Volume versus ℓ^2-Betti numbers

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Motivation

Gromov's main inequality

For every dimension d there is a constant $C(d) > 0$ such that every closed d-dimensional Riemannian manifold M satisfies

$$\|M\| \leq C(d) \cdot \mathrm{vol}(M)$$

provided the Ricci curvature is bounded from below by -1.

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Conjecture – sometimes question (Gromov)

For every dimension d there is a constant $C(d) > 0$ such that every closed d-dimensional **aspherical** manifold M satisfies

$$\beta_p^{(2)}(M) \leq C(d)\|M\| \quad \text{for every } p \geq 0.$$

Status & Goals

- No conceptual strategy for proving the conjecture – so far.
- Focus on (conjectural) corollaries instead.
- Expand scope to other invariants (ℓ^2-torsion, homology growth)
- Expand scope by relaxing geometric conditions.

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Prototypical result

For every dimension d there is a constant $C(d) > 0$ such that every closed aspherical d-dimensional Riemannian manifold M satisfies

$$\beta_p^{(2)}(M) \text{ or other homological invariant } \leq C(d) \cdot \mathrm{vol}(M)$$

provided some curvature condition holds.

A method for bounding homology of M

1. **Cover M by open balls** $\mathcal{U}$ (using geometry of M).
2. Control Lipschitz constant of nerve map $f\colon M \to \mathrm{nerve}(\mathcal{U})$.

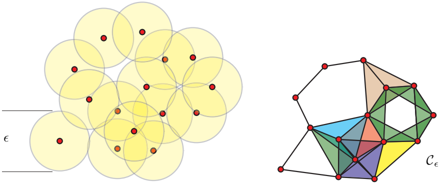

Figure: R. Ghrist: *Barcodes: The persistent topology of data*

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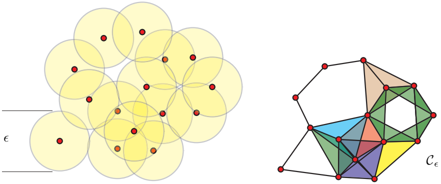

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3. Homotope f to d-skeleton keeping **Lipschitz control**.
4. Number of d-simplices hit by f is $\leq \text{Lip(f)}^d \cdot \text{vol}(M)$.
5. Using asphericity we construct:

$$M \underset{g}{\overset{f}{\rightleftarrows}} \text{nerve}(\mathcal{U}) \qquad g \circ f \simeq \text{id}_M$$

6. Betti numbers of M bounded by $\leq \text{Lip(f)}^d \cdot \text{vol}(M)$.

Adjusting the method to ℓ^2-Betti numbers

Differences

- For ℓ^2-Betti number we have to work equivariantly on the universal covering $\widetilde{M}$. This will be harder.
- For finding a left homotopy inverse this makes life slightly easier.

Covers versus packings

- Our covers often arise from maximal packings on $\widetilde{M}$ by balls (e.g. of a fixed radius r) by taking concentric balls 3 times as big.
- No equivariant packing by r-balls if $r >$ injectivity radius!

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Randomization

- Consider **equivariant random covers**, i.e. a $\pi_1(M)$-invariant probability measure on the space of covers of $\widetilde{M}$ and the resulting **random field of nerves**.
- Then use **Gaboriau's theory** to push through the method before.

Two theorems based on this method

Theorem

For every dimension d there is a constant $C(d) > 0$ such that every closed d-dimensional aspherical Riemannian manifold M satisfies

$$\beta_p^{(2)}(M) \leq C(d) \cdot \mathrm{vol}(M) \quad \text{for every } p \geq 0.$$

provided the Ricci curvature is bounded from below by -1.

Theorem

For every dimension d there is a constant $\epsilon(d) > 0$ such that every closed d-dimensional aspherical Riemannian manifold M with $\mathrm{vol}(M) < \epsilon(d)$ satisfies

$$\beta_p^{(2)}(M) = 0 \;\; \text{for } p \geq 0$$

provided the Ricci curvature is bounded from below by -1.

Equivariant random covers

First Theorem

- Let (X, μ) be any probability space with an essentially free, measure-preserving action of $\pi_1(M)$.
- Take maximal equivariant packing of $X \times \widetilde{M}$ by sets of the form (Borel set)$\times$ 1-ball. This is also maximal non-equivariantly!
- Take push-forward of μ under $X \to \{\text{Packings by 1-balls on } \widetilde{M}\}$.

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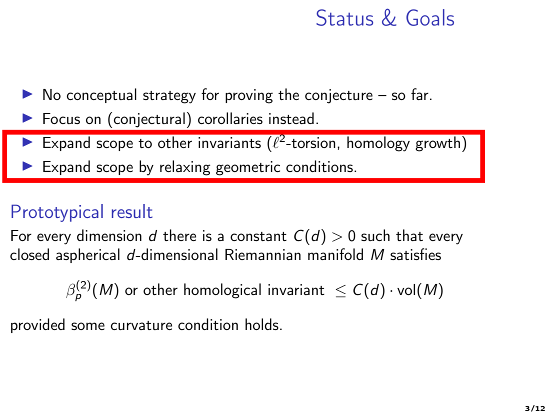
Second Theorem

- Margulis lemma for Ricci curvature: M is covered by amenable (virtually nilpotent) subsets U_i with multiplicity $\leq d$.
- Assemble packings on each $X \times \mathrm{pr}^{-1}(U_i)$.
- May assume $\pi_1(M)$ amenable. Then take packing of $X \times \widetilde{M} \sim X \times \pi_1(M)$ from Ornstein-Weiss-Rokhlin lemma.

More recent developments

Next we want to expand the scope by

1. by relaxing the Ricci curvature condition,
2. by considering torsion homology growth.

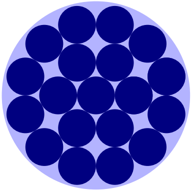


The role of curvature

Sectional (metric, macroscopic)

Ricci (metric-measure, macroscopic)

- The Ricci curvature of a tangent vector is an average of sectional curvatures.
- Bishop-Gromov inequality
  $\rightsquigarrow$ Packing inequality (1-balls in 5-ball)
  $\rightsquigarrow$ Bound on dimension of nerve



Scalar (measure, microscopic)

- Scalar curvature at a point is an average of Ricci curvatures.
- Volume of small balls:

$$\mathrm{vol}(B(r;p)) = \mathrm{vol}(B^e(r))\Big(1 - \frac{\mathrm{scal}(p)}{6(d+2)}r^2 + o(r^2)\Big)$$

- Conjecture: scalar curvature version of main inequality.

Macroscopic scalar curvature and ℓ^2

Macroscopic scalar curvature

The **macroscopic scalar curvature at** $p \in M$ **at scale** r is the real number S such that the r-ball in the (scaled) model space ($\mathbb{H}^d$, $\mathbb{E}^d$, $\mathbb{S}^d$) with scalar curvature S has the same volume as the r-ball around $\tilde{p}$ in $\widetilde{M}$.

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The general case of the following theorem will appear in the PhD thesis of Sabine Braun.

Theorem

For every dimension d there is a constant $C(d) > 0$ such that for every closed aspherical Riemannian manifold M we have

$$\beta_p^{(2)}(M) \leq C(d) \cdot \mathrm{vol}(M) \quad \text{for every } p \geq 0.$$

provided the macroscopic scalar curvature at scale 1 is ≥ -1.

On the proof

Good covers

A ball $B(r)$ is **good** if

1. $\mathrm{vol}(B(100r)) \leq 10^{4(d+3)} B(100^{-1}r)$,
2. $\mathrm{vol}(B(r)) \leq V(1) r^{d+3}$,
3. $r \leq 1/100$.

Apply Vitali covering lemma to the set of all good balls ($\rightarrow$ Gromov).

Some features

- Randomized equivariant version.
- Random field of nerve which are metric cube complexes.
- Field of nerve maps is Lipschitz-controlled on a high volume set ($\rightarrow$ **Guth**).

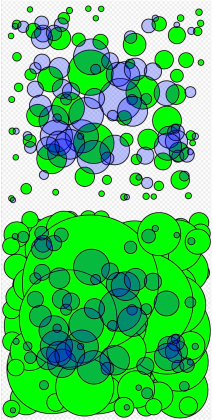

Figure: created by Claudio Rocchini

Torsion

All the theorems before possess a version where you replace $\beta_p^{(2)}(M)$ by

$$\lim_{i\to\infty} \frac{\log|\operatorname{tors} H_p(M_i;\mathbb{Z})|}{\deg(M_i \to M)}$$

with (M_i) being a residual tower of regular finite coverings of M.

Conjecture

All theorems are true when one replaces ℓ^2-Betti numbers by ℓ^2-torsion in the case of ℓ^2-acyclic manifolds.