Volume versus $\ell^2$-Betti numbers

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Motivation

Gromov’s main inequality

For every dimension $d$ there is a constant $C(d) > 0$ such that every closed $d$-dimensional Riemannian manifold $M$ satisfies

$$\|M\| \leq C(d) \cdot \text{vol}(M)$$

provided the Ricci curvature is bounded from below by $-1$. 
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Conjecture – sometimes question (Gromov)
For every dimension $d$ there is a constant $C(d) > 0$ such that every closed $d$-dimensional aspherical manifold $M$ satisfies

$$\beta_p^{(2)}(M) \leq C(d)\|M\| \quad \text{for every } p \geq 0.$$
Status & Goals

- No conceptual strategy for proving the conjecture – so far.
- Focus on (conjectural) corollaries instead.
- Expand scope to other invariants ($\ell^2$-torsion, homology growth)
- Expand scope by relaxing geometric conditions.
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Prototypical result
For every dimension $d$ there is a constant $C(d) > 0$ such that every closed aspherical $d$-dimensional Riemannian manifold $M$ satisfies

$$\beta_p^{(2)}(M) \text{ or other homological invariant} \leq C(d) \cdot \text{vol}(M)$$

provided some curvature condition holds.
A method for bounding homology of $M$

1. **Cover** $M$ by open balls $\mathcal{U}$ (using geometry of $M$).
2. Control Lipschitz constant of nerve map $f : M \rightarrow \text{nerve}(\mathcal{U})$.

**Figure:** R. Ghrist: *Barcodes: The persistent topology of data*
A method for bounding homology of $M$

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**Figure:** R. Ghrist: *Barcodes: The persistent topology of data*

3. Homotope $f$ to $d$-skeleton keeping **Lipschitz control**.
4. Number of $d$-simplices hit by $f$ is $\leq \text{Lip}(f)^d \cdot \text{vol}(M)$.
5. Using asphericity we construct:

$$M \xrightarrow{f} \text{nerve}(\mathcal{U}) \xleftarrow{g} g \circ f \simeq \text{id}_M$$

6. Betti numbers of $M$ bounded by $\leq \text{Lip}(f)^d \cdot \text{vol}(M)$.
Adjusting the method to $\ell^2$-Betti numbers

Differences

- For $\ell^2$-Betti number we have to work equivariantly on the universal covering $\tilde{M}$. This will be harder.
- For finding a left homotopy inverse this makes life slightly easier.

Covers versus packings

- Our covers often arise from maximal packings on $\tilde{M}$ by balls (e.g. of a fixed radius $r$) by taking concentric balls 3 times as big.
- No equivariant packing by $r$-balls if $r >$ injectivity radius!
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Randomization

- Consider **equivariant random covers**, i.e. a $\pi_1(M)$-invariant probability measure on the space of covers of $\tilde{M}$ and the resulting **random field of nerves**.
- Then use **Gaboriau’s theory** to push through the method before.
Two theorems based on this method

**Theorem**
For every dimension $d$ there is a constant $C(d) > 0$ such that every closed $d$-dimensional aspherical Riemannian manifold $M$ satisfies

$$\beta_p^{(2)}(M) \leq C(d) \cdot \text{vol}(M) \quad \text{for every } p \geq 0.$$  

provided the Ricci curvature is bounded from below by $-1$.

**Theorem**
For every dimension $d$ there is a constant $\epsilon(d) > 0$ such that every closed $d$-dimensional aspherical Riemannian manifold $M$ with $\text{vol}(M) < \epsilon(d)$ satisfies

$$\beta_p^{(2)}(M) = 0 \quad \text{for } p \geq 0$$

provided the Ricci curvature is bounded from below by $-1$. 
Equivariant random covers

First Theorem

- Let \((X, \mu)\) be any probability space with an essentially free, measure-preserving action of \(\pi_1(M)\).
- Take maximal equivariant packing of \(X \times \tilde{M}\) by sets of the form (Borel set)\(\times 1\)-ball. This is also maximal non-equivariantly!
- Take push-forward of \(\mu\) under \(X \to \{\text{Packings by 1-balls on } \tilde{M}\}\).
Equivariant random covers

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► Take push-forward of \(\mu\) under \(X \rightarrow \{\text{Packings by 1-balls on } \tilde{M}\}\).

Second Theorem

► Margulis lemma for Ricci curvature: \(M\) is covered by amenable (virtually nilpotent) subsets \(U_i\) with multiplicity \(\leq d\).

► Assemble packings on each \(X \times \text{pr}^{-1}(U_i)\).

► May assume \(\pi_1(M)\) amenable. Then take packing of \(X \times \tilde{M} \sim X \times \pi_1(M)\) from Ornstein-Weiss-Rokhlin lemma.
More recent developments

Next we want to expand the scope by
1. by relaxing the Ricci curvature condition,
2. by considering torsion homology growth.

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provided some curvature condition holds.
The role of curvature

Sectional (metric, macroscopic)

Ricci (metric-measure, macroscopic)

- The Ricci curvature of a tangent vector is an average of sectional curvatures.
- Bishop-Gromov inequality
  - Packing inequality (1-balls in 5-ball)
  - Bound on dimension of nerve

Scalar (measure, microscopic)

- Scalar curvature at a point is an average of Ricci curvatures.
- Volume of small balls:

\[
\text{vol}(B(r; p)) = \text{vol}(B^e(r)) \left(1 - \frac{\text{scal}(p)}{6(d + 2)} r^2 + o(r^2)\right)
\]

- Conjecture: scalar curvature version of main inequality.
Macroscopic scalar curvature and $\ell^2$

Macroscopic scalar curvature

The **macroscopic scalar curvature at** $p \in M$ **at scale** $r$ is the real number $S$ such that the $r$-ball in the (scaled) model space $(\mathbb{H}^d, \mathbb{E}^d, \mathbb{S}^d)$ with scalar curvature $S$ has the same volume as the $r$-ball around $\tilde{p}$ in $\tilde{M}$. 
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The general case of the following theorem will appear in the PhD thesis of Sabine Braun.

**Theorem**

For every dimension $d$ there is a constant $C(d) > 0$ such that for every closed aspherical Riemannian manifold $M$ we have

$$\beta_p^{(2)}(M) \leq C(d) \cdot \text{vol}(M) \quad \text{for every } p \geq 0.$$ 

provided the macroscopic scalar curvature at scale 1 is $\geq -1$. 


On the proof

Good covers

A ball $B(r)$ is **good** if

1. $\text{vol}(B(100r)) \leq 10^{4(d+3)}B(100^{-1}r)$,
2. $\text{vol}(B(r)) \leq V(1)r^{d+3}$,
3. $r \leq 1/100$.

Apply Vitali covering lemma to the set of all good balls ($\rightarrow$ Gromov).

Some features

- Randomized equivariant version.
- Random field of nerve which are metric cube complexes.
- Field of nerve maps is Lipschitz-controlled on a high volume set ($\rightarrow$ Guth).

Figure: created by Claudio Rocchini
All the theorems before possess a version where you replace $\beta_p^{(2)}(M)$ by

$$\lim_{i \to \infty} \frac{\log |\text{tors } H_p(M_i; \mathbb{Z})|}{\deg(M_i \to M)}$$

with $(M_i)$ being a residual tower of regular finite coverings of $M$.

**Conjecture**

All theorems are true when one replaces $\ell^2$-Betti numbers by $\ell^2$-torsion in the case of $\ell^2$-acyclic manifolds.