Minicourse:

Introduction to arithmetic groups and their cohomology

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An arithmetic group is, roughly speaking, a group of integer valued matrices. A typical example is the group $\mathrm{SL}_n(\mathbb{Z})$ of integer matrices of determinant one. More precisely, every arithmetic group Γ is, by definition, a subgroup of a linear algebraic group G over the field of rational numbers and automatically a discrete subgroup of the associated real Lie group $G(\mathbb{R})$. If the Lie group $G(\mathbb{R})$ is semi-simple, then Γ acts on the associated Riemannian symmetric space X and the quotient $\Gamma\backslash X$ is a locally symmetric space of finite volume. The cohomology of $\Gamma\backslash X$ is a very interesting object since it is equipped with additional structure which relates it to automorphic forms and number theory.

In this course we introduce arithmetic groups and the associated locally symmetric spaces. We explain how the cohomology of $\Gamma\backslash X$ can be described in terms of relative Lie algebra cohomology. This gives rise to an interesting decompsition of the cohomology which is related to automorphic representations. Finally, we introduce the action of Hecke operators on the cohomology and mention some conjectures and results concerning applications to number theory. Where possible we will focus on concrete examples.

References

Günter Harder is writing an introduction to the cohomology of arithmetic groups. Parts of the book can be found on Harder's website:
`http://www.math.uni-bonn.de/people/harder/Manuscripts/buch/`

The following survey articles (in chronological order) are a good starting point to learn about the cohomology of arithmetic groups.

• 1974 - A. Borel, *Cohomology of Arithmetic Groups*, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 1.

• 1990 - J. Schwermer, *Cohomology of arithmetic groups, automorphic forms and L-functions*, Cohomology of arithmetic groups and automorphic forms (Luminy-Marseille, 1989), 1–29, Lecture Notes in Math., 1447, Springer, Berlin, 1990.

• 2006 - A. Borel, *Introduction to the Cohomology of Arithmetic Groups*, Lie groups and automorphic forms, 51–86, AMS/IP Stud. Adv. Math., 37, Amer. Math. Soc., Providence, RI, 2006.

• 2010 - J. Schwermer, *Geometric cycles, arithmetic groups and their cohomology*, Bull. Amer. Math. Soc. (N.S.) **47** (2010), no. 2, 187–279.

A standard source for facts about arithmetic groups is:

• A. Borel, *Introduction aux groupes arithmétiques*, Hermann, Paris, 1969.

A basic reference for relative Lie algebra cohomology is the book:

• A. Borel, N. Wallach, *Continuous cohomology, Discrete Subgroups, and Representations of Reductive Groups*, 2nd Ed., Mathematical Surveys and Monographs, Vol. 67, AMS.