MOD-p METHODS AND THE p-GRADIENT

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Abstract. This is a preliminary draft of the course "Mod-p methods and the p-gradient" imparted during the school of the thematic program "L^2-invariants and their analogues in positive characteristic" (Madrid, 19 Febraury - 15 June, 2018).

Let G be a group, K a field and A a n by m matrix over the group ring $K[G]$. Let $G = G_1 > G_2 > G_3 \cdots$ be a sequence of normal subgroups of G of finite index with trivial intersection. The multiplication on the right side by A induces linear maps

$$\begin{array}{rccc} \phi^A_{G/G_i}: & K[G/G_i]^n & \to & K[G/G_i]^m \\ & (v_1,\dots,v_n) & \mapsto & (v_1,\dots,v_n)A. \end{array}$$

We are interested in properties of the sequence $\{\frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|}\}$. These numbers appear naturally in the study of dimensions of the homology groups $H_p(G_i\backslash X, K)$ where X is G-CW-complex of finite type. In particular, we would like to address the following questions:

(1) Is there the limit $\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|}$?
(2) If the limit exists, how does it depend on the chain $\{G_i\}$?
(3) How can we express $\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|}$ in terms of G?

It turns out that the answers on these three questions are known if we assume that K is of characteristic 0.

In this course I will present what is known in the case where K is of positive characteristic.

Contents

1. The p-gradient, examples and motivations

In this note p will always denote a prime natural number. Let G be a finitely generated group and let $G = G_1 > G_2 > G_3 > \dots$ be a chain of subgroups. In this course we will always assume that all the subgroups G_i are of *finite index* in G. We will say that the chain $\{G_i\}$ is **normal** if all the subgroups G_i are normal in G, **subnormal** if for all $i \geq 1$, G_{i+1} is normal in G_i and **exhausting** if the intersection of G_i is trivial. A normal chain $\{G_i\}$ is called **pro-p** if the profinite completion of G with respect to $\{G_i\}$ is a pro-p group. In the same way we can define a **p-adic** chain or a **virtually pro-p** chain.

The first homology group of G with coefficients in a field K is defined as

$$H_1(G, K) = K \otimes_{\mathbb{Z}} G/[G,G].$$

The **p-gradient of G with respect to a chain** $\{G_i\}$ is the limit

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} H_1(G_i, \mathbb{F}_p)}{|G : G_i|},$$

if such limit exists. We will denote it by $RG_p(G, \{G_i\})$. The notion of p-gradient was introduced by Marc Lackenby in his study of 3-manifold groups. He was first who realized that this invariant may reflect structural properties of finitely presented groups. Let us show one such example.

Recall that a group G is called **large** if a subgroup of finite index of G maps onto a non-abelian free group. The following interesting criterion of largeness was proved by M. Lackenby in [14].

Theorem 1.1. *Let G be a finitely presented group and let $\{G_i\}$ be a subnormal chain of subgroups of G of p-power index. Assume that*

1. *G does not have property (τ) with respect to $\{G_i\}$ and*
2. *$RG_p(G, \{G_i\}) > 0$.*

Then G is large.

As an immediate consequence of the previous result we obtain the following well-known statement.

Corollary 1.2. *A finitely presented group is large if and only if a subgroup of finite index maps on the wreath product $\mathbb{Z} \wr C_p$.*

Proof. The "if only" part is clear. Let us show the "if" part.

Assume that a finitely presented group G maps onto $\mathbb{Z} \wr C_p$. Then it maps on $\mathbb{Z}$. Let G_i be the preimage of $p^i\mathbb{Z}$. It is clear that G does not have property (τ) with respect to $\{G_i\}$ and an easy calculation in the wreath product $\mathbb{Z} \wr C_p$ shows that $RG_p(G, \{G_i\}) \geq 1$. Thus, by the previous theorem G is large. $\square$

This example shows the importance of the p-gradient. Unfortunately, its calculation is not always an easy task. I will present now one example, which motivates a lot of research in the subject.

Let G be an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$. These groups will be studied in more detail in the second half of the school in the courses of Steffen Kionke and Haluk Sengun. For our purposes we can think that G is $\mathrm{SL}_2(\mathbb{Z}[i])$ or the fundamental group of the figure eight knot (which can be embedded as a subgroup of finite index in $\mathrm{SL}_2(\mathbb{Z}[\sqrt[3]{1}])$).

Using the arithmetic structure of G we can define the subgroups

$$G_i = G(p^i) = \{A \in G : A \equiv I \pmod{p^i}\}.$$

Then $\{G_i\}$ is a p-adic exhausting chain. In [5] F. Calegari and M. Emerton conjectured that $RG_p(G, \{G_i\}) = 0$. Their interest in this invariant comes from the theory of automorphic forms and Galois representations [6] (this will be explained in more detail in the course of Haluk Sengun). Another motivation is the paper [11], where it is shown that if the Calegari-Emerton conjecture holds, then the congruence kernel of any arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ is a projective profinite group. There exists some heuristic explanation for the equality $RG_p(G, \{G_i\}) = 0$ (see for example [3]). In the last section we will come back to this conjecture and will provide an algebraic "evidence" for this equality.

The study or the p-gradient of a finitely presented group has the following natural generalization which we will describe now. Let G be a finitely presented group and let $\{G_i\}$ be a normal exhausting chain. Any presentation of a group G with d generators and r relations induces a resolution of the trivial (left) $\mathbb{Z}[G]$-module $\mathbb{Z}$, which we can use to calculate $H_1(G, K)$ (for some field K).

$$\mathbb{Z}[G]^r \stackrel{\phi_A}{\to} \mathbb{Z}[G]^d \to \mathbb{Z}[G] \to \mathbb{Z} \to 0. \tag{1}$$

Here ϕ_A denotes the homorphism realizing the multiplication by a matrix A. After tensoring (1) with $K[G/G_i]$ over $\mathbb{Z}[G]$, we obtain the sequence

$$K[G/G_i]^r \stackrel{\phi^A_{G/G_i}}{\to} K[G/G_i]^d \stackrel{\alpha_i}{\to} K[G/G_i] \to K \to 0$$

and $H_1(G, K) \cong \ker \alpha_i / \operatorname{Im} \phi^A_{G/G_i}$. A direct calculation implies that

$$\dim_K H_1(G_i, \mathbb{F}_p) = \dim_K \ker \phi^A_{G/G_i} + (d - r - 1)|G : G_i| - 1.$$

In particular, when $K = \mathbb{F}_p$,

$$RG_p(G, \{G_i\}) = d + r - 1 + \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \ker \phi^A_{G/G_i}}{|G : G_i|}.$$

From now on, we will consider $\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G : G_i|}$ for a general matrix $A \in \mathrm{Mat}_{n\times m}(K[G])$ over $K[G]$ (we also will not assume that G is finitely presented). However, we would like to warn the reader that any approach in the study of the p-gradient, that does not take into account additional properties of matrices proceeding from a presentation of the group G, may loss some useful information that can be essential, for example, in the solution of the Calegari-Emerton conjecture.

In this general situation we would like to consider the following three questions.

(1) Is there the limit $\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G : G_i|}$?

(2) If the limit exists, how does it depend on the chain $\{G_i\}$?

(3) How can we express $\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|}$ in terms of G?

It turns out that the answers on these three questions are known if we assume that K is of characteristic 0. The limit is always exists and it does not depend on the chain. Since A has only finitely many entries, we may assume that K is a finitely generated filed. Consider any embedding of K into $\mathbb{C}$, so A becomes a matrix over $\mathbb{C}$. Then we also have the equality

$$\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|} = \dim_G \ker \phi^A_G.$$

where $\dim_G$ is the von Neumann dimension of G-Hilbert module $\ker \phi^A_G$ and $\phi^A_G : l^2(G)^n \to l^2(G)^m$ is the morphism induced by the right multiplication by A.

This was proved when $K = \mathbb{Q}$ by W. Lück [16], when $K = \bar{\mathbb{Q}}$ by J. Dodziuk, P. Linnell, V. Mathai, T. Schick and S. Yates [8] and when K is an arbitrary field of characteristic 0 by A. Jaikin-Zapirain [12]. The fourth advanced course of the thematic program will be dedicated to the proof of this result.

We conjecture that the two first questions have the same answers in positive characteristic as in the characteristic 0. This already will imply the Calegari-Emerton conjecture. Indeed, let G be an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ and let $G_i = G(p^i)$. By recent advances in the theory of 3-manifolds [2, 22], we know that G is virtual finitely generated by cyclic. This, in particular, implies that there exists a normal exhausting chain $\{G'_i\}$ in G such that $RG_p(G, \{G'_i\}) = 0$. (In fact, the existence of such chain can be also showing using less sophisticated methods which are presented in [11]). Thus, the independence of the p-gradient of the normal exhausting chain would imply that $RG_p(G, \{G_i\}) = 0$ as well.

At this moment we can prove the existence of the limit $\lim_{i\to\infty} \frac{\dim_K \ker \phi^A_{G/G_i}}{|G:G_i|}$ and its independence of the normal exhausting chain $\{G_i\}$ only when G is amenable. This will be explained in Section 3. Also we can show that if the chain $\{G_i\}$ is virtually pro-p then the limit in the first question always exists. We will see it in Section 4. It is not clear what should be the answer in general on the third question when K is of positive characteristic. We will answer the third question when G is elementary amenable such that there exists a bound of the orders of finite subgroups (see Section 3) or the chain $\{G_i\}$ is p-adic (see Section 4). We have a good candidate for the answer on the third question when G is a free group. In Section 5 we will explain how this is already enough in order to solve the Calegari-Emerton conjecture.

2. Preliminaries

In this section we collect some basic notions and results that will be used later on in the paper.

2.1. **Amenable and elementary amenable groups.** The group G is **amenable** if there exists a family of finite subsets $\{F_i\}$ such that for every $g \in G$,

$$\lim_{i\to\infty} \frac{|F_i \cap F_i g|}{|F_i|} = 1.$$

Elementary amenable groups form a large subclass of amenable groups. They form the smallest subclass of the class of all groups that closed under isomorphisms, contains all finite and all abelian groups, closed under the operations of taking subgroups, forming quotients, and forming extensions and closed under directed unions.

2.2. **Sofic groups.** Let F be a free finitely generated group and assume that it is freely generated by a set S. Recall that an element w of F has **length** n if w can be expressed as a product of n elements from $S\cup S^{-1}$ and n is the smallest number with this property.

Let N be a normal subgroup of F and $G=F/N$. We say that G is **sofic** if there is a family $\{X_k\}_{k\in\mathbb{N}}$ of finite F-sets (F acts on the right) such that if we put

$$T_{k,s}=\{x\in X_k:\ x=x\cdot w \text{ if } w\in B_s(1_F)\cap N, \text{ and } \ x\neq x\cdot w \text{ if } w\in B_s(1_F)\setminus N\},$$

then for every s,

$$\lim_{k\to\infty}\frac{|T_{k,s}|}{|X_k|}=1.$$

The family of F-sets $\{X_k\}$ is called a **sofic approximation** of G.

This is one of many equivalent definitions of soficity for a finitely generated group. We reccommend [21] where many different definitions of soficity can be found.

Our definition has the following geometric meaning. The action of F on X_k converts X_k in an $S^{\pm 1}$-labeled graph. Let $T'_{k,s}$ be the set of vertices x of X_k such that the s-ball $B_s(x)$ in X_k and the s-ball $B_s(1_G)$ in G are isomorphic as $S^{\pm1}$-labeled graphs. It is clear that

$$T'_{k,s}\subseteq T_{k,s}\subseteq T'_{k,2s}.$$

Thus, the soficity condition says that for every s most of the vertices of X_k are in $T'_{k,s}$ when k tends to infinity.

For an arbitrary group G we say that G is **sofic** if every finitely generated subgroup of G is sofic. Amenable groups and residually finite groups are sofic. It is important to note that no nonsofic group is known at this moment.

2.3. **The Ore rings of fractions.** In this subsection we recall the definition of the left Ore condition and the construction of the Ore ring of fractions.

An element $r\in R$ is a **non-zero-divisor** if there exists no non-zero element $s\in R$ such that $rs=0$ or $sr=0$. Let T be a multiplicative subset of non-zero-divisors of R. We say that (T,R) satisfies the **left Ore condition** if for every $r\in R$ and every $t\in T$, the intersection $Tr\cap Rt$ is not empty. If T consists of all the non-zero-divisors we simply say that R satisfies the **left Ore condition**.

The goal is to construct the **left Ore ring of fractions** $T^{-1}R$. Let us recall briefly this construction. For more details the reader may consult [18, Chapter 2]. As a set, $T^{-1}R$ coincides with the set of equivalence classes in $T\times R$ with respect to the following equivalence relation:

$$(t_1,r_1)\equiv(t_2,r_2) \text{ if and only if there are } r'_1,r'_2\in R \text{ such that } \ r'_1t_1=r'_2t_2\in T \text{ and } \ r'_1r_1=r'_2r_2.$$

The equivalence class of (t,a) is denoted by $t^{-1}a$. Note that there is no obvious interpretation for the sum $s^{-1}a+t^{-1}r$ and the product $(t^{-1}r)(s^{-1}a)$ $(a,r\in R,$

$s, t \in T$). In order to sum $s^{-1}a$ and $t^{-1}r$, we observe that for every $s, t \in T$ there exists $s', t' \in R$ such that $s's = t't \in T$. Hence,

$$s^{-1}a + t^{-1}r = (s's)^{-1}s'a + (t't)^{-1}t'r = (s's)^{-1}(s'a + t'r)$$

In order to multiply $s^{-1}a$ and $t^{-1}r$, we rewrite rs^{-1} as a product $(s_0)^{-1}r_0$ with $r_0 \in R$ and $s_0 \in T$. The condition $Tr \cap Rs$ is not trivial implies exactly the existence of $s_0 \in T$ and $r_0 \in R$ such that $s_0 r = r_0 s$, and so $rs^{-1} = (s_0)^{-1}r_0$. Hence,

$$(t^{-1}r)(s^{-1}a) = (t^{-1})(s_0)^{-1}r_0 a = (s_0 t)^{-1}r_0 a.$$

When T consists of all the non-zero-divisors of R and (T, R) satisfies the left Ore condition, we denote $T^{-1}R$ by $Q_l(R)$ and we call it **the left classical ring of fractions** of R.

2.4. **Sylvester module rank functions.** A **Sylvester rank function** dim is a function that assigns a non-negative real number to each finitely presented R-module and satisfies the following conditions.

(SMod1) $\dim(\{0\}) = 0$, $\dim(R) = 1$;
(SMod2) $\dim(M_1 \oplus M_2) = \dim M_1 + \dim M_2$;
(SMod3) if $M_1 \to M_2 \to M_3 \to 0$ is exact then

$$\dim M_1 + \dim M_3 \geq \dim M_2 \geq \dim M_3.$$

Let K be a field. If G is a group and N a normal subgroup of G of finite index, then we denote by $\dim_{G/N}$ the Sylvester module rank function on $K[G]$ by means of

$$\dim_{G/N}(M) = \frac{\dim_K K[G/N] \otimes_{K[G]} M}{|G : N|}.$$

Then we can reformulate the first question raised in Section 1 in the following way. Let $\{G_i\}$ be a normal exhausting chain in G and let M be a finitely presented $K[G]$-module Is there the limit $\lim_{i\to\infty} \dim_{G/G_i}(M)$? Observe if such limit exists for all M, then $\lim_{i\to\infty} \dim_{G/G_i}$ is again a Sylvester module rank function on $K[G]$. Thus, we can expect that if the limit does not depend on the chain $\{G_i\}$, this Sylvester module rank function should be "canonical".

There exists a natural generalization of the notion $\dim_{G/N}$. Assume G acts (on the right) on a finite set X. Then if M is a finitely presented $K[G]$-module we put

$$\dim_X(M) = \frac{\dim_K K[X] \otimes_{K[G]} M}{|X|}.$$

The set of Sylvester module rank functions of a ring R is denoted by $\mathbb{P}(R)$. For any morphism $\alpha : R \to S$ we denote by $\alpha^{\#} : \mathbb{P}(S) \to \mathbb{P}(R)$ the map defined by means of

$$\alpha^{\#}(\dim)(M) = \dim(S \otimes_R M).$$

In this note we will call a Sylvester module rank function dim **dimension** if it satisfies

(SMod3′) given a surjection $\phi : M \twoheadrightarrow N$ between two finitely presented R-modules,

$$\dim M - \dim N = \inf\{\dim L : \ L \twoheadrightarrow \ker \phi \text{ and } L \text{ is finitely presented}\}.$$

For example, a simple Artinian ring has a unique Sylvester module rank function that is also a dimension.

An equivalent way to introduce a Sylvester rank functionon a ring R is to present a **Sylvester matrix rank function**, which is a function rk that assigns a non-negative real number to each matrix over R and satisfies the following properties.

(SMat1) $\mathrm{rk}(A) = 0$ if A is any zero matrix and $\mathrm{rk}(1) = 1$;

(SMat2) $\mathrm{rk}(A_1A_2) \le \min\{\mathrm{rk}(A_1), \mathrm{rk}(A_2)\}$ for any matrices A_1 and A_2 which can be multiplied;

(SMat3) $\mathrm{rk}\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} = \mathrm{rk}(A_1) + \mathrm{rk}(A_2)$ for any matrices A_1 and A_2;

(SMat4) $\mathrm{rk}\begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix} \ge \mathrm{rk}(A_1) + \mathrm{rk}(A_2)$ for any matrices A_1, A_2 and A_3 of appropriate sizes.

Given a Sylvester module rank function dim on a ring R, the associated Sylvester matrix rank function rk is defined as follows. If $A \in \mathrm{Mat}_{n\times m}(R)$ then

$$\mathrm{rk}(A) = m - \dim(R^m/(R^n)A).$$

2.5. **Cohn's theory of epic division R-rings.** Let R be a ring. A R-ring is a homomorphism $f : R \to S$ of rings. If f is clear from the context, we will simply say that S is a R-ring. We will say that two R-rings (S_1, f_1) and (S_2, f_2) are isomorphic if there exists an isomorphism $\alpha : S_1 \to S_2$ for which the following diagram is commutative:

$$\begin{array}{lll} R & \to^{\mathrm{Id}} & R \\ \downarrow f_1 & & \downarrow f_2 \\ S_1 & \to^{\alpha} & S_2. \end{array}$$

We will say that $f : R \to D$ is a an epic division R-ring if D is generated by $f(R)$ as a division algebra. (The reader may consult [13] for explanation of the word epic in this definition.)

If R is a commutative ring, then there exists a natural bijection between $\mathrm{Spec}(R)$ and the isomorphism classes of division R-rings: a prime ideal $P \in \mathrm{Spec}(R)$ corresponds to the field of fractions $Q(R/P)$ of R/P and $f : R \to Q(R/P)$ is defined as $f(r) = r + P$ for any $r \in R$.

If R is a domain and satisfies the left Ore condition then its classical left ring of fractions $Q_l(R)$ is a division ring. Moreover, as in the commutative case, the division R-ring $Q_l(R)$ is the unique (up to R-isomorphism) faithful division R-ring. Thus, if R is a left Noetherian ring, then there exists a natural bijection between the **strong prime** ideals of R (ideals P such that R/P is a domain) and the isomorphism classes of division R-rings.

For an arbitrary ring R, P. Cohn proposed the following approach to classify division R-rings. If D is a division ring, let $\dim_D$ be the dimension on D-modules. If D is also a R-ring, abusing the notation, we denote by $\dim_D$ the Sylvester module rank function on R defined as follows

$$\dim_D(M) = \dim_D(D \otimes_R M).$$

We denote by rk_D the Sylvester matrix rank function associated with $\dim_D$-

Theorem 2.1. [7, Theorem 4.4.1] *Let (D_1, f_1) and (D_2, f_2) be two epic division R-rings. Then the following is equivalent*

(1) *(D_1, f_1) and (D_2, f_2) are isomorphic.*

(2) *For each matrix A over R*

$$\mathrm{rk}_{D_1}(A) = \mathrm{rk}_{D_2}(A).$$

(3) *For each finitely presented R-module M*

$$\dim_{D_1}(M) = \dim_{D_2}(M).$$

(4) *For each matrix A over R, $f_1(A)$ is invertible over D_1 if and only if $f_2(A)$ is invertible over D_2.*

It is clear that a Sylvester rank functions associated with a division R-algebra takes integer values. P. Malcolmson [19] proved the converse of this assertion.

Given a ring R and a n by m matrix over A over R, the **inner rank** of A is the smallest k such that $A = BC$ where $B \in \mathrm{Mat}_{n\times k}(R)$ and $C \in \mathrm{Mat}_{k\times m}$. In general the inner rank is not a Sylvester matrix rank function. The rings where it happens are called **Sylvester domains**. In this case this Sylvester matrix rank function is associated with an embedding of the ring R into a division algebra D, that is called **universal** division R-ring. If R is the free K-algebra on the set X, then R is a Sylvester domain and its universal division algebra is isomorphic to a free division K-algebra over X:

2.6. **Limit with respect to an ultrafilter.** Given a set X, an **ultrafilter** on X is a set ω consisting of subsets of X such that

(1) the empty set is not an element of ω;
(2) if A and B are subsets of X, A is a subset of B, and A is an element of ω, then B is also an element of ω;
(3) if A and B are elements of ω, then so is the intersection of A and B;
(4) if A is a subset of X, then either A or $X \setminus A$ is an element of ω.

If $a \in X$, we can define $\omega_a = \{A \subseteq X : a \in A\}$. It is a ultrafilter, called a **principal** ultrafilter. It is a known fact that if X is infinite, then the axiom of choice implies the existence of a non-principal ultrafilter.

Let ω be a ultrafilter on X and $\{a_i \in \mathbb{R}\}_{i\in X}$ a family of real numbers. We write $a = \lim\limits_{\omega} a_i$ if for any $\epsilon > 0$ the set $\{i \in X : |a - a_i| < \epsilon\}$ is an element of the ultrafilter w. It is not difficult to see that for any bounded family $\{a_i \in \mathbb{R}\}_{i\in X}$ there exists a unique $a \in \mathbb{R}$ such that $a = \lim\limits_{\omega} a_i$.

The limit with respect to an ultrafilter will be used in several situations in this paper. For example, let $\{X_i\}_{i\in\mathbb{N}}$ be a collection of finite G-sets. Then, in general, there is no $\dim_{i\to\infty} \dim_{X_i}$. However, if ω is a ultrafilter on $\mathbb{N}$ the $\lim\limits_{\omega} \dim_{X_i}$ make sense and it is a Sylvester module rank function on $K[G]$.

3. The case of amenable groups

In this section we explain the proof of the following theorem.

Theorem 3.1. *Let K be a field and F a finitely generated free group. Let $\{X_k\}_{k\in\mathbb{N}}$ be a family of finite F-sets. Assume that $\{X_k\}$ approximates an amenable group $G = F/N$.*

(1) *For every finitely presented module M of $K[F]$, there exists the limit*

$$\lim_{k\to\infty} \dim_{X_k}(M),$$

which does not depend on the sofic approximation $\{X_k\}$ of G.

(2) *Assume, in addition, that G is elementary amenable and the orders of finite subgroups of G are bounded. Then $K[G]$ satisfies the left Ore condition, the ring $Q = Q_l(K[G])$ is Artinian and there exists a dimension* $\dim_Q$ *on Q such that*

$$\lim_{k\to\infty} \dim_{X_k}(M) = \dim_Q(Q \otimes_{K[F]} M).$$

Observe that if $\alpha : K[F] \to K[G]$ denotes the natural homomorphism Then, $\alpha^{\#} : \mathbb{P}(K[G]) \to \mathbb{P}(K[F])$ is injective and $\lim_{k\to\infty} \dim_{X_k} \in \operatorname{Im} \alpha^{\#}$.

The first part of Theorem 3.1 is [13, Theorem 7.1] and the second part is [13, Corollary 9.4]. See [13] for some more details on the history behind this result. In this note we will only consider the first part of Theorem 3.1 .

3.1. **Sofic approximations of amenable groups.** The main idea of the proof of the first part of Theorem 3.1 is to show that any two sofic approximations of a given amenable group are very similar. This was proved by G. Elek and E. Szabó in [9]. Let us formulate their result.

Let X be a finite set. The **Hamming distance** on $\operatorname{Sym}(X)$ is defined as follows.

$$\mathrm{d_H}(\sigma,\tau) = \frac{|\{x \in X :\ \sigma(x) \neq \tau(x)\}|}{|X|}.$$

Assume now that F is a finitely generated free group and let $\{X_i\}$ be a sofic approximation of $G = F/N$. Fix a non-principal ultrafilter on $\mathbb{N}$ and let d_ω be the pseudo-distance on $\prod_i \operatorname{Sym}(X_i)$:

$$\mathrm{d}_\omega((\sigma_i),(\tau_i)) = \lim_\omega \mathrm{d_H}(\sigma_i,\tau_i).$$

We put $N_\omega = \{\sigma \in \prod_i \operatorname{Sym}(X_i) :\ \mathrm{d}_\omega(\sigma,1) = 0\}$ and $\Sigma_\omega = \prod_i \operatorname{Sym}(X_i)/N_\omega$. The actions of F on X_i induce a homomorphism $\psi_{\{X_i\},\omega} : F \to \Sigma_\omega$. Clearly $\ker \psi_{\{X_i\},\omega} = N$.

Now, let $\{X_i^1\}$ and $\{X_i^2\}$ be two sofic approximations of $G = F/N$. We put $Y_i^1 = Y_i^2 = X_i^1 \times X_i^2$ and let F act on Y_i^1 by acting only on the first coordinate and F act on Y_i^2 by acting only on the second coordinate. Then $\{Y_i^1\}$ and $\{Y_i^2\}$ are two approximations of F/N.

Theorem 3.2. ([9, Theorem 2]) *The representations $\psi_{\{Y_i^1\},\omega}$ and $\psi_{\{Y_i^2\},\omega}$ are conjugate.*

The proof of this theorem uses in an essential way the results of a fundamental work of D. Ornstein and B. Weiss [20] on amenable groups.

3.2. **Proof of the first part of Theorem 3.1.** Observe that an infinite subfamily of a family that approximates a group G also approximates G. Thus, if the limit $\lim_{k\to\infty} \dim_{X_k}(M)$ does not exists or it depends on the sofic approximation $\{X_i\}$, we will be able to find two families $\{X_i^1\}_{i\in\mathbb{N}}$ and $\{X_i^2\}_{i\in\mathbb{N}}$ such that the limits $\lim_{i\to\infty} \dim_{X_i^1}(M)$ and $\lim_{i\to\infty} \dim_{X_i^2}(M)$ exist but they are different.

Let us use the notation of Theorem 3.2. Then clearly

$$\dim_{X_i^1} = \dim_{Y_i^1} \ \text{ and } \ \dim_{X_i^2} = \dim_{Y_i^2}.$$

On the other hand, Theorem 3.2 implies that

$$\lim_\omega \dim_{Y_i^1}(M) = \lim_\omega \dim_{Y_i^2}(M)$$

for any non-principal ultrafilter ω on $\mathbb{N}$ (prove this! or look at the proof of [13, Theorem 7.1]). Thus,

$$\lim_{i\to\infty} \dim_{X_i^1}(M) = \lim_{\omega} \dim_{Y_i^1}(M) = \lim_{\omega} \dim_{Y_i^2}(M) = \lim_{i\to\infty} \dim_{X_i^2}(M).$$

We have obtained a contradiction.

4. Virtually pro-p and p-adic analytic chains

In this section we will see that the first question asked in Section 1 has a positive answer for virtually pro-p chains if char$K = p$. This suggests that probably in the case where char$K = p$, then the second and third questions should be also considered under this additional condition.

4.1. The case of virtually pro-p chains.

Let K be a field. Let G be a profinite group and N a normal open subgroup of G. Then the Sylvester module rank function $\dim_{G/N}$ can be thought also as a Sylvester module rank function on $K[[G]]$:

$$\dim_{G/N}(M) = \frac{\dim_K K[G/N] \otimes_{K[[G]]} M}{|G : N|}.$$

In this subsection we show the following.

Theorem 4.1. *Let K be a field of characteristic p. Let G be virtually pro-p group and $G = G_0 > G_1 > G_2$ an exhausting chain of open normal subgroups of G. Then for every finitely presented $K[[G]]$-module M, there exists $\lim_{i\to\infty} \dim_{G/G_i}(M)$ and this limit does not depend on the chain $\{G_i\}$.*

The proof of Theorem 4.1 depends on the following well-known fact.

Lemma 4.2. *Let K be a field of characteristic p and let A be a finitely generated $K[C_p]$-module. Then*

$$\frac{\dim_K A}{|C_p|} \le \dim_K K \otimes_{K[C_p]} A.$$

Proof. Decompose A as a direct sum of cyclic $K[C_p]$-modules. Since K has characteristic p, $\dim_K K \otimes_{K[C_p]} A$ is equal to the number of non-trivial cyclic summands. Since the dimension of each cyclic $K[C_p]$-module is at mpst p, we obtain the inequality

$$\dim_K A \le p \cdot \dim_K K \otimes_{K[C_p]} A.$$

□

Corollary 4.3. *Let K be a field of characteristic p and let P be a finite p-group. Let A be a finitely generated $K[P]$-module. Then*

$$\frac{\dim_K A}{|P|} \le \dim_K K \otimes_{K[P]} A.$$

Proof. We prove the statement by the induction on $|P|$. By Lemma 4.2, it holds if $|P| \le p$. Assume that $|P| > p$. Let Q be a normal subgroup of P of order p. Then

$$\dim_K A \overset{\text{Lemma 4.2}}{\le} p \cdot \dim_K K \otimes_{K[Q]} A.$$

Observe that $\dim_K K \otimes_{K[Q]} A$ is a $K[P/Q]$-module. Hence using the induction hypothesis, we obtain

$$\dim_K A \leq p \cdot |P/Q| \dim_K K \otimes_{K[P]} A = |P| \dim_K K \otimes_{K[P]} A.$$

□

Corollary 4.4. *Let G be a profinite group and let $N_1 \leq N_2$ two normal open subgroups of G. Assume that N_2/N_1 is a p-group. Then for every finitely presented $\mathbb{F}_p[[G]]$-module,*

$$\dim_{G/N_1}(M) \leq \dim_{G/N_2}(M).$$

Proof of Theorem 4.1. By Corollary 4.4, the limit $\lim_{i\to\infty} \dim_{G/G_i}(M)$ exists. If $\{H_i\}$ is another exhausting chain of open normal subgroups of G, then for every i there exists j such that $G_j \leq H_i$ and $H_j \leq G_i$. Hence, Corollary 4.4 implies also that $\lim_{i\to\infty} \dim_{G/G_i}(M) = \lim_{i\to\infty} \dim_{G/H_i}(M)$. □

In view of Theorem 4.1, if G is virtually pro-p group, we will denote by $\dim_G$ the Sylvester module rank function on $\mathbb{F}_p[[G]]$ that is equal to the limit $\lim_{i\to\infty} \dim_{G/G_i}$, where $\{G_i\}$ is a exhausting chain of open normal subgroups in G.

4.2. **Interpretations of $\dim_G$ in terms of division $K[[G]]$-algebras.** Let Γ be a group and let $\Gamma = \Gamma_1 > \Gamma_2 > \Gamma_3 > \dots$ be a exhausting virtually pro-p chain. Let G be the completion of Γ with resect to the chain $\{\Gamma_i\}$. In this subsection we will give a general criterion which relates $\dim_G$ with an embedding of $\mathbb{F}_p[\Gamma]$ into a division algebra.

We say that a homomorphism $V : D^* \to \mathbb{Z}$ of the multiplicative group of a division K-algebra D is a **valuation** on D if for any $a, b \in D$, $V(a+b) \geq \max\{V(a), V(b)\}$ and $V(k) = 0$ for every $k \in K^*$. We will also put $V(0) = +\infty$.

Remark. *It is useful to extend the valuation from D to $E = Q_l(D[t])$ by defining*

$$V(\sum a_i t^i) = \max_i \{V(a_i) + i\} (a_i \in D) \text{ and } V(\frac{a}{b}) = V(a) - V(b) \ (a, 0 \neq b \in E).$$

For every $i \geq 0$ we put $R_i = \{a \in D : V(a) \geq i\}$. Then R_i are principal ideals in R_0. Moreover, R_0 is a local field and R_1/R_0 is a division algebra.

Theorem 4.5. *Let K be a field of characteristic p. Let D be a division K-algebra and let V a be valuation on D. Let Γ be a finitely generated subgroup of $1 + R_1$. Let M be a finitely presented $K[\Gamma]$-module. Put $\Gamma_i = (1 + R_i) \cap \Gamma$.*

1. *$\{\Gamma_i\}$ is a exhausting pro-p-chain in Γ.*
2. *Denote by G the profinite completion of Γ with respect of $\{\Gamma_i\}$. Then*

$$\dim_G(M) \leq \dim_D(M).$$

3. *Let D_i denotes the division subalgebra of D generated by Γ_i. If for every $i \geq 1$, $\dim_{D_i} D_1 = |\Gamma : \Gamma_i|$, then*

$$\dim_G(M) = \dim_D(M).$$

Proof. (1) The first claim is clear.

(2) We say that a n by n matrix A over D of full D-rank if $\mathrm{rk}_D(A) = n$. Observe that for an arbitrary matrix A over D, $\mathrm{rk}_D(A)$ is equal to the maximal size of a square matrix having full D-rank y obtained from A by removing some of its rows

or columns. Thus, in order to prove (2) it is enough to show that if $\dim_D(M) = 0$, then $\dim_G(M) = 0$.

Thus, from now on we assume that $\dim_D(M) = 0$. In particular, $R_0 \otimes_{K[\Gamma]} M$ is a R_0-module of finite length $l = l_{R_0}(R_0 \otimes_{K[\Gamma]} M)$. Observe that

$$\dim_{\Gamma/\Gamma_i}(M) = \frac{\dim_{R_0/R_1}(R_0/R_1 \otimes_{K[\Gamma_i]} M)}{|\Gamma : \Gamma_i|}.$$

Thus, we are done once we prove the following two claims.

Lemma 4.6. *Let H be a normal subgroup of Γ of p-power index.*

(a) $\frac{l_{R_0}(R_0 \otimes_{K[H]} M)}{|\Gamma : H|} \le l_{R_0}(R_0 \otimes_{K[\Gamma]} M)$.

(b) *If $H \le \Gamma_i$, then* $l_{R_0}(R_0 \otimes_{K[H]} M) \ge i(\dim_{R_0/R_1}(R_0/R_1 \otimes_{K[H]} M)$.

Proof. (a) We will prove the statement by induction on $|\Gamma : H|$. Assume first that $|\Gamma : H| = p$ and let $g \in \Gamma \setminus H$. Put $L = R_0 \otimes_{K[H]} M$. Let ϕ be an R_0-endomorphism of N defined as follows

$$\phi(a \otimes b) = ag \otimes g^{-1}b - a \otimes b.$$

It is well defined (exercise!) and $\phi^p = 0$. Since $\phi^i(L)/\phi^{i+1}(L)$ is a quotient of $L/\phi(L) \cong R_0 \otimes_{K[\Gamma]} M$ and l_{R_0} is additive on torsion R_0-modules, we obtain that

$$l_{R_0}(R_0 \otimes_{K[H]} M) \le p l_{R_0}(R_0 \otimes_{K[\Gamma]} M).$$

This proves the base of the induction.

Now assume that $|\Gamma : H| > p$. Let H_0 be a normal subgroup of Γ of index p containing H. Then we obtain the following

$$l_{R_0}(R_0 \otimes_{K[H]} M) \overset{\text{Induction}}{\le} |H : H_0| l_{R_0}(R_0 \otimes_{K[H_0]} M) \overset{\text{Induction}}{\le} |G : H| l_{R_0}(R_0 \otimes_{K[\Gamma]} M).$$

(b) As we have explained at the beginning of the subsection, we can assume that R_0 contains a central element t such that $V(t) = 1$. Thus, $R_1 = (t)$.

There exists a matrix $A \in \mathrm{Mat}_{n\times m}(K[H])$ such that $M \cong K[H]^m/(K[H]^n)A$ as $K[H]$-modules. Since $\dim_D(M) = 0$, $m \le n$. Observe also that

$$L = R_0 \otimes_{K[H]} M \cong R_0^m/(R_0^n)A.$$

Clearly, the choice of A is not unique. In fact, we can choose A such that

$$A = \begin{pmatrix} I_a + B_1 & B_2 \\ B_3 & B_4 \end{pmatrix},$$

where $I_a \in \mathrm{Mat}_a(K)$ is the identity matrix and B_1, B_2, B_3 and B_4 are matrices with entrance in the augmentation ideal of $K[H]$. Observe that

$$\dim_{R_0/R_1}(R_0/R_1 \otimes_{K[H]} M) = m - a.$$

We can express as A as a product of two matrices

$$\begin{pmatrix} I_a + B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} I_a + B_1 & B_2' \\ B_3 & B_4' \end{pmatrix} \begin{pmatrix} I_a & 0 \\ 0 & t^i I_{m-a} \end{pmatrix} =: A_1 A_2,$$

where B_2' and B_4' are matrices over R_0. Thus $R_0^m/(R_0^m)A_2$ is a quotient of L. Hence

$$l_{R_0}(L) \ge i(m-a) = i \dim_{R_0/R_1}(R_0/R_1 \otimes_{K[H]} M).$$

□

(3) Observe that

$$\dim_{\Gamma/\Gamma_i}(M) = \frac{\dim_{R_0/R_1}(R_0/R_1 \otimes_{K[\Gamma_i]} M)}{|\Gamma : \Gamma_i|} \geq \frac{\dim_D(D \otimes_{K[\Gamma_i]} M)}{|\Gamma : \Gamma_i|} = \frac{\dim_{D_i}(D_i \otimes_{K[\Gamma_i]} M)}{|\Gamma : \Gamma_i|} = \frac{\dim_{D_i}((D_i \otimes_{K[\Gamma_i]} K[\Gamma]) \otimes_{K[\Gamma]} M)}{|\Gamma : \Gamma_i|}.$$

Now, the condition $\dim_{D_i} D_1 = |\Gamma : \Gamma_i|$, implies that $D_i \otimes_{K[\Gamma_i]} K[\Gamma]$ is isomorphic to D_1 as $(D_i, K[\Gamma_i])$-bimodule (exercise!). Hence

$$\dim_{\Gamma/\Gamma_i}(M) \geq \frac{\dim_{D_i}(D_1 \otimes_{K[\Gamma]} M)}{|\Gamma : \Gamma_i|} = \dim_{D_1}(D_1 \otimes_{K[\Gamma]} M) = \dim_D(M).$$

□

The following pro-p version of the previous theorem can be proved using similar arguments.

Theorem 4.7. *Let D be a division ring of characteristic p and let V a be valuation on D. Let G be a profinite subgroup and assume that $\mathbb{F}_p[[G]]$ is a subring of R_0. Put $G_i = G \cap 1 + R_i$ and assume that they are open in G. Let M be a finitely presented $\mathbb{F}_p[[G]]$-module. Then*

$$\dim_G(M) \leq \dim_D(M).$$

If, moreover, $\dim_{D_i} D_0 = |G : G_i|$, where D_i denotes the division subalgebras of D generated by G_i, then

$$\dim_G(M) = \dim_D(M).$$

4.3. ***P*-adic chains.** A profinite group G is said to be **p-adic** if it is a subgroup of $\mathrm{GL}_n(\mathbb{Z}_p)$ for some n. Any finitely generated group which is linear over a field of characteristic 0 can be embedded in a p-adic profinite group. The completed group algebra $\mathbb{F}_p[[G]]$ of a p-adic profinite group G is Noetherian and satisfies the left Ore condition. We put $\Omega_G = Q_l(\mathbb{F}_p[[G]])$.

Any p-adic subgroup contains a torsion-free pro-p group P. The completed group algebra $\mathbb{F}_p[[P]]$ is a domain. Thus, Ω_P is a division algebra. We define by $\dim_{\Omega_G}$ the dimension function on $\mathbb{F}_p[[G]]$-modules associated with the embedding of $\mathbb{F}_p[[G]]$ into Ω_G:

$$\dim_{\mathbb{F}_p[[G]]}(M) = \frac{\dim_{\Omega_P} \Omega_P \otimes_{\mathbb{F}_p[[P]]} M}{|G : P|}.$$

This definition does not depend on the choice of P.

Theorem 4.8. *Let G be p-adic profinite group. Then $\dim_{\mathbb{F}_p[[G]]} = \dim_G$. In particular, $\dim_G$ takes only rational values and if, moreover, G is torsion free, then $\dim_G$ takes integer values.*

Proof. Recall that a finitely generated pro-p group P is called **uniform** if P satisfies $[P, P] \leq P^{2p}$ and P is torsion free. Every p-adic profinite group contains a uniform subgroup.

If P is uniform, then Ω_P has a valuation that can be defined as follows. Let I be the augmentation ideal of $\mathbb{F}_p[[P]]$, then we put $V(a) = k$ if $0 \neq a \in I^k \setminus I^{k+1}$ and $V(ab^{-1}) = V(a) - V(b)$ for an arbitrary non-zero element ab^{-1} of Ω_P.

Then we can apply Theorem 4.7 and obtain that $\dim_P = \dim_{\Omega_P}$. This implies that $\dim_G = \dim_{\Omega_G}$.

□

Remark. *The last part of the theorem is the characteristic p analog of a result of Farcas and Linnell* [10], *where they prove the Atiyah conjecture for torsion-free p-adic pro-p groups.*

4.4. **Free pro-p groups.** In this section we prove the following.

Theorem 4.9. *Let F be a finitely generated free pro-p. Then rk_F is equal to the inner rank on $\mathbb{F}_p[[F]]$.*

Proof. Let $\{g_1, \dots, g_n\}$ be free generators of F. Let $X = \{x_1, \dots, x_n\}$ be a set and let E be the universal division $\mathbb{F}_p\langle X\rangle$-algebra. Put $D = E((t))^{-1}$ and define a valuation V on D such that $V(e) = 0$ for every $e \in E$ and $V(t) = 1$. Then $R_0 = E[[t]]$. We can embed $\mathbb{F}_p[[F]]$ into R_0 by sending g_i to $1 + tx_i$. By results on universal division algebras [7], we know that rk_D restricted on matrices over $\mathbb{F}_p[[F]]$ is equal to its inner rank. Thus, we have only to show that $\mathrm{rk}_D = \mathrm{rk}_F$ as Sylvester matrix rank functions on $\mathbb{F}_p[[F]]$. This follows from Theorem 4.7.

□

Let G be a pro-p group. By induction we denote $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. If F is a free pro-p group, then $F/\gamma_n(F)$ is a torsion-free p-adic group. Combining Theorem 4.8 and Theorem 4.9 we obtain the following corollary.

Corollary 4.10. *Let F be a finitely generated free pro-p. Then $\lim_{i\to\infty} \mathrm{rk}_{\mathbb{F}_p[[F/\gamma_n(F)]]}$ is equal to the inner rank of $\mathbb{F}_p[[F]]$.*

5. The case of free groups and posible applications to the Calegari-Emerton conjecture

As a consequence of Theorem 4.9 we obtain.

Corollary 5.1. *Let F be a finitely generated free group and let I be the augmentation ideal of $\mathbb{F}_p[F]$. For any $k \geq 1$, we denote $F_k = F \cap 1 + I^k$. Then there exists the limit*

$$\lim_{i\to\infty} \mathrm{rk}_{F/F_i}$$

and it is equal to the inner rank of matrices over $\mathbb{F}_p[F]$.

Proof. Observe that the profinite completion of F with respect of $\{F_i\}$ is a free pro-p group. Now, the corollary follows from the fact that the inner rank of a matrix over $\mathbb{F}_p[F]$ is the same as a matrix over $\mathbb{F}_p[[F]]$ and Theorem 4.9.

Also one can argue as in the proof Theorem 4.9. and use Theorem 4.5 instead of Theorem 4.7. □

This corollary suggests the following conjecture.

Conjecture 1. *Let F be a finitely generated free group and let $\{F_i\}$ be an exhausting p-adic chain in F. Then there exists the limit*

$$\lim_{i\to\infty} \mathrm{rk}_{F/F_i}$$

and it is equal to the inner rank of matrices over $\mathbb{F}_p[F]$.

We finish this note with a result that relates the previous conjecture with the Calegari-Emerton conjecture.

Theorem 5.2. *Conjecture 1 implies the Calegari-Emerton conjecture*

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