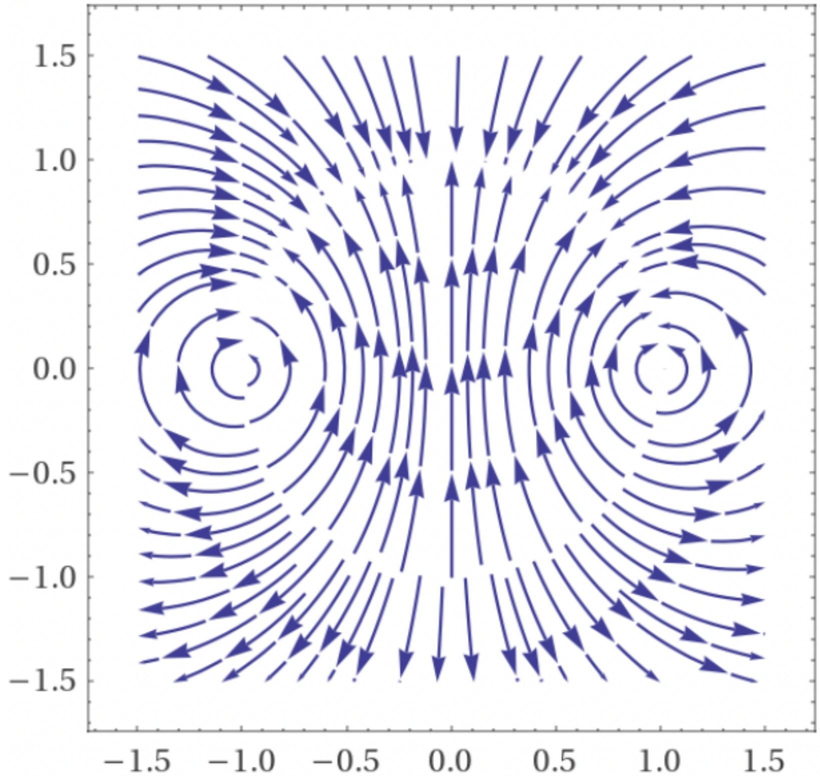

FIGURE 1. Illustration of the Hopf b-vector field on the $y = 0$ section. It is essentially the smooth Hopf vector field rescaled as it approaches $r = 1$, see [17].

Given a 3-dimensional Riemannian manifold (M, g), the curl of a vector field $X \in \mathfrak{X}(M)$ is defined to be the vector field that satisfies $d\iota_X g = \iota_{\nabla\times X}\mu$, where μ is a volume form (possibly the volume form associated to the metric g).

Definition 3.20. *A vector field is said to be Beltrami if it is tangent to its own curl, that is*

$$\nabla \times X = fX \tag{17}$$

for some $f \in C^\infty(M)$. A Beltrami vector field is said to be rotational if $f \neq 0$, meaning it has a nonzero curl.

Beltrami vector fields have their origin in fluid dynamics, as they are time-independent solutions to Euler's equation for a perfect incompressible fluid, see [15].

The so-called ABC vector fields give an example of Beltrami vector fields.

Example 3.21. *Consider the flat Riemannian metric on the 3-torus $\mathbb{T}^3$. The vector field*

$$X(x, y, z) = [A \sin z + C \cos y]\frac{\partial}{\partial x} + [B \sin x + A \cos z]\frac{\partial}{\partial y} + [C \sin y + B \cos x]\frac{\partial}{\partial z}$$

for $A, B, C \in \mathbb{R}$ are real parameters is a Beltrami vector field ***Exercise****. Furthermore, $\alpha = g_{flat}(X, \cdot)$ is a contact form and the integral curves of its Reeb vector field and of X coincide.*

The fact that the above Beltrami vector field is a Reeb vector field (up to reparametrization) is not a coincidence.

For time-independent Euler vector fields that are not everywhere tangent to its curl (so not Beltrami), the famous structure theorem of Arnold [2] says that the manifold can be into solid tori where the tori are invariant sets for the flow of X. Beltrami vector fields do not satisfy this, and geometrically, the condition (17) says that the vector fields twist about themselves. This is reminiscent of the notion of the everywhere 'twisting' plane field given by a contact structure. Etnyre–Ghrist formalized this observation: a rotational Beltrami vector field is a Reeb vector field for some contact form on M!

Theorem 3.22 (Theorem 2.1 in [15])**.** *Any rotational Beltrami field on M is a Reeb vector field (up to rescaling) for some contact form on M. Conversely, given a contact form α with Reeb vector field R_α, any nonzero rescaling of X is a rotational Beltrami field for some metric and volume form on M.*

This theorem thus opens the door to apply known techniques from contact topology to the field of Beltrami vector fields, and vice-versa.

Given a 3-dimensional manifold with boundary $(M, \partial M)$ and a vector field X that is Beltrami on $M \setminus \partial M$, a natural condition is to ask that the Beltrami vector field is tangent to the boundary. But this is precisely what b-vector fields were defined for! In what follows, we will thus describe the above description for b-manifolds. Spoiler-alert: Theorem 3.22 still holds in this set-up. To do this, we need to describe the metrics to take into consideration.

Definition 3.23 (b-metrics)**.** *A b-metric is a bilinear positive-definite form $\Gamma({}^bT^*M \otimes {}^bT^*M)$.*

A b-metric naturally induces a b-form of a maximal degree called b-volume form.

Definition 3.24. *A b-Beltrami vector field X is a vector field on a Riemannian b-manifold (M, Z, g) such that* $\operatorname{curl} X = \lambda X$*, for some nonzero constant λ, where the curl operator is defined with respect to the b-metric g.*

Here is an example of a b-Beltrami vector field on $\mathbb{T}^3$.

Example 3.25. *Consider the b-manifold $(\mathbb{T}^3, \mathbb{T}^2)$ with b-metric given by $g = dx^2 + dy^2 + \frac{dz^2}{\sin^2 z}$. The b-vector field*

$$X(x, y, z) = C \cos y \frac{\partial}{\partial x} + B \sin x \frac{\partial}{\partial y} + [C \sin y + B \cos x] \sin z \frac{\partial}{\partial z}$$

for $A, B \in \mathbb{R}$ real parameters is a b-Beltrami vector field. ***Exercise***

Note that this vector field looks similar to the Reeb vector field in Example 3.16. Once more, this is not a coincidence, as similar to the smooth case, any non-vanishing b-Beltrami vector field is a reparametrization of the Reeb field associated with a b-contact form. More precisely, we have the following:

Theorem 3.26 ([8])**.** *Let (M, Z) be a b-manifold of dimension three. Any b-Beltrami vector field that is non-vanishing as a section of ${}^bTM$ on M is a Reeb field (up to rescaling) for some b-contact form on (M, Z). Conversely, given a b-contact form α with Reeb field X, then any nonzero rescaling of X is a b-Beltrami vector field for some b-metric and b-volume form on M.*

When looking closely at the proof of Theorem 3.26, one sees that if X is a b-Beltrami vector field on (M, Z, g), the Reeb field associated to the b-contact form $\alpha := g(X, \cdot)$ is given by $\frac{1}{\|X\|^2} X$, where the norm is computed using the b-metric g.

The b-Beltrami vector field in Example 3.25 is thus the Reeb vector field b-contact form on $\mathbb{T}^3$ given by $\alpha = g(X, \cdot)$. The dynamics of this vector field can be explicitly computed, similar to the computations we did in Example 3.16. **Exercise:** Prove that there are 8 singular periodic orbits in Example 3.25 (see Example 1.2 in [41]). This example gives thus further evidence to the singular Weinstein conjecture.

3.5. **The existence of escape orbits.** The first step in studying the singular Weinstein conjecture is to examine the semi-local behavior of the b-Reeb vector field around the critical set. We will consider a more general type of orbit, which will be proven to exist.

Definition 3.27. *Let (M, Z, α) be a b-contact manifold. An escape orbit is an integral curve of R_α such that at least one of the semi-orbits has a stationary limit point on Z.*

Here, we include a comic about spos and escape orbits.

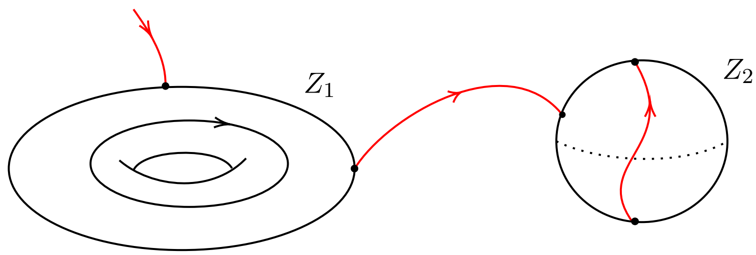

FIGURE 2. Examples of an escape orbit (on the left, tending to a point on the critical torus) and two singular periodic orbits. The critical set is a disjoint union of a torus Z_1 and a sphere Z_2, and a b-Reeb orbit on the critical torus is depicted in black.

It is clear from the definition that the existence of escape orbits is a necessary condition for the proof of the singular Weinstein conjecture. We will prove the following theorem.

Theorem 3.28 (Theorem 3.1 in [17]). *Let α be a b-contact form on a 3-dimensional manifold (M, Z) without boundary, with Z a closed embedded surface in M. Then there exists a b-contact form C^∞-close to α, such that the associated b-Reeb vector field has either*

(1) infinitely many escape orbits if $b_1(Z) > 0$, or
(2) at least $2N$ escape orbits if $b_1(Z) = 0$, where N is the number of connected components of Z.

Moreover, the set of b-contact forms exhibiting these properties is open in the C^∞-topology.

Here $b_1(Z)$ denotes the first Betti number of the critical surface Z.

Proof. By Equation (10), a b-contact form in a tubular neighbourhood around the critical set Z, denoted by $\mathcal{N}(Z)$, is given by

$$\alpha = f\frac{dz}{z} + \beta,$$

where $f \in C^\infty(\mathcal{N}(Z))$ and $\beta \in \Omega^1(\mathcal{N}(Z))$.

By a C^∞-small perturbation, we can assume that the function f in the above decomposition for α restricts to a Morse function on Z. Indeed, we choose a b-contact form that is C^∞-close to α as

$$\widetilde{\alpha} := (f + \epsilon h)\frac{dz}{z} + \beta, \tag{18}$$

h is a C^∞-small function, supported in a tubular neighborhood of the critical set, such that $(f + \epsilon h)|_Z$ is a Morse function. For ϵ is small enough, this is still a b-contact form, as this being contact is an open condition. The reason for perturbing f to become Morse will become apparent from the analysis carried out around a tubular neighborhood of the associated b-Reeb vector field. Thus, we will assume that α satisfies this condition.

The associated b-Reeb vector field in this tubular neighborhood is given by

$$R_\alpha = gz\frac{\partial}{\partial z} + Y,$$

where $g \in C^\infty(\mathcal{N}(Z))$ and $Y \in \mathfrak{X}(\mathcal{N}(Z))$ such that $\iota_Y(dz) = 0$.

As in Equation (14), the restriction of the smooth 2-form

$$\Theta := fd\beta + \beta \wedge df$$

to Z is symplectic. This implies that at a critical point $p \in Z$ of $f|_Z$, $f(p) \neq 0$, i.e. 0 is a regular value of $f|_Z$.

By Proposition 3.13, $R_\alpha|_Z$ is a Hamiltonian vector field with respect to $\omega|_Z$, and the exceptional Hamiltonian is given by $H := -f|_Z$. We denote this Hamiltonian vector field by

$$\overline{R} := R_\alpha|_Z = Y|_Z.$$

It follows that at a critical point $p \in Z$ of H, we have a zero of the b-Reeb vector field, that is, $(R_\alpha)_p = \overline{R}_p = 0$. Thus, by the assumption that Z is closed, R admits at least two zeroes (corresponding to the maximum and minimum values of H). Furthermore, at a critical point $p \in Z$, we have $g(p) \neq 0$ because the Reeb condition $\alpha(R_\alpha) = 1$ yields

$$1 = \alpha(R_\alpha)|_p = f(p)g(p) + \beta_p(Y_p) = f(p)g(p).$$

We now study the linear stability of R_α around the critical points. At a critical point p, the differential of R_α is given by

$$DR(p) = \begin{pmatrix} D\overline{R} & * \\ 0 & g \end{pmatrix}\Bigg|_p.$$

We choose a Darboux chart around the critical point in Z so that in local coordinates with $\omega = -dx \wedge dy$, we obtain

$$DR(p) = \begin{pmatrix} H_{xy} & H_{yy} & * \\ -H_{xx} & -H_{xy} & * \\ 0 & 0 & g \end{pmatrix}\Bigg|_p.$$

It is now easy to determine the linear stability at p by looking at the eigenvalues of this matrix. The eigenvalues are λ_+, λ_- and λ_z, where λ_+ and λ_- are eigenvalues of the first 2×2 minor,

$$\lambda_\pm = \pm\sqrt{-\operatorname{Hess} H(p)},$$

and $\lambda_z = g(p) \neq 0$. Notice $\lambda_\pm \neq 0$ because we assume that $f|_Z$ (and hence H) is a Morse function.

There are two situations to consider, according to the sign of $\operatorname{Hess} H(p)$:

- $\operatorname{Hess} H(p) < 0$: In this case, the critical point of R is hyperbolic, and there is a two-dimensional stable or unstable (depending on the sign of $g(p)$) manifold at p that is transverse to Z.
- $\operatorname{Hess} H(p) > 0$: In this case, the critical point of R is non-hyperbolic, and there is a one-dimensional stable or unstable (depending on the sign of $g(p)$) manifold at p that is transverse to Z, the center manifold being Z.

When the transverse invariant manifold is of dimension two, all of the orbits lying within it (of which there are infinitely many) are escape orbits with limit point p (see Figure 3). A transverse invariant manifold of dimension one guarantees two escape orbits (one on each side of Z) with limit point p.

Let C_k be the number of critical points of H of index k on Z and b_k the k-th Betti number. We will use the Morse inequality

$$C_k \geq b_k(Z) \tag{19}$$

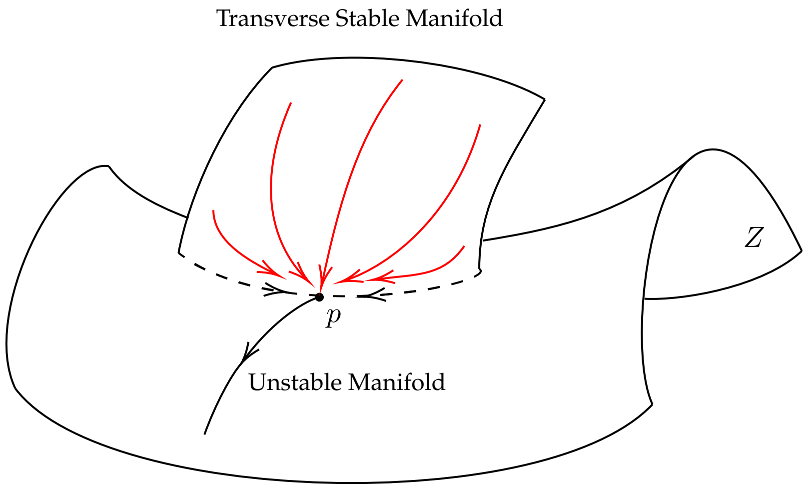

FIGURE 3. Example of case Hess $H(p) < 0$ and $g(p) < 0$, so there is a transverse 2-dimensional stable manifold containing infinitely many escape orbits, which are colored red.

to conclude the proof.

Case $b_1(Z) > 0$. In this case, there is at least one critical point of H of index one (in fact at least two because the first Betti number is even), so there is a saddle point and, therefore, infinitely many escape orbits.

Case $b_1(Z) = 0$. This corresponds to Z consisting of $N \geq 1$ disjoint surfaces all diffeomorphic to $\mathbb{S}^2$. In this case, there are at least two escape orbits for each critical point (one escape orbit on each side of the corresponding sphere), some of which may coincide to form singular periodic orbits. In any case, since there are at least $2N$ critical points, there must be at least $2N$ distinct escape orbits. Note that it can still be that the exceptional Hamiltonian has a saddle point on Z, in which case there would be infinitely many escape orbits.

Finally, notice that Morse and the sign conditions presented above are open in the C^∞-topology, so we conclude that the set of b-forms for which the theorem applies is not only dense but also open. $\square$

A remark on the chronology is in order: previous to Theorem 3.28, it was proved that given a b-Beltrami vector field on a b-manifold equipped with a b-metric that can be written around Z as

$$g = P^*h + \frac{dz^2}{z^2}, \tag{20}$$

where h is a smooth metric on Z, the b-Beltrami vector field $X = Y + X_z z\frac{\partial}{\partial z}$ ($Y \in \mathfrak{X}(Z)$, and X_z smooth function) satisfies that $X_z|_Z$ is an eigenvalue of the Laplacian with respect to the metric h. **Exercise**. Eigenfunctions of the Laplacians are well-studied, and generically, they are Morse functions as was proved by Uhlenbeck [48]. The same analysis as in the proof of Theorem 3.28 holds, but the genericity is less general (as in the proof of Theorem 3.28, the b-contact form is directly perturbed), whereas the in [41], metric h is perturbed. However, splitting the metric under the form of Equation (20) is a strong assumption.

Theorem 3.28 gives, as previously said, further evidence to the singular Weinstein conjecture: around the critical set, there do exist (at least generically) escape orbits, which are a generalization of spos (=*singular periodic orbits*). However, as will be seen in the following subsection, this result tricked us: in fact, we will construct an example where the singular Weinstein conjecture does not hold, i.e., a compact b-contact manifold, without periodic orbits away from Z, nor spos.

3.6. **A counter-example to the singular Weinstein conjecture.** Recall that the singular Weinstein conjecture (Conjecture 3.19) claims that on a compact b-compact manifold, there exists always a singular periodic Reeb orbit or a periodic Reeb orbit away from the critical set. The conjecture was supported by several examples presented in Subsection 3.3 and a 'semi-local' version, namely Theorem 3.28. We will revise in this section Example 3.18: in this example, we showed that there are no periodic orbits away from Z and that there exists *exactly* two spos. Furthermore, one can see from Figure 1 that any orbit away from Z (except for the two spos) tend in positive or negative time to a non-trivial periodic orbit in $Z = S^2$. We will see that using results from smooth contact topology, each of the spos can be *broken*. By this, we mean that the b-contact structure of Example 3.18 can be changed outside Z to a different b-contact structure, such that there are no spos and no periodic orbits away from Z.

We thus obtain the following:

Theorem 3.29 (Theorem 4.3 in [16]). *There is a b-contact form on the b-manifold* $(\mathbb{S}^3, \mathbb{S}^2)$*, which has no singular periodic orbits and no periodic orbits away from the critical set.*

The strategy will be to perturb the contact form in a Darboux neighborhood away from Z where the spo goes through. The spo will be *deviated* by choosing a suitable perturbation. The spo will thus be broken in this process, and instead of the spo, we will obtain two orbits $\gamma^{\pm}$ such that $\lim_{t\to\pm\infty} \gamma^{\pm}(t) = p \in Z$ be a singular point, and $\gamma^{\pm}$ will have α-set given by a circle (periodic orbit) on Z.

Proof. Let $p \in M\setminus Z$ be a point on a singular periodic orbit, and U a Darboux neighborhood containing p and intersecting no other escape orbits. We assume that p is sufficiently close to the origin but *not* on the z-axis of this chart. Endowing U with cylindrical coordinates (and after rescaling the z direction), the contact form $\alpha|_U$ has the expression

$$\alpha|_U = \frac{1}{2}dz + r^2 d\varphi\,,$$

and its associated Reeb vector field in this chart is $R_\alpha = 2\partial_z$, so all orbits are vertical lines (we refer to $\alpha|_U$ as simply α to simplify notation). In particular, the singular periodic orbit passing through p is a vertical line in U that does not contain the origin. Now consider another contact form on U,

$$\alpha' = zrdr + \frac{1}{2}(1 + z^2 - r^2)dz + r^2 d\varphi\,, \tag{21}$$

whose Reeb vector field is proportional to $\partial_z + \partial_\varphi$. We will use this contact form to perturb α. The result of this perturbation is shown in Figure 4.

To this end, it is convenient to introduce a compact set $K \subset U$, with $p \in K$, of the form $D_\delta \times [-\delta, \delta]$ in cylindrical coordinates, where D_δ is a closed 2-disk of radius $\delta > 0$. Take a smooth bump function $f : \mathbb{R} \to [0, 1]$ which is equal to 1 in $I_{ct} := [-\delta/2, \delta/2]$, and whose support is contained in $(-\delta, \delta)$. Without loss of generality, we assume that the singular periodic orbit is a vertical line whose r-coordinate is in the interior of the interval I_{ct}. For any small $\varepsilon > 0$, consider the 1-form

$$\tilde{\alpha} = (1 - \varepsilon f(r)f(z))\,\alpha + \varepsilon f(r)f(z)\alpha',$$

defined on M, which coincides with α on $M\backslash K$, and is C^∞-close to α on K provided that ε is small enough. Obviously, $\tilde{\alpha}$ is a b-contact form on M.

We claim that $\tilde{\alpha}$ has one singular periodic orbit less than α. We compute the Reeb field $R_{\tilde{\alpha}}$ to show this. Since we are only interested in the singular periodic orbit, we can restrict our study to the set $D_{\delta/2} \times (-\delta, \delta) \subset K$, which contains the point p. A straightforward computation shows that on the set $D_{\delta/2} \times (-\delta, \delta)$ we have

$$\tilde{\alpha} = \varepsilon f(z) z r dr + \frac{1}{2}(1 + \varepsilon f(z)(z^2 - r^2))dz + r^2 d\varphi,$$

and

$$\begin{aligned} d\tilde{\alpha} &= \varepsilon(f'z + f) r dz \wedge dr - \varepsilon f r dr \wedge dz + 2r dr \wedge d\varphi = \\ &= \varepsilon(f'z + 2f) r dz \wedge dr + 2r dr \wedge d\varphi. \end{aligned}$$

To find the corresponding Reeb vector field $R_{\tilde{\alpha}} = \tilde{R}^r \partial_r + \tilde{R}^\varphi \partial_\varphi + \tilde{R}^z \partial_z$, we see immediately that $\tilde{R}^r = 0$, and that $\tilde{R}^\varphi$ and $\tilde{R}^z$ satisfy

$$\varepsilon(\frac{1}{2} f'z + f)\tilde{R}^z = \tilde{R}^\varphi.$$

If ε is small enough, we can compute the vector field $R_{\tilde{\alpha}}$ in terms of an ε-expansion, which yields:

$$\tilde{R}^r = 0\,, \qquad \tilde{R}^\varphi = \varepsilon(f'(z)z + 2f(z)) + O(\varepsilon^2)\,, \qquad \tilde{R}^z = 2 + O(\varepsilon)\,.$$

Therefore, we can integrate approximately the integral curves of $R_{\tilde{\alpha}}$. In the particular case of the trajectory that corresponds to the singular periodic orbit of R_α, taking as initial condition the point $r_0 = \delta_0 < \delta/2$, $\varphi_0 = 0$, $z_0 = -\delta$, we obtain

$$r(t) = r_0\,, \qquad \varphi(t) = \frac{1}{2}\varepsilon \int_{-\delta}^{-\delta+2t} \Big(f'(s)s + 2f(s)\Big) ds + O(\varepsilon^2 t)\,, \qquad z(t) = -\delta + (2 + O(\varepsilon))t\,.$$

Since the time taken from the perturbed orbit to go from $\{z = -\delta\}$ to $\{z = \delta\}$ is $T = \delta + O(\varepsilon)$, we finally conclude that

$$\varphi(T) = \frac{1}{2}\varepsilon \int_{-\delta}^{\delta} \Big(f'(s)s + 2f(s)\Big) ds + O(\varepsilon^2 \delta) = \frac{1}{2}\varepsilon \int_{-\delta}^{\delta} f(s) ds + O(\varepsilon^2 \delta)\,,$$

where we have integrated by parts and used that $f(z) = 0$ near $z = \pm\delta$. It then follows from the fact that f is a non-negative function that

$$\varphi(T) = \frac{C}{2}\varepsilon\delta + O(\varepsilon^2\delta) > 0,$$

for some $C > 0$, and hence the continuation of the singular periodic orbit along the flow of $R_{\tilde{\alpha}}$ rotates slightly within a small cylindrical neighborhood of the origin, in addition to moving upwards (see Figure 4).

The upshot is that the semi-orbit of the singular periodic orbit coming into the cylindrical set K from below no longer matches the semi-orbit coming out from above. Thus, the singular periodic orbit is broken into two orbits such that each of those new orbits limits in positive (resp. negative) time to a non-trivial periodic orbit in Z. Furthermore, these semi-orbits do not coincide with any other semi-orbits of escape orbits, as the neighborhood we had taken had no other escape orbits.

□

Summarizing, in this proof, we traded the spo with two orbits such that for $t \to \infty$ (respectively $t \to -\infty$) tend to a singular point of the Reeb vector field on Z, but for $t \to -\infty$ (resp. $t \to \infty$), they spiral around a periodic orbit of the Reeb vector field on Z; more precisely, the α (or ω-)set is given by a non-trivial periodic orbit on Z. We call such

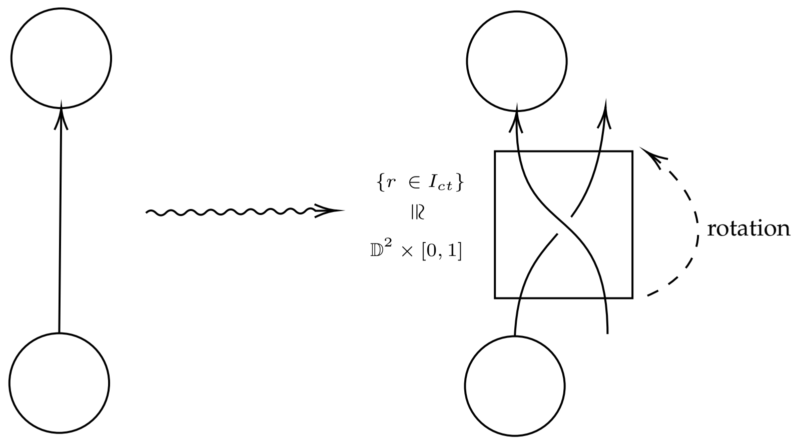

FIGURE 4. Illustration of the perturbation used to break singular periodic orbits. The circles represent the critical spheres. On the left, a singular periodic orbit is shown. On the right, it is broken into two escape orbits after a small perturbation of the b-contact form.

an orbit a generalized singular periodic orbit. We include a cartoon of the different types of orbits in the cartoon given by Figure 5.

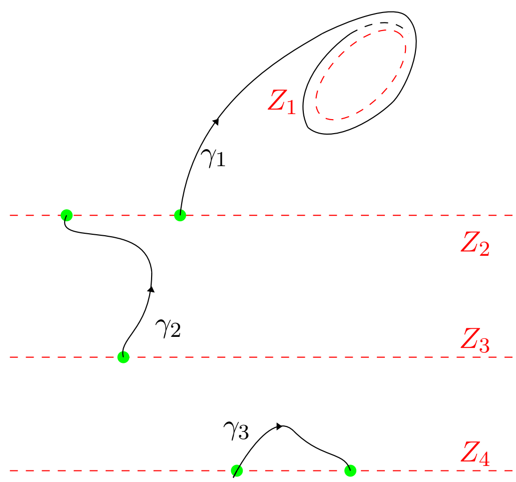

FIGURE 5. Different types of escape and singular periodic orbits: γ_1 is a generalized singular periodic orbit, γ_2, γ_3 are singular periodic orbits

The proof of Theorem 3.29 inspired us to prove that given a smooth contact manifold of dimension 3, we can change the contact form to a b-contact form that has critical sets given by the disjoint union of S^2 where the number of singular periodic orbits is controlled. The precise statement is the following:

Theorem 3.30 (Theorem 3.1 in [16]). *Let (M, ξ) be a co-orientable 3-dimensional contact manifold. Then, for any integers $0 \leq k \leq N$, there exists a b-contact form on M whose critical set Z consists of N components diffeomorphic to $\mathbb{S}^2$ and the number of singular periodic orbits is exactly k. The associated b-contact structure coincides with ξ outside a neighborhood of the balls enclosed by Z. Furthermore, there is an infinite number of generalized escape orbits converging to each component of Z (i.e., orbits whose α- or ω-limit sets are on Z but are not singular points).*

The proof consists of inserting N copies of S^2 in Darboux neighborhoods, where the Reeb dynamics is given around each of the S^2 as in the Example 3.18. We will not include the proof here.

3.7. **A guide through b-contact geometry.**

We finish this section with a guide through the literature of other known results in b-contact geometry.

Contrary to b-symplectic, respectively log-symplectic geometry literature as resumed in Subsection 2.7, the literature related to b-contact structures has a relatively manageable size.

- To learn more about the local geometry of b-contact manifolds and their associated Jacobi structures, see [40].
- The singular Weinstein conjecture was formulated in [39].
- The Beltrami-contact correspondence in the set-up of b-contact forms was proved in [8].
- The existence of escape orbits was proved first in [41]. The results were later generalized in [17].
- Regularizations of b^k-contact manifolds are explored in [14]
- The topology of b-contact manifolds and the h-principle are investigated in [9].
- The equivariant version of the Beltrami-contact correspondence can be found in [18] - and its relation to the Euler–Kepler flow.
- Finally, the counterexample to the singular Arnold conjecture can be found in [16].
- Related to the material is the preprint [6] on folded symplectic forms in contact topology.

4. BACK TO INFINITY...

In this section, we will briefly go back to the RPC3BP. Recall that in Subsection 1.4, we introduced in (3) the McGehee change of variables

$$r := \frac{2}{x^2}, \quad x \in \mathbb{R}^+,$$

which pulls back the standard symplectic form to the b^3-symplectic form given as in Equation (4) by

$$\omega = -4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha,$$

and the domain is given by $\mathbb{R}^+ \times S^1 \times \mathbb{R}^2$. In Theorem 1.14, the authors considered regular level-sets of the Hamiltonian Σ_c for energies small enough (more precisely, for c below the energy of the first Lagrange point). Recall that these level-sets where not compact due to possible collisions with the one of the two massive bodies, the projection to the position space of these level-sets (which are called the Hill's region) are compact sets in $\mathbb{R}^2$.

This section will consider different level-sets, namely, whose Hill's region is *non-compact*, i.e., the satellite can escape to infinity. This is the case when the energy is positive, that is, when $H = c > 0$. First, let us consider the vector field $Y = \sum_{i=1}^2 p_i \frac{\partial}{\partial p_i}$ in the cotangent bundle of $\mathbb{R}^2$. We will see that this vector field is transverse to Σ_c for positive energies.

Lemma 4.1 (Lemma 4.3. in [39])**.** *The vector field $Y = p\frac{\partial}{\partial p}$ is a Liouville vector field and is transverse to Σ_c for $c > 0$.*

Proof. The vector field Y is a Liouville vector field as $\mathcal{L}_Y \left(\sum_{i=1}^2 dp_i \wedge dq_i\right) = \omega$ and is transverse to Σ_c for $c > 0$. Indeed

$$Y(H)=|p|^2+p_1q_2-p_2q_1=\frac{|p|^2}{2}+\frac{1-\mu}{|q-q_E|}+\frac{\mu}{|q-q_M|}+H(q,p).$$

Hence $Y(H)|_{H=c}=\frac{|p|^2}{2}+\frac{1-\mu}{|q-E|}+\frac{\mu}{|q-M|}+c$, which is a sum of positive terms when $c>0$. $\square$

We now prove that the vector field Y is also transverse to the level sets of the Hamiltonian at infinity. This strategy is to do the McGehee change of coordinates and to check that the vector field defined in Lemma 4.1 is still transverse to the level-set of the Hamiltonian.

Theorem 4.2. *After the McGehee change, the Liouville vector field $Y=p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c>0$ and the level-sets $(\Sigma_c,\iota_Y\omega)$ for $c>0$ are b^3-contact manifolds. Topologically, the critical set is a cylinder, and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.*

Proof. First, let us compute the Hamiltonian of the PRC3BP, given by Equation (1), in polar coordinates. We will then perform the McGehee change of coordinate. The polar coordinates are defined by the position $q=(r\cos\alpha,r\sin\alpha)$, $(r,\theta)\in\mathbb{R}^+\times S^1$, and the momenta $p=(P_r\cos\alpha-\frac{P_\alpha}{r}\sin\alpha,P_r\sin\alpha+\frac{P_\alpha}{r}\cos\alpha)$, $(P_r,P_\alpha)\in\mathbb{R}^2$. Under this coordinate change, the resulting Hamiltonian is given by the following expression:

$$\begin{aligned}&H\left(r,\alpha,P_r,P_\alpha\right)\\=&\frac{1}{2}\left(P_r^2-\left(\frac{P_\alpha}{r}\right)^2\right)-\frac{1-\mu}{r^2-2\mu r\cos\alpha+\mu^2}-\frac{\mu}{r^2-2\left(1-\mu\right)r\cos\alpha+\left(1-\mu\right)^2}-P_\alpha.\end{aligned}$$

The coordinate change is symplectic, and therefore, the symplectic form is given by $dr\wedge d\alpha+dP_r\wedge dP_\alpha$, and the Liouville vector field writes down $Y=P_r\frac{\partial}{\partial P_r}+P_\alpha\frac{\partial}{\partial P_\alpha}$.

After the McGehee change of coordinates $r=\frac{2}{x^2}$, the Hamiltonian is given by

$$\begin{aligned}&H\left(x,\alpha,P_r,P_\alpha\right)\\=&\frac{1}{2}\left(P_r^2-\frac{1}{4}x^4P_\alpha^2\right)-x^4\frac{1-\mu}{4-4\mu x^2\cos\alpha+\mu^2x^4}-x^4\frac{\mu}{4-4x^2\left(1-\mu\right)\cos\alpha+\left(1-\mu\right)^2x^4}-P_\alpha.\end{aligned}$$

The Liouville vector field does not change under the McGehee change of coordinates. Still, instead of a symplectic form, the underlying geometric structure is a b^3-symplectic structure with a critical set given by $\{x=0\}$ given by $\omega=-4\frac{dx}{x^3}\wedge dPr+d\alpha\wedge dP_\alpha$. We already checked that the Liouville vector field is everywhere transverse to the level-set of H, and we now check that it is also transverse to the critical set.

On the critical set, the Hamiltonian is given by $H=\frac{1}{2}P_r^2-P_\alpha$, so that $Y(H)=P_r^2-P_\alpha$. On the level-set $H=c>0$, we obtain $Y(H)=\frac{1}{2}P_r^2+c>0$. Hence, it is transverse to the critical set as well, and therefore, the induced b^3-contact form on the critical set is given by $\alpha=(P_r\frac{dx}{x^3}+P_\alpha d\alpha)|_{H=c}$.

The critical set of the b^3-contact manifold is defined by the equations

$$Z=\{(x,\alpha,P_r,P_\alpha)\,|x=0,\frac{1}{2}P_r^2-P_\alpha=c\}.$$

Topologically, the critical set is a cylinder, as solutions for $\frac{1}{2}P_r^2-P_\alpha=c$ are given by $P_\alpha=\frac{1}{2}P_r^2-c:=P_\alpha\left(P_r\right)$. The cylinder is described by $Z=\{0,\alpha,P_r,P_\alpha\left(P_r\right)\}$ and hence non-compact.

According to the decomposition lemma, the b^3-contact form decomposes as $\alpha=u\frac{dx}{x^3}+\beta$ and by Proposition 3.13, the Reeb vector field on the critical set is Hamiltonian for the Hamiltonian function u. The Hamiltonian function here is given by P_r. As the Hamiltonian vector field is contained in the level-set of the Hamiltonian, we obtain that both cylinders are foliated by non-trivial periodic orbits away from $P_r = 0$. □

A reformulation of Theorem 4.2 from the viewpoint of a dynamical system is the following:

Corollary 4.3. *After the McGehee change in the RPC3BP, there are infinitely many non-trivial periodic orbits at the manifold at infinity for energy values of $H = c > 0$ (hyperbolic motion).*

We conclude the lecture notes with a series of open questions and problems.

5. Open questions

5.1. **More general singular contact structures.** In these notes, we developed techniques to study the Reeb dynamics for contact forms over the b-tangent bundle. The b-tangent bundle can be constructed thanks to the b-vector fields satisfying Serre–Swan's theorem; that is, they consist of a finitely generated $C^\infty(M)$-module.

Question 5.1. *Can one study the Reeb dynamics over more general Lie algebroids? What are the corresponding dynamics?*

Of course, many other $C^\infty(M)$-modules are finitely generated. The geometry of such vector bundles has been initiated in [42], and the dynamics have been studied in some cases in [14]. Still, the dynamics are far from being understood.

5.2. **Hofer's method.** As we saw in this mini-course, compact b-contact manifolds can be seen as *non-compact* contact manifolds, equipped with an additional behavior on the boundary. This non-compactness is the troublemaker when trying to apply general results on Reeb dynamics. In general, very little is known about the Reeb dynamics when the manifold is open: for instance, the linear dynamics in $\mathbb{R}^3$ is, of course, Reeb (the Darboux contact form). The situation we are facing here is different because the ambient manifold is *compact*, but $M \setminus Z$ is not. Furthermore, we have the additional geometric structure on the boundary (see Subsection 3.2). It is reasonable to think that more can be said about the dynamics on $M \setminus Z$, at least when $M \setminus Z$ is *overtwisted* contact. As is proved in [39], for overtwisted b-contact manifold, which satisfies additionally that the contact form is invariant with respect to the action of a transverse contact vector field. There exists a 1-parametric family of periodic Reeb orbits on $M \setminus Z$. The condition of the invariant b-contact form is strong; therefore, one may ask if this condition can be removed.

Question 5.2. *Do overtwisted b-contact manifolds admit a periodic Reeb orbit away from Z?*

This would imply that Examples 3.11 and 3.12 would be tight b-contact manifolds.

5.3. **Generalized Weinstein conjecture.** Periodic Reeb orbits yield topological invariants of contact manifolds. This is the content of contact homology, a homology generated by the periodic Reeb orbits. Compact three-dimensional b-contact manifolds do not always admit periodic orbits - and therefore, constructing a homology for compact b-contact manifolds is hopeless. The singular Weinstein conjecture was the first attempt to 'enlarge' the class of generators for a possible candidate for contact homology. However, as we saw in Theorem 3.29, there are also counter-examples to the singular Weinstein conjecture, and thus the quest for a b-contact homology failed - again! The question, therefore, is whether b-contact manifolds do admit dynamical invariants, and if so, if they can be used to construct a homology out of them.

Question 5.3. *Are there dynamical invariants on compact b-contact manifolds?*

Given a positive answer, the natural continuation is therefore:

Question 5.4. *Can one construct a b-contact homology?*

5.4. Traps and plugs. TBD

5.5. Applications to celestial mechanics.
As we have seen in the previous section, in the RCP3BP, after the McGehee changes, the induced geometric structure on the level-set for positive energy is equipped with a b^3-symplectic structure. The McGehee is only one of many possible regularizations used to study the dynamics of the three problems. We mentioned Moser's regularization, as was used to study the contact geometry in the RPC3BP in [1]; other ones are, for instance, the Levi-Civita and Kustaanheimo-Stiefel regularization.

Question 5.5. *Which singular geometric geometric appears when applying the regularizations in celestial mechanics? Can one use the techniques developed in Question 5.1 to this setup?*

In particular, given a potential regularization that yields a compact three-dimensional b-manifold, one could apply the results from Subsection 3.5 regarding the existence of escape orbits.

Question 5.6. *Can the examples of the existence of escape orbits be applied to the problems in celestial mechanics?*

5.6. The singular Arnold conjecture.
This section here will only be dealt with in the mini-course if all of the previous sections are handled in the mini-course. Many interesting open questions still remain to be fully understood, for instance.

Question 5.7. *Can the Floer homology as defined in* [7] *be shown to be independent of the choice of almost complex structure, t, class of Hamiltonian?*

Question 5.8. *Is there a more general definition of Floer homology (i.e., b-Floer homology for general b-Hamiltonians, or smooth Hamiltonian)?*

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BCAM BILBAO, MAZARREDO, 14. 48009 BILBAO BASQUE COUNTRY - SPAIN
Email address: `coms@bcamath.org`