Notes on “Introduction to L^2-invariants II” at the introductory school of the thematic program “L^2-invariants and their analogues in positive charactersitic” at ICMAT Madrid

Thomas Schick*
Mathematisches Institut
Georg-August-Universität Göttingen
Germany

March 12, 2018

Abstract

Short notes on the contents of the course “Introduction to L^2-invariants II” at ICMAT March 2018 in Madrid.

1 Introduction

In the first week, Lukasz Grabowski introduces L^2-invariants, with a focus on L^2-Betti numbers, discussing in particular aspects of the Atiyah conjecture about possible values of these L^2-Betti numbers, and focusing on the combinatorial aspects of these invariants.

The part II will complement this introduction focusing in particular on:

(1) Differential topological: L^2-invariants defined as invariants of the Laplace-Beltrami operator.

(2) Dodziuk's L^2-Hodge-de Rham theorem: combinatorial and analytic L^2-invariants coincide

(3) finer invariants beyond the dimension of the kernel: L^2-Betti numbers, Novikov-Shubin invariants, spectral density functions and their dilation class

(4) even finer invariants: L^2-determinants and L^2-torsion: definitions, properties, applications, computations (not the analytic versions)

(5) not covered: a glimpse on the equality of analytic and combinatorial L^2-torsion

*e-mail: thomas.schick@math.uni-goettingen.de
www: http://www.uni-math.gwdg.de/schick

(6) only hinted: Computations of L^2-invariants for locally symmetric spaces: a (more or less difficult) exercise in harmonic analysis

(7) if time permits: Analytic proof of the integrality of L^2-Betti numbers for space with free fundamental group due to Linnell

2 Spectral content near zero

2.1 Remark. Traces. Have used

(1) trace property

(2) positivity

(3) normality

(4) normalization

There is a universal such trace for our group von Neumann algebras, the center valued trace, taking values in the center of the von Neumann algebra and being the identity for elements of the center.

For ICC-groups, that center is trivial!

Otherwise: part of center comes from finite normal subgrou (there is a maximal one if the torsion is bounded). This contains a lot of information.

Rest comes from elements with infinite order with finitely many conjugates. Tendency that those don't see too much.

2.2 Definition. Spectral density function

$$F(\lambda) := \mathrm{tr}_\Gamma\, \chi_{[0,\lambda]}(\Delta).$$

2.3 Definition. Dilation domination and dilation equivalence:

$F \leq G$ if there are $C, \epsilon >$ such that

$$F(\lambda) \leq G(C\lambda) \qquad \forall \lambda \in [0, \epsilon].$$

$F \sim G$ if and only if $F \leq G$ and $G \leq F$.

Fredholm: $F(\lambda) < \infty$ for some $\lambda > 0$.

2.4 Definition. Small refinement for a chain complex: instead of Δ_p use

$$d_p \colon \mathrm{im}(d_{p-1})^\perp \to D_{p+1}.$$

2.5 Definition. Novikov-Shubin invariant of a spectral density function

$$\alpha(F) = \liminf_{\lambda \to 0^+} \frac{\log(F(\lambda) - F(0))}{\log(\lambda)} \in [0, \infty]$$

if $F(\lambda) > F(0)$ for all $\lambda > 0$, otherwise $\alpha(F) := \infty^+$.

Dually, define the capacity $c(F)$ as the inverse of the Novikov-Shubin invariant. It is essentially the smallest α such that $F(\lambda) - F(0) \leq c\lambda^\alpha$ for small λ.

2.6 Exercise. Show that the Novikov-Shubin invariant depends only on the dilation equivalence class of the spectral density function.

2.7 Exercise. Use elements in $L^\infty(S^1)$ to produce essentially arbitrary spectral density functions, in particular arbitrary Novikov-Shubin invariants.

2.8 Theorem. *If two Hilbert Γ-chain complexes are chain homotopy equivalent then their spectral densitiy functions are mutually dilation equivalent.*

Proof. Use the mapping cone of the chain homotopy equivalence. It is contractible.

Prove the

2.9 Lemma. *A Hilbert Γ-chain complex is contractible if and only if all Laplacians are invertible if and only if all L^2-Betti numbers are zero and all capacities are 0^- (and it is Fredholm).*

Produce then the short exact sequences

$$0 \to C_* \to cyl(f) \to cone(f) \to 0 \qquad 0 \to D_* \to cyl(f) \to cone(\mathrm{id}_C) \to 0$$

Prove (from the chain contractions of the cones) that they split as chain complexes: the middle one is the direct sum of left and right.

So $C_* \oplus cone(f) \cong D_* \oplus cone(\mathrm{id}_C)$. Using obvious formulas for the spectral density of direct sums and that the cone doesn't contribute concludes the proof. □

2.10 Definition. A chain homotopy equivalence (of Hilbert $L\Gamma$-chain complexs) is a chain map $f\colon C_* \to D_*$ which admits a chain homotopy inverse $g_*\colon D_* \to C_*$ together with chain homotopies $g \circ f - \mathrm{id} = hc + dh$ and $f \circ g - \mathrm{id} = kd + ck$.

For now, we do this with finitely generated Hilbert $L\Gamma$-modules, so all $L\Gamma$-morphisms are automatically bounded.

A chain contraction is the chain homotopy for the chain equivalence to the zero chain complex: $hc + ch = 1$.

2.11 Lemma. *If a finitely generated Hilbert $L\Gamma$-chain complex is contractible, the spectral density functions of all Laplacians are dilation equivalent to the zero function.*

Proof. functional analytic input: there is an abstract Hodge decomposition (for any Hilbert chain complex):

$$C_k = \ker(\Delta_k) \oplus \ker(d_k)^\perp = \overline{\mathrm{im}(d_k^*)} \oplus \overline{\mathrm{im}(d_{k+1})}.$$

Abstract functional analysis (open mapping theorem) says: d_k has closed image (is surjective onto $\overline{\mathrm{im}(d_k)}$) if and only if $d_k^* d_k$ has a spectral gap near zero.

Contractibility implies that $\mathrm{im}(d_k) = \ker(d_{k-1})$ which is closed. This also implies that there is no room for $\ker(\Delta_k)$: $\ker(\Delta_k) = \{0\}$. The assertion follows. □

2.12 Lemma. *The converse to Lemma 2.11 also holds.*

Proof. We have to construct the chain contraction. The abstract functional analysis shows $d_k\colon \ker(d_k)^\perp \to \operatorname{im}(d_k) = \ker(d_{k-1})$ is an isomoprhism. We set h_k to be the inverse (extended by 0 on the complement). □

2.13 Definition. Mapping cone and mapping cylinder of chain map:
cone:

$$C(f)_k = C_{k-1} \oplus D_k; \qquad \begin{pmatrix} -c_{k-1} & 0 \\ f_k & d_k \end{pmatrix}$$

cylinder:

$$C_{n-1} \oplus C_n \oplus D_n; \qquad \begin{pmatrix} -c_{n-1} & 0 & 0 \\ -\operatorname{id} & c_n & 0 \\ f_{n-1} & 0 & d_n \end{pmatrix}$$

Pretty much by definition we have the extension

$$0 \to C_* \to cyl(f) \to cone(f) \to 0$$

$$0 \to D_* \to cyl(f) \to cone(\operatorname{id}_C) \to 0$$

2.14 Lemma. *If* $0 \to C_* \xrightarrow{f_*} D_* \xrightarrow{g_*} E_* \to 0$ *is exact sequence of Hilbert* Γ*-chain complexs and* E_* *is contractible, the sequence splits, i.e.* $D_* \cong C_* \oplus E_*$ *as chain complexes.*

Proof. E_* being contractible means that it is a direct sum of isomorphisms $\ker(e_k)^\perp \xrightarrow{e_k} \operatorname{im}(e_k) = \ker(e_{k-1})$.

Moreover, as Hilbert Γ-modules D_* splits: $D_k = (\operatorname{im}(f_k) = \ker(g_k)) \oplus \ker(g_k)^\perp$ and f_k implements an isomorphism between c_k and $\operatorname{im}(f_k)$, whereas g_k implements an isomorphism between $\ker(g_k)^\perp$ and E_k.

On $\operatorname{im}(e_k)^\perp = \ker(e_{k-1})^\perp$ we define the splitting as the inverse of the restriction of g_{k-1}, and on its image, the missing complementary summand in such a way that the map becomes a chain map. This is the required splitting as one immediately checks. □

2.15 Lemma. *The spectral density function of a direct sum is the sum of the spectral density functions.*

A bit more difficult: if u is an isomorphism (bounded with bounded inverse), the spectral density functions of f, fu, and uf are dilation equivalent (but not equal!).

Proof. To prove this, it is useful to have an expression of the spectral density function which does not need the functional calculus:

$$F_f(\lambda) = \sup\{\dim_\Gamma(L) \mid |f(x)| \leq \lambda\, |x|\, \forall x \in L\}.$$

□

Putting all this together, the homotopy invariance of the dilation class of the spectral density functions follows.

3 L^2-determinant

3.1 Definition. Define the (log) Fuglede-Kadison determinant of an invertible Hilbert Γ-morphism between finitely generated Hilbert Γ-modules

$$D(F) := \frac{1}{2}D(F^*F) = \int_{0^+}^{\infty} \log(\lambda) dE_{F^*F}(\lambda) \in [-\infty, \infty).$$

Use the same formula for arbitrary (not necessarily invertible) morphisms. Then the value $-\infty$ can actually occur.

Denote F of *determinant class* if the determinant is $> -\infty$.

3.2 Lemma. *$D(f) > -\infty$ if and only if $\int_{0^+}^{a} \frac{1}{\lambda}(F_f(\lambda) - F_f(0))\, d\lambda < \infty$. Then*

$$D(f) = -\int_{0^+}^{a} \frac{1}{\lambda}(F_f(\lambda) - F_f(0))\, d\lambda + \log(a)(F(a) - F(0)).$$

Properties of log det:

3.3 Theorem. *(1) Invariance under conjugation*

(2) $D(f) = D(f^*) = \frac{1}{2}D(f^*f) = \frac{1}{2}D(ff^*)$

(3) f injective positive, then $D(f + \epsilon) \xrightarrow{\epsilon \to 0^+} D(f)$.

(4) f injective positive, $f \le g$ implies $D(f) \le D(g)$.

(5) $D(f \oplus g) = D(f) + D(g)$.

3.4 Theorem. *$D(fg) = D(f) + D(g)$ if g has dense image and f is injective.*
Block sum formula if the diagonal terms are injective with dense image.

Proof. This looks at first glance expected, as this is a basic property of determinants. At second glance, it is quite remarkeable: the spectrum of fg (or rather $(fg)^*fg$) has only superficial relation to the spectrum of f^*f and g^*g.

We prove it for honestly invertible operators first, the general case is obtained by a limiting argument.

Because we want to work with positive operators throughout, the key is to consider $f, g > \epsilon > 0$ and to compute $D(gf^2g) = D(g^2) + D(f^2)$.

We can scale the operators (and know that the Determinant behaves as expected) so that we can really work with $1-f$ and $1-g$ with $0 < f, g < 1-\epsilon < 1$. Then the logarithm is given by the usual power series.

We look at $tr(\log((1-g)(1-tf)(1-g)))$ and we are interested in this for $t = 1$. Instead of computing this directly (hard!) we look at how it changes as t varies from 0 to 1.

We compute the t-derivative of $\mathrm{tr}(\log(1 - v(t)))$, it is

$$\mathrm{tr}((1 - v(t))^{-1} v'(t))$$

by evaluating the power series expansion of

$$\begin{aligned} \log(1 - v(t)) &= \sum \frac{1}{k} v(t)^k \\ &= \log(v(0)) + t \sum_{k=1}^{\infty} \frac{v(0)^{k-1}v'(0) + v(0)^{k-2}v'(0)v(0) + \dots}{k} + O(t^2) \end{aligned}$$

This formula has no good simplification on the nose, but upon taking the trace, because of its invariance under cyclic permutations, it all works out and gives the desired formula. □

3.5 Proposition. *For a matrix f over $\mathbb{Z}\Gamma$ with Γ finite,*

$$D(f) \geq 0 : \qquad \exp(|\Gamma| \cdot D(f)) \in \mathbb{N}_{>0}.$$

Proof. The determinant is the "usual" determinant upto normalization by the order of the finite group, namely the product of the non-zero eigenvalues of f^*f. This is a coefficient of the characteristic polynomial, therefore an integer. □

3.6 Conjecture. *Determinant conjecture: For f a matrix over $\mathbb{Z}\Gamma$,*

$$D(f) \geq 0.$$

In particular, f is of determinant class.

3.7 Theorem. *The determinant conjecture holds whenever Γ is a sofic group.*

3.8 Proposition. *If the Novikov-Shubin invariant is positive then the operator is of determiant class.*

3.9 Conjecture. *Lück's determinant approximation conjecture: if f is a matrix over $\mathbb{Z}\Gamma$ and we have a residual chain, the determinants converge.*

4 L^2-torsion

4.1 Definition. Let $\cdots \to C_2 \xrightarrow{c_2} C_1 \xrightarrow{c_1} C_0$ be a finite Hilbert Γ-chain complex.

It is of *determiant class* if $D(\Delta_p) > -\infty$ for all p. In this case we define its L^2-determinant as

$$\rho(C_*) := \sum_{p=0}^{\infty} p(-1)^p D(\Delta_p).$$

An easy computation shows that this coincides with

$$\rho(C_*) = \sum_{p=1}^{\infty} (-1)^p D(c_p^* c_p)|_{\ker(c_)^\perp}$$

which is perhaps the more intuitive formula: we take the relevant part (where it is non-zero) of each differential exactly once into account.

This invariant has better properties in case all L^2-Betti numbers of the chain complex vanish, which is what we typically assume. Applied to the cellular L^2-chain ocmplex of a Γ-finite free Γ-CW-complex it defines its L^2-determinant.

4.2 Theorem. *The L^2-determinant for an L^2-acyclic Γ-CW-complex X has the following properites:*

(1) $\rho(X)$ is independent of the CW-decomposition

(2) if $X = X_1 \cup X_2$ with $X_1, X_2, X_0 := X_1 \cap X_2$ L^2-acyclic then $\rho(X) = \rho(X_1) + \rho(X_2) - \rho(X_0)$.

(3) *Let Y be another Γ-CW complex. Then $X \times Y$ is l^2-acyclic and of determinant class and*

$$\rho(X \times y) = \rho(X) \cdot \chi(Y).$$

4.3 Remark. The classical counterpart to L^2-torsion, the Reidemeister torsion, is a very delicate invariant which can be used to distinguish lens spaces which are honotopy invariant, but not diffeomorphic.

In contrast, it is conjectured (and known in many cases) that the L^2-determinant is a homotopy invariant.

4.4 Exercise. Let $\Gamma = \{1\}$ and $\dots C_2 \to C_2 \to C_0$ be finite chain complex of free abelian groups.

Relate the torsion of $C_* \otimes \mathbb{C}$ to the torsion subgroups of the homoloty of the $\mathbb{Z}$-chain complex in the case thtat $H_*(C_* \otimes \mathbb{C}) = 0$. Start with a chain complex of length 1.

5 "Analytic L^2-Betti numbers via the Hodge-de Rham operator

We consider the following situation:

- (M, g) a compact smooth manifold, possible with non-empty boundary (but you can assume that the boundary is empty if you like this better) with a Riemannian metric g
- $\bar{M} \to M$ a normal covering with deck transformation group Γ (normal means that $\bar{M}/\Gamma = M$). If $\bar{M}$ is connected then $\Gamma \cong \pi_1(M)/\pi_1(\bar{M})$. We pull back the metric g to a metric $\bar{g}$ on $\bar{M}$, then Γ acts by isometries.
- the metric defines differential form Laplace operators $\bar{\Delta}_p$ acting to start with on smooth differential p-forms with compact support on $\bar{M}$. This operator commutes with the action of Γ.
- If M (and then $\bar{M}$) is oriented, we can also form the Hodge-$*$-operator $*\colon \Omega^p \to \Omega^{m-p}$ which is Γ-invariant. The de Rham differential d has the formal adjoint $\delta = \pm * d*$ (this is a local formula which is always defined, because locally we can choose $*$ with only a sign ambiguity which cancels out in $\pm * d*$). We have $\Delta = (d+\delta)^* = d\delta + \delta d$. Note that this is the *geometers convention*: Δ is a positive formally self-adjoint operator.
- In ase $\partial M \neq$ we have to impose *boundary conditions*. For us, there are two important types:
  - *absolute boundary conditions*: $\iota^*\omega = 0$ and $\iota^*\delta\omega = 0$
  - or *relative boundary conditions*: $\iota^*(*\omega) = 0$ and $\iota^*(*d\omega) = 0$ (note that again this is well defined as a local condition even if M is not orientable).

  The meaning of "imposing boundary conditions" is: one restricts the operators to the subspace of those forms satisfying the boundary conditions.

- The Riemannian metric defines a measure on $M, \bar{M}$ and a fiberwise scalar product on the differential form bundles. We get an inner product on the smooth forms with compact support:

$$\langle \omega, \eta \rangle = \int_{\bar{M}} \langle \omega_x, \eta_x \rangle_x \, d\mu_{\bar{g}}(x).$$

  The completion with respect to this innerproduct is the Hilbert space of square integrable forms $L^2\Omega^*(\bar{M})$. The deck transformation group Γ acts unitarily on this.

- Of course, there is also a concrete description as (equivalence classes of) measurable sections (upto nullsets) which are square integrable.

- It is not hard to find a measurable *fundamental domain* $F \subset \bar{M}$ for the covering $\pi\colon \bar{M} \to M$, i.e. a subset such that the restriction of π is a measurable equivalence. Even better, we can choose F open and such that $\pi|F\colon F \to M$ is injective. Then all the Γ-translates of F are disjoint, and there union covers $\bar{M}$ upto a subset of measure 0. Furthermore, we can achieve that the closure still is a measurable fundamental domain: $\pi|\colon \bar{F} \to M$ is surjective, but the Γ-translates of F intersect only in sets of measure 0. For the construction: start with a smooth triangulation of M and take the interiors of top-dimensional simplices and their closures.

- From a measurable perspective, the covering $\bar{M} \to M$ is trivial, i.e. isomorphic to $F \times \Gamma \to F$ (and $F \cong M$). It follows that we get induces unitary isomorphisms as Γ-representations

$$L^2\Omega^p(\bar{M}) \cong L^2\Omega^p(F) \otimes l^2(\Gamma).$$

  By definition, this makes $L^2(\Omega^p(\bar{M}))$ a Hilbert Γ-module. Note that these Hilbert modules are not finitely generated at all (if M is not 0-dimensional).

- The operators d, δ, Δ are defined on smooth forms with compact support (satisfying in addition our chosen boundary conditions) which commute with Γ. These are dense subspaces of $L^2\Omega^p(\bar{M})$, but the operators can't be extended to bounded operators on the whole space.

  Howver, such operators come up a lot: in PdE, mathematical physics (quantum mechanics), and functional analysis has developped a whole theory around them:

  In the case at hand, their graphs have a closure which again is the graph of an operator (called the closure of the initial operator), and this closure is very nice in the sense of functional analysis: for Δ (and $d + \delta$) we obtain *self-adjoint operators*. This is a serious and non-trivial and somewhat technical condition. Caveat: on manifolds with boundary one has to impose boundary conditions, and the operators do depend on these. But once these are chosen, there is a unique self-adjoint extension, which is as good as it can get.

- Another result from functional analysis: we have a functional calculus to self-adjoint unbounded operators, allowing to form $f(\Delta)$ for every (bounded or even unbounded, continuous or only measurable) function $f\colon \mathbb{R} \to \mathbb{C}$. If f is bounded this will give a bounded operator which commutes with the action of Γ. The assignment $f \mapsto f(\Delta)$ is a C^*-algebra homomorphism from the C^*-algebra of bounded functions (pointwise operators, sup-norm) to the C^*-algebra of bounded operators commuting with Γ. This means in particular: if $f_n \xrightarrow{n\to\infty} f$ in sup-norm, then $f_n(\Delta) \xrightarrow{n\to\infty} f(\Delta)$ in operator norm.

Additionally: if $f_n \xrightarrow{n\to\infty} f$ pointwise (and there is a uniform bound $abs f_n \le C$ for all n) then $f_n(\Delta) \xrightarrow{n\to\infty} f(\Delta)$ pointwise —meaning in the strong operator topology: $f_n(\Delta)f \xrightarrow{n\to\infty} f(\Delta)v$ for all $v \in L^2\Omega^p(\bar{M})$.

The favorite functions we are using are $\chi_{[0,T]}(\Delta)$, giving the spectral projectors for Δ, and $e^{-T\Delta}$ for $T \ge 0$, giving the *heat kernel* of Δ.

- Important additional fact: $f(\Delta) = g(\Delta)$ if f, g coincide on the *spectrum of* Δ where

  $\sigma(\Delta) = \{\lambda \in \mathbb{C} \mid (\Delta-\lambda)\colon D(\Delta) \to L^2$ is not surjective with bounded inverse$\}$.

- This way, Δ and its friends canonically are operators which are affiliated to the von Neumann algebra of Γ-invariant bounded operators on $L^2\Omega^p(\bar{M})$. This von Neumann algebra is (isomorphic to) $B(L^2\Omega^p(F)) \otimes L\Gamma$.

  It has a *semifinite* trace: the trace is defined on all positive self-adjoint operators in the algebra, but can take the value $+\infty$ there, and is just the tensor product of the usual trace on bounded operators with the canonical trace on $L\Gamma$. From this, one defines the subset of Γ-trace class operators: finite linear combinations of self-adjoint operators with finite trace. This is a dense ideal.

- There is an analytic formula for the Γ-trace for particularly nice operators (using a Rellich type theorem in the context at hand):

  If a bounded operator A is given by integration against a continuous integral kernel $k(x, y)$ on $\bar{M} \times \bar{M}$ which is Γ-equivariant (for the diagonal action) and which is sufficiently rapidly decaying (necessary to define a bounded operator on L^2 in the first place) then it is of Γ-trace class and

$$\operatorname{tr}_\Gamma(B) = \int_F \operatorname{tr}_x k(x, x)\, d\mu_g(x).$$

5.1 Definition. The assignment $A \mapsto \chi_A(\Delta)$ from the set of measurable subsets of $\mathbb{R}$ to the set of Γ-invariant projectors defines what is called the *projection valued σ-additive spectral measure*. Writing down the axioms for a projection valued σ-additive measure is straightforward.

5.2 Example. The standard example for a fundamental domain is the unit square in $\mathbb{R}^2$ (open or closed) for the covering $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = T^2$.

Drawings of Escher show similarly nice fundamental domains for $\bar{M}$ the hyperbolic plane.

One has to proof the analytic properties of the Laplacian (and its friends) we listed above. This can be done transparently with standard methods from the theory of PdE, in particular a priori energy estimates. With such, Chernoff proves an important property of generalized Laplace operators, proved for Dirac type operators (with essentially the same proof) in [7]:

5.3 Theorem. *On a complete Riemannian manifold, the solutions to the wave equation satisfy* unit propagation speed*: if $f(x,t)$ is compactly supported smooth and satisfies*

$$\frac{\partial}{\partial t} f(x,t) = i(d+\delta) f(x,t)$$

then $\operatorname{supp}_x f(\cdot,|T|) \subset U_{|T|}(\operatorname{supp}_x f(\cdot,0))$*: the support at time $T \in \mathbb{R}$ is contained in the $|T|$-neighborhood of the support at time 0.*

This works in general for Dirac type operators, for us $D = (d+\delta)$. It implies without too much work that these operators have unique self-adjoint extensions and allows to construct the heat kernel $e^{-T\Delta}$ as a (family) of particularly nice operators: for each $T > 0$ the operator $e^{-T\Delta}$ is given by integration against a smooth integral kernel $K_T(x,y)$ (which depends also smoothly on T) which is Γ-invariant for the diagonal action, and which is exponentially decaying off the diagonal.

For each $T > 0$, the image of $e^{-T\Delta}$ consists of smooth (square integrable) forms.

Proof. The fact that $e^{-T\Delta}$ has a smooth integral kernel is elliptic regularity and is strictly speaking not directly following from the finite propagation of the wave operator, but is another standard analytic fact of the Laplacian (or any other elliptic operator).

The rapid decay is a consequence of finite propagation. Functional calculus allows us to write, using Fourier transform

$$e^{-T(d+\delta)^2} = \int_{\mathbb{R}} g_T(\xi) e^{i\xi(d+\delta)}\, d\xi$$

where $g_T(\xi)$ is the Fourier transform of $x \mapsto e^{-Tx^2}$ which is itself a Gaussian in ξ. The contribution to the integral for $|\xi| \geq C >> 0$ therefore is very small (as $e^{i\xi(d+\delta)}$ has norm bounded by 1), and the contribution of $|\xi| \leq C$ has propagation $\leq C$ as integral of operators of propagation $\leq C$.

If we now consider an element $e^{-T\Delta}\omega(x) = \int_{\bar{M}} k_t(x,y)\omega(y)\, dy$ in the image of $e^{-T\Delta}$, using the integral kernel $k_T(x,y)$ describing this operator, we see that by smoothness in x, and using rapid decay off the diagonal to interchange differentiation and integration that the image indeed can be arbitrarily often differentiated. □

5.4 Theorem. *For every $T \geq 0$ the operator $\chi_{[0,T]}(\bar{\Delta})$ is of Γ-trace class (it is a positive operator, so this means that its trace is finite). The same is true for $e^{-T\Delta}$ as long as $T > 0$ (note that $e^{0\Delta} = \mathrm{id}$ which is* not *of Γ-trace class). The image of $\chi_{[0,T]}(\bar{\Delta}) = e^{-\Delta} \cdot \exp|_{[0,T]}(\Delta)$ is contained in the image of $e^{-\Delta}$ and therefore consists of smooth forms.*

The analytic L^2-Betti numbers are defined as

$$b_p^{(2)}(\bar{M};\Gamma) := \dim_\Gamma(\ker(\bar{\Delta}_p)) = \operatorname{tr}_\Gamma(\chi_{\{0\}}(\bar{\Delta}_p)) = \lim_{T\to\infty} \operatorname{tr}_\Gamma(e^{-T\bar{\Delta}_p}).$$

The last equation follows from the fact that tr_Γ is normal and that $e^{-T\bar{\Delta}_p}$ converges strongly to $\chi_{\{0\}}(\bar{\Delta}_p)$ and is uniformly bounded by a Γ-trace class operator: same phenomenom as in measure theory!.

Proof. We just argued that the heat kernel $e^{-T\bar{\Delta}_p}$ is given by a smooth integral kernel which is exponentially (rapidly enough) decaying off the diagonal, so that the Rellich type theorem shows that it is of Γ-trace class. As $\chi_{[0,T]}(x) \le e^T \exp(-x)$, by positivity of the trace also $\chi_{[0,T]}(\Delta_p)$ has a finite trace. □

5.5 Remark. This is the original definition of Atiyah [1] of L^2-Betti numbers.

5.6 Definition. The L^2-de Rham complex of $\bar{M}$ is the complex

$$dom(d_0) \xrightarrow{d_0} dom(d_1) \xrightarrow{d_1} \dots$$

where in each step we use the domain of the closure of d_k inside $L^2\Omega^k(\bar{M})$. The L^2-de Rham cohomology is defined as $H^k_{2,dR} := \ker(d_k)/\overline{\operatorname{im}(d_{k-1})}$. This is a Hilbert Γ-module, isomorphic to the orthogonal complement of $\operatorname{im}(d_{k-1})$ inside $\ker(d_k)$.

Standard easy results from functional analysis show that the inclusion gives a canonical isomorphism of Hilbert Γ-modules

$$\ker(\Delta_k) = \ker(d_k) \cap \ker(\delta_k) \cong H^k_{2,dR}(\bar{M};\Gamma).$$

It seems that it was suggested by Atiyah and Singer to Dodziuk to investigate whether there is a more combinatorial way to define and investigate the L^2-Betti numbers, which then Dodziuk did:

5.7 Theorem. *Let (M,g) be a compact Riemannian manifold with a smooth triangulation K, giving rise to a lifted triangulation $\bar{K}$ of $\bar{M}$.*

The "usual" maps between the simplical cochain complex and the de Rham complex (integration over the simplices to go from de Rham to simpicial, a Whitney map to go back) induce Γ-equivariant bounded cochain homotopy equivalences and hence in particular isomorphisms of Hilbert Γ-modules between the L^2-de Rham cohomology and the simplicial de Rham cohomology.

Proof. We make this argument rigorous by introducing intermediate chain complexes associated to the de Rham complex: for each $\Lambda > 0$ we consider the "spectrally cut" subcomplexes consisting of $\chi_{[0,\Lambda]}(\Delta_p) \subset L^2\Omega^p(\bar{M})$.

Standard spectral theory shows that these are indeed subcomplexes (exercise) and that the inclusion and the orthogonal projection back provide chain homotopy equivalences between the original de Rham complex and the spectrally cut subcomplex.

The usual de Rham integration map I is defined on this subcomplex (consisting of smooth L^2-forms), and we obtain by copying one standard proof of the Hodge-de Rham isomorphism a Γ-chain homotopy between the composition of I and a chain homototpy inverse "Whitney map" W and the inclusion of the spectrally cup in the full de Rham complex.

This homotopy involves the integration map from the Poincaré lemma. It would be interesting to write out more details of this.

The other composition of W and I is the identity on the nose. □

5.8 Remark. Our argument shows that the comment made by Dodziuk [4, p. q58] that some parts of the standard proof of the de Rham theorem (for the injectivity of the de Rham map) are global and therefore can't used in the setting of L^2-Betti numbers is not justified.

The analytic definition Atiyah gave of the Γ-dimension works for kernels of any elliptic differential operator, in particular any Dirac type operator. This was actually Atiyah's initial motivation. He proved the L^2-index theorem:

5.9 Theorem. *Let $D\colon \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator between sections of two Hermitean bundles E, F on M. An example is $d+\delta\colon \Omega^{ev}(M) \to \Omega^{odd}(M)$.*

It has an index

$$\operatorname{ind}(D) := \dim(\ker(D)) - \dim(\ker(D^*\colon \Gamma(F) \to \Gamma(E))) \in \mathbb{Z}.$$

Lift everything to the covering $\bar{M}$, giving rise to unbounded elliptic operators the Hilbert Γ-modules $L^2(\bar{E})$ and $L^2(\bar{F})$.

Then $\dim_\Gamma(\ker(\bar{D}))$ and $\dim_\Gamma(\ker(\bar{D}^*))$ are finite and we define the Γ-index

$$\operatorname{ind}_\Gamma(\bar{D}) := \dim_\Gamma(\ker(\bar{D})) - \dim_\Gamma(\ker(\bar{D}^*)).$$

Then it holds $\operatorname{ind}(D) = \operatorname{ind}_\Gamma(\bar{D})$.

Proof. The trace property and functional calculus allows to prove the McKean-Singer formula:

$$\operatorname{tr}_\Gamma(e^{-T\tilde{D}^*\tilde{D}}) - \operatorname{tr}_\Gamma(e^{-T\tilde{D}\tilde{D}^*})$$

is constant in T, and correspondingly downstairs. Its limit for $T \to \infty$ is by normality of the trace precisely the (L^2-) index.

On the other hand, if we make T sufficiently small (and if we have unit propagation speed as for Dirac type operators) the operator $e^{-T\bar{D}^*\bar{D}}$ is the lift of e^{-TD^*D}, in particular the restriction to the diagonal is the lift, and therefore by the trace formula the Γ-trace and the trace downstairs coincide. A small extra argument allows to apply the same argument to general elliptic differential operators (what Atiyah does). □

This lead Atiyah to his famous question: the L^2-index is always an integer, what are the possible values of the dimensions of the kernels. In particular

5.10 Question. What are the possible values of L^2-Betti numbers? If Γ is torsion-free, are they always integers? Are they always rational numbers? We know that the inverse of the orders of finite subgroups could occur (as 0-th L^2-Betti number).

5.11 Question. We have very strong forms of approximation theorems for L^2-Betti numbers by "smaller" L^2-Betti numbers (amenable approximation, residually finite appriximation, sofic approximation, approximation in residually elementary amenable towers). And we have the perfect approximation theorem for L^2-indices (equality). By now, this is generalized from combinatorial (and therefore analytic) Laplacians to matrices over the group ring with algebraic numbers and even complex numbers as coefficients.

What about the kernel of other elliptic operators? Do we have approximation theorems for the L^2-dimension of the kernel itself?

The proof Lukasz described would work the same way, provided we had, for the given operator D, the famous function $f\colon (0,\infty) \to [0,\infty)$ with $\operatorname{tr}(\chi_{(0,e)}(D_i)) \le f(e)$ for all the approximating operators D_i involved, and with $f(e) \xrightarrow{e\to 0} 0$.

It seems we get this spectral control only in very special situations.

On the other hand, to construct a counterexample (which might well exist) requires to find an elliptic differential operator D such that $\ker(\bar{D})$ is "big", and this for no good (index theoretic) reason. But the generic behaviour seems to be that there is no discrete spectrum if there is no good reason for it to exist. On top of that, explicit computations are hard anyway.

The analytic definition might seem more complicated than the combinatorial one: who would like to compute the harmonic L^2-integrable forms and the trace of the projector onto this space. However, it turns out that analytic methods are often extremely useful. Most of the results of Damien's list of known L^2-Betti numbers of groups are obtained analytically.

5.12 Theorem. *If $(\bar{M}, \bar{g})$ has a transitive isometry group (i.e. M is a homogeneous space) then, due to the isometry invariance of $\bar{\Delta}$ and therefore of $f(\bar{\Delta})$ for any function f, all the expressions we encounter when computing L^2-invariants are obtained from universal constants of the homogeneous metric. Γ-traces are obtained from this universal constant by integration over the fundamental domain, i.e. we just have to multiply this constant with the volume of the fundamental domain, which is the volume of M. If $(\bar{M}, \bar{g})$ is, in particular, a symmetric space, harmonic analysis can be employed to compute the relevant universal constant (the Laplacian is related to the Casimir operator, twisted with suitable representations), compare in particular* [9] *where this is carried out in some detail, following earlier work of Lott (and Borel).*

More generally, the argument proves a proportionality principle (due to Cheeer and Gromov): if $(\bar{M}, \bar{g})$ and $(\bar{N}, \bar{h})$ are isometric, the L^2-Betti numbers of M and N are proportional with constant of proportionality the quotient of the volumes of M and of N.

Another situation where only an analytic proof is available for the computation of L^2-Betti numbers is the following result of Gromov [6]:

5.13 Theorem. *Assume that M is compact* Kähler hyperbolic, *meaning by definition that M admits a Kähler metric with Kähler form $\omega \in \Omega^2(M)$ such that the pull-back of ω to the universal covering $\bar{M}$ is the differential of a bounded* 1-*form (this is the case e.g. if M admits a potentially different) metric of negative sectional curvature.*

Recall that a Kähler form is a closed 2-*form ω which is non-degenerate ($\omega^{m/2}$ is a volume form) such that there is a compatible complex structure J on TM: $J \in \mathrm{End}(TM)$ with $J^2 = -1$ integrable such that $\omega(v, Jw)$ is positive definite. Equivalently (Riemannian only): ω is parallel for the Riemannian metric.*

In this case, all L^2-Betti numbers of $\bar{M}$ vanish except for the middle dimensional one. This vanishing result uses the algebraic structure involved (a Lefschetz theorem about injectivity of multiplication with ω). Moreover, the middle dimensional one is definitely positive. This non-vanishing relies on a rather non-trivial index theoretic argument.

6 A more detailed account of the de Rham isomorphism theorem

6.1 Poincaré Lemma chain contractions for $\mathbb{R}^n$

Recall that the de Rham $\Omega^*(\mathbb{R}^n)$ (just all smooth forms, no further condition), augmented by $\Omega^{-1}(\mathbb{R}^n) := \mathbb{R}$ is by the Poincaré lemma a contractible cochain complex. Explicitly, the chain contraction

$$J_p\colon \Omega^p(\mathbb{R}^n) \to \Omega^{p-1}(\mathbb{R}^n)$$

is given by integrayting along straight line segments starting from the origin. Moreover, set $J_0(f) = f(0)$.

Similarly, also the de Rham complex with compact supports $\Omega^*_c(\mathbb{R}^n)$ is contractible, or rather the subcomplex $\Omega^*_{c,0}(\mathbb{R}^n)$ with $\int^n_{\mathbb{R}} \omega = 0$ for $\omega \in \Omega^n_c(\mathbb{R}^n)$.

Again, there is an explicit formula for the chain contraction $J_{p,c}\colon \Omega^p_{c,0}(\mathbb{R}^n) \to \Omega^{p-1}_{c,0}(\mathbb{R}^n)$ involving integration, which is however slightly more complicated. Fix $f\colon \mathbb{R} \to \mathbb{R}$ with compact support and with $\int_{\mathbb{R}} f = 1$.

Let $I_k\colon \Omega^p_c(\mathbb{R}^n) \to \Omega^{p-1}(\mathbb{R}^n)$ be given by integrating along lines paralell to the k-th coordinate axis

$$I_k\omega(x_1, \dots, x_n) = \int_{-\infty}^{x_k} \omega(x_1, \dots, t, x_{k+1}, \dots).$$

We also need $I'_k\colon \Omega^p(\mathbb{R}^n) \to \Omega^{p-1}(\mathbb{R}^{n-1})$ which completely integrates out the k-th coordinate and we set

$$I''_k\omega(x_1, \dots, x_n) := I'_k\omega(x_1, \dots, \hat{x}_k, \dots, x_n) \cdot f(x_k)\, dx_k.$$

The chain contraction is then defined as

$$J_{k,c} := I_1 + I''_1 I_2 + I''_1 I''_2 I_3 + \dots \colon \Omega^k_{c,0}(\mathbb{R}^n) \to \Omega^{k-1}_c(\mathbb{R}^n).$$

For $k = n$ we need $\int_{\mathbb{R}^n} \omega = 0$ to achieve that $J_{n,c}\omega$ indeed is compactly supported.

It is standard that these maps indeed are the required chain contractions, compare e.g. [2].

6.2 Chain contractions for partially compactly supported forms

We also need a mix of compactly supported and arbitrary differential forms. More precisely, consider

$$\Omega^*_c(\mathbb{R}^k) \otimes \Omega^*(\mathbb{R}^{n-k}) := \{\omega \in \Omega^*(\mathbb{R}^n) \mid \exists K \subset \mathbb{R}^k \text{ compact, } \operatorname{supp}(\omega) \subset \mathbb{K} \times \mathbb{R}^{n-k}\}.$$

Note that this is a suitable completion of the algebraic tensor product.

6.1 Lemma. *If $p > k$ and ω is a form of degree p in $\Omega^*_c(\mathbb{R}^k) \otimes \Omega^*(\mathbb{R}^{n-k})$ is closed, then*

$$d(\mathrm{id} \otimes I_p)(\omega) = \omega,$$

where $I_p \otimes \mathrm{id}$ is defined by applying the radial integration (I_p) in the last $n-k$ components.

Similarly, if $p \leq k$, ω is a form of degree p in $\Omega_c^*(\mathbb{R}^k) \otimes \Omega^*(\mathbb{R}^{n-k})$ with $d\omega = 0$ and which satisfies in addition, if $p = k$, that $\int_{\mathbb{R}^k} \omega(\cdot, x) = 0$ for one and then all $x \in \mathbb{R}^{n-k}$ (the latter by Stoke's theorem as $d\omega = 0$) then

$$d(J_{p,c} \otimes \mathrm{id})\omega = \omega.$$

Again, $\mathrm{id} \otimes J_{p,c}$ means we apply the integrations defining $J_{p,c}$ in the first k coordinates.

Proof. This follows (by passage to appropriate limites) from the elementary fact that a contractible cochain complex tensored with any other cochain complex remains contractible, with new cochain contraction given by the old cochain contraction tensored with the identity. The relevant calculation for this is

$$(d_k \otimes 1 + (-1)^{k+l} 1 \otimes c_l) \circ K_{k+1} \otimes 1 + K_* \otimes 1 \circ (d_{k+1} \otimes 1 + (-1)^{(k+1)+l} 1 \otimes c_l)$$
$$= (dK+Kd)\otimes 1 + (-1)^{k+l} K_{k+1} \otimes c_l + (-1)^{(k+1)+l} K_{k+1} \otimes c_l = (dK+Kd)\otimes 1 = 0.$$

In our case, the condition $p > k$ in the first formula implies that we don't see the difference between the non-contractible chain complex $\Omega^*(\mathbb{R}^{n-k})$ and its contractible augmentation: there is no contribution of the form $\alpha \otimes \beta$ with $\deg(\beta) = 0$ to ω, as then $\deg(\alpha) = p > k$ which implies $\alpha = 0 \in \Omega^p(\mathbb{R}^k) = \{0\}$.

For the second formula, the degree condition $p \leq k$ and integral condition means the $\omega \in \Omega_{c,0}^*(\mathbb{R}^k) \otimes \Omega^*(\mathbb{R}^k)$ so that indeed we work with the chain contraction of the contractible chain complex $\Omega_{c,0}^*(\mathbb{R}^k)$.

The chain homotopy condition applied to a closed form then boils down to precisely the claimed formulas. □

6.3 Working on the manifold M

Let M be a smooth manifold (without boundary) and T a smooth triangulation of M (with some conditions to be added below). We recall the standard de Rham isomorphism maps.

6.2 Definition. The de Rham map $R\colon \Omega^*(M) \to C^*(T)$ is defined by

$$\omega \mapsto (\sigma \mapsto \int_\sigma \omega).$$

Whitney defined an explicit chain homotopy inverse as follows: choose a smooth partition of unity $(\phi_\sigma)_{\dim \sigma = 0}$ indexed by the 0-dimensional simplices of the partition and such that the support of ϕ_σ is contained in the open star of σ, i.e. in the union of the interiors of all simplices which contain σ.

We assume that we have a partial ordering on the vertices of T which restricts to a total ordering on each subset spanning a vertex, e.g. by passing to the barycentric subdivision of T.

The Whitney map $W\colon C^*(T) \to \Omega^*(M)$ is now defined by sending a simplex σ spanned by the (ordered) vertices $(v_0, \dots, v_p)$ to

$$W\sigma := p! \sum_{i=0}^{p} (-1)^i \phi_i d\phi_0 \wedge \cdots \wedge \widehat{d\phi_i} \wedge \cdots \wedge d\phi_p,$$

where $\hat{a}$ means as usual that the corresponding term is left out.

We know well that both are chain maps and $R \circ W = \mathrm{id}$. The task is to construct explicitly the chain homotopy $K\colon \Omega^*(M) \to \Omega^{*-1}(M)$ between WR and the identity.

Above, we claimed that the "classical" proof of Whitney [16] of the de Rham isomorphism theorem works essentially without change for the L^2-case, using as additional tool the "spectral cut".

Whithney does not state his proof in a particularly explicit way. Because we need the details of this construction to check that it works in our case we give some additional details here.

In addition to the manifold M and the triangulation T we need and choose the following setup:

- for each open k-simplex σ of the triangulation an open neighborhood U_σ which is contained in the union of the interiors of all simplices which contain σ as a face. We choose a chart diffeomorphism $U_\sigma \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ sending the open k-simplex to $\mathbb{R}^k \times \{0\}$.
- Inductively on the dimension k, we also construct a slightly smaller compact subset L_σ of σ with image L'_k in $\mathbb{R}^k$ under the chart. We require that our chart, restricted to $L'_\sigma \times \mathbb{R}^{m-k}$ maps the higher dimensional faces of the triangulation into subspaces of $\mathbb{R}^m$ containing $\mathbb{R}^k \times \{0\}$.
- Moreover, we choose a smooth compactly supported cutoff function *alpha*$_\sigma$ (with values in $[0,1]$) on $\mathbb{R}^m$ which is identically equal to 1 on a neighborhood of $L'_\sigma \times \{0\}$. We identify α_σ with a corresponding function on M under the chart.
- The union over the faces τ of σ of those points where α_τ is identically equal to 1 is a neighborhood of the boundary of σ and we require (inductviely) L_σ to contains its complement in σ.

6.3 Definition. Let σ be a k-simplex of T, $\omega \in \Omega^p(M)$ such that its restriction to U_σ, under the chart chosen above, is supported in $L'_\sigma \times \mathbb{R}^{m-k}$. In particular, the restriction belongs to the (topological) tensor product $\Omega^*_c(\mathbb{R}^k) \otimes \Omega^*(\mathbb{R}^{m-k})$.

On this, we have the standard chain contraction map $\mathrm{id} \otimes J_p$ from Subsection 6.2 which is given by integrating radially from the origin. Here, we use the chart to identify $U_\sigma \subset M$ with $\mathbb{R}^k \times \mathbb{R}^{m-k}$.

Define $I_{p,\sigma}(\omega) := \alpha_\sigma \cdot \mathrm{id} \otimes J_p \omega$.

Note that $I_{p,\sigma}$ is defined on p-forms which are zero in the appropriate neighborhood of the boundary of σ and produces a $p-1$-form which is supported near σ (on the support of α_σ) and which remains zero in the neighborhood of the boundary of σ.

6.4 Lemma. *Assume that $\omega \in \Omega^p(M)$, σ is a k-simplex with $p > k$ satisfy the condition on the support of Definition 6.3 and $d\omega = 0$.*

Then

$$dI_{p,\sigma}\omega = \omega \qquad \text{on } \{x \mid \alpha_\sigma(x) = 1\}.$$

This follows directly from the chain contraction property of Lemma 6.1. Note that, as $p > k$ we do not encounter contributions $\Omega^p_c(\mathbb{R}^k) \otimes \Omega^0(\mathbb{R}^{m-k})$ where the second chain complex would actually not be quite contractible.

We need the following property of $I_{p,\sigma}$.

6.5 Lemma. *If $p > k$ and τ is a p-simplex which touches σ then $\int_\tau dI_{p,\sigma}\omega = 0$.*

Proof. By Stokes theorem, the integral equals $\int_{\partial\tau} \alpha_\sigma J_p\omega$, where J_p integrates the p-form ω radially along lines which, by our choice of the coordinates, lie inside the face τ (if not $\tau = \sigma$, then the restriction of $J_p\omega$ to σ is 0 by construction). In any case, by this very procedure the pullback of $J_p\omega$ to the face τ is identically zero, proving the claim. □

)

6.6 Definition. If $\omega \in \Omega^p(M)$ andσ is a k-simplex with $p \leq k$, then we use the standard chain contraction of $\Omega_{c,0}(\mathbb{R}^k)$ instead, tensored with the identity on $\Omega^*(\mathbb{R}^{m-k})$.

We require again that the restriction of ω to U_σ is supported in $L'_\sigma \times \mathbb{R}^{m-k}$.

If $k = p$ we assume in addition that the form ω, restricted to $\mathbb{R}^k \times \{x\}$ has total integral 0 (i.e. lives on the subcomplex of $\Omega^*_{c,0}(\mathbb{R}^k)$ which is actually contractible). Note that, if $d\omega = 0$ then by Stokes theorem this holds for all $x \in \mathbb{R}^{m-k}$ if it holds for one, e.g. for $x = 0$.

Define then $I_{p,\sigma}(\omega) := \alpha_\sigma \cdot (J_{p,c} \otimes \mathrm{id})\omega$.

As before, $I_{p,\sigma}$ is defined on certain p-forms which vanish in a neighborhood of the boundary of σ and produces such a form which in addition is supported near σ, namely on the support of α_σ.

6.7 Lemma. *If ω and σ satisfy the conditions of Definition 6.6 and $d\omega = 0$ then*

$$dI_{p,\sigma}\omega = \omega \qquad \text{on } \{x \in M \mid \alpha_\sigma(x) = 1\}$$

6.4 Construction of the chain homotopy K between WR and id

We now give the explicit construction of the desired chain homotopy K^* between id and $W \circ R$.

This we do inductively on the degree, starting with m-forms. For each fixed degree p, we construct the map K^p as a sum of m maps, achieving the required chain homotopy properties first on a neighborhood of the zero skeleton, then the 1-skeleton etc. Indeed, our map essentially is constructed by Whitney in [16], who does however not use the language of chain homotopies. We will construct K^p such that $K^pWR = 0$: the chain homotopy lives only on the complement of the projection WR, where we use $WRWR = W \,\mathrm{id}\, R = WR$.

The inductive construction of K on p-forms (assuming it is already defined on $p+1$-forms) is done as follows:

Given $\omega' \in \Omega^p(M)$, consider

$$\omega'' := (1 - WR)\omega' \text{ and } \omega := \omega'' - K^{p+1}d\omega''.$$

Because $RW = \mathrm{id}$, $R\omega'' = 0$, i.e. the integral of ω'' over each p-simplex is 0. The corresponding statement is true by the same argument for $d\omega'' = d\omega' - WRd\omega'$. By the chain homotopy property of K^{p+1},

$$dK^{p+1}(d\omega'') = d\omega'' \qquad \Longrightarrow\ d\omega = 0.$$

We construct $K^p\omega'$ as a sum of different terms $K_0\omega$, $K_1\omega$,... we assign one after the other to auxiliary p-forms $B_0\omega, B_1\omega, \ldots$ which measure the success already made, i.e. with $B_j\omega = \omega - B_0\omega - B_1\omega \cdots - B_{j-1}\omega$. In particular, for the start of the induction, $B_0\omega := \omega$. Inductively, we require that $dB_j\omega = 0$, $RB_j\omega = 0$, i.e. the integral of $B_j\omega$ over the p-simplices vanishes and for each j-simplex σ the form $B_j\sigma$ vanishes on the neighborhood of the boundary of σ such that $I_{p,\sigma}(B_j\omega)$ is defined if $p > j$ and $J_{p,\sigma}(B_j\omega)$ is defined if $p \le j$. Recall for the condition of vanishing of $\mathbb{R}^k$-integrals if $p = k$ that it suffices, as $dB_j\omega = 0$, that $RB_j\omega = 0$.

Set inductively for $j \ge 0$

$$K_j^p(\omega) := \begin{cases} \sum_{\dim\sigma=j} I_{p,\sigma}(B_j\omega) & p > j \\ \sum_{\dim\sigma=j} J_{p,\sigma}(B_j\omega) & p \le j. \end{cases}$$

By Lemma 6.4, $dK_j^p\omega = B_j\omega$ on the neighborhood of the j-skeleton where the α_σ with $\dim\sigma \le j$ are identically 1.

Define

$$B_{j+1}\omega := B_j\omega - dK_j^p\omega = \omega - dK_0^p\omega - dK_1^p\omega \cdots - dK_{j+1}^p\omega.$$

Then still $dB_{j+1}\omega = 0$ and by Lemma 6.5 also still $RB_{j+1}\omega = 0$. Moreover, by the support condition $B_{m+1}\omega = 0$.

Set now

$$K^p\omega' := \sum_{j=0}^{m} K_m^p\omega.$$

Note that in particular $K^pWR = 0$ as $\omega = (1 - WR)\omega'$.

Rewriting our previous computations we have

$$\begin{aligned} dK^p\omega' = \omega = \omega' - WR\omega' - K^{p+1}d\omega' + \underbrace{K^{p+1}WR}_{=0}d\omega' \\ \implies dK^p + K^{p+1}d = \mathrm{id} - WR. \end{aligned}$$

7 The L^2-de Rham chain homotopy

The construction of the chain homotopy for the de Rham isomoprhism of the previous section is entirely local. It involves:

(1) the stars of the simplices and certain chart diffeomorphisms on them

(2) certain integration maps on these stars of simplices

(3) cutoff functions and multiplication with them, again on stars of simplices

(4) the de Rham differential

(5) the Whitney map in terms of Whitney functions on the stars of the vertices

(6) the de Rham integration map

It is immediate that all of this can be lifted in a Γ=-equivariant way to a Γ-covering.

It remains to argue why the maps are actually defined and bounded on a suitable version of the L^2-de Rham complex. This is not completely trivial, as the whole L^2-de Rham complex contains all forms in the domain of the closure of d. This includes many forms which can not be restricted to simplices of positive codimension.

The main trick is that we use the spectrally cut subcomplex of the de Rham complex:

7.1 Definition. For $\Lambda > 0$ we consider the "spectrally cut" subcomplex $\Omega^*_\Lambda(\bar{M})$ of the L^2-de Rham complex consisting of $\chi_{[0,\Lambda]}(\Delta_p) \subset L^2\Omega^p(\bar{M})$.

Proof. We have to argue why this really is a subcomplex. For this, use the polar decomposition $u\,|d|| : \ker(d)^\perp \to \overline{\operatorname{im}(d)}$.

With this restriction of domain and range (implicit throughout) the operator is injective with dense image, therefore u is an honest unitary, and $|d|| = \sqrt{d^*d|}$.

Note furthermore that under the orthogonal decomposition $L^2\Omega^p(\bar{M}) = \ker(d)^\perp \oplus \ker(\Delta_p) \oplus \overline{\operatorname{im}(d)}$ the Laplacian decomposes as direct sum of self-adjoint operators $d^*d| \oplus 0 \oplus d|d|^*$.

It suffices therefore that $d|\chi_{[0,\Lambda]}(d^*d|) = \chi_{[0,\Lambda]}(d|d|^*)d|$. Now, using the above polar decomposition, $d|^* = |d||\,u^*$, so $\chi_{[0,\Lambda]}(d|d|^*) = \chi_{[0,\Lambda]}(ud^*d|u^*) = u\chi_{[0,\Lambda]}(d^*d|)u^*$.

Consequently,

$$d\chi_{[0,\Lambda]}(d^*d) = u\,|d|\,\chi_{[0,\Lambda]}(|d|^2) = u\chi_{[0,\Lambda]}(|d|^2)u^*u\,|d| = \chi_{[0,\Lambda]}(dd^*)d.$$

□

7.2 Proposition. *The inclusion $\Omega^*_\Lambda \to L^2\Omega^*$ is a chain homotopy equivalence with chain homotopy inverse the orthogonal projection onto the subspace.*

The restrictions of the spectral density functions of the respective Laplacians to $[0,\Lambda]$ are identical.

Proof. The latter statement is evident and indeed is a strong version of the homotopy equivalence statement. The relevant chain homotopy (between the identity and $\chi_{(\Lambda,\infty)}(\Delta)$ on $L^2\Omega^*(\bar{M})$ is given by the inverse of d restricted to $\operatorname{im}\chi_{(\Lambda,\infty)}(d^*d|_{\ker(d)^\perp}$ on which subspace d is invertible with bounded inverse (norm of the inverse bounded by Λ^{-1}). □

We now observe that $\chi_{[0,\Lambda]}(\Delta)$ is evidently contained in the domain of $(1+\Delta)^s$ for each $s > 0$, and $(1+\Delta)^s$ restricted to this subspace is bounded with bounded inverse.

But these domains are precisely the Sobolev spaces and $|(1+\Delta)^s\omega|_{L^2}$ the Sobolev norm. This means that our spectrally cup subcomplex is contained in every Sobolev space, and that restricted to this subspace the L^2-norm is equivalent to any Sobolev norm.

Now we can invoke the standard results about Sobolev spaces, in particular the Sobolev embedding theorems: $\chi_{[0,\Lambda]}(\Delta)$ is contained in $C^k\Omega^p(\bar{M})$ for each k and the inclusion map is continuous (for the L^2-norm on the left).

This implies without difficulty (similar to the considerations of Dodziuk [4] for the full Sobolev spaces) that all maps involved in the definition of W, R, K (lifted Γ-equivariantly to $\bar{M}$), and therefore those maps themselves, are well defined and continuous. It is not garanteed that they take value in $\Omega^*_\Lambda(\bar{M})$, and in general not true. But we can simply compose with $\chi_{[0,\Lambda]}(\Delta)$ to correct that, and in light of Proposition 7.2 the homological properties won't be changed.

This finishes the argument: the correct "local" proof of the de Rham theorem can be applied to the L^2-de Rham complex (after spectrally cutting the latter).

As a consequence we get the strong form of the L^2-de Rham theorem:

7.3 Theorem. *The dilation class of the spectral densitiy functions of the L^2-de Rham complex and of the L^2-simplicial complex of a triangulation coincide.*

This comes from the explicit chain homotopy equivalence obtained by spectrally cutting the de Rham complex and then applying the de Rham map and the Whitney map.

8 Lück's dimension function and a naive application to L^2-de Rham theory

Making systematic the approach of Cheeger-Gromov to L^2-Betti numbers of arbitrary groups (or spaces with a Γ-action), Lück proved the following:

8.1 Theorem. *There is a well-behaved dimension function on arbitrary modules (in the algebraic sense) for the ring $L\Gamma$. It satisfies:*

(1) additivity: the dimension is additive for short exact sequences

(2) normalization: $\dim_\Gamma(L\Gamma) = 1$

(3) continuity: for an increasing union, the dimension is the sup of the dimensions of the constituents.

This allows to define the L^2-Betti numbers of completely arbitrary spaces with a Γ-action (even non-free): use the singular integral chain complex, which consists of $\mathbb{Z}\Gamma$-modules, and tensor up to $L\Gamma$. The homology consists of algebraic $L\Gamma$-modules whose Γ-dimensions are the L^2-Betti numbers.

There are now several ways to use this in the context of differnetial forms:

If M is a smooth manifold with a Γ-action, we can take the de Rham complex and just algebraically tensor over $\mathbb{C}\Gamma$ to $L\Gamma$, arriving at a complex of $L\Gamma$-modules. Does this compute the same $L^{(2)}$-Betti numbers as the singular cochain complex?

There is another variant: for a smooth manifold M with fundamental group Γ (or more generally with a homomorphism $\pi_1(M) \to \Gamma$) we can twist the de Rham complex with the flat $L\Gamma$ module bundle $\tilde{M} \times_\Gamma L\Gamma$. This is a bit less algebraic, but clearly one will get de Rham cohomology groups of $L\Gamma$-modules. Again the question is: what are the resulting $L^{(2)}$-Betti numbers.

Note that we don't want to put in too much further topology. As in classical de Rham theory, we might therefore want to look at just all differential forms (no "growth condition" at all). The de Rham theorem says its cohomology is just the singular cohomology of the space. On the other extreme, we can take the de Rham complex with compact supports, isomorphic to singular cohomology with compact supports.

8.2 Question. This suggests a number of questions: what is the meaning of these analytic L^2-Betti numbers, the relation between the different versions and to the singular versions?

References

[1] M. F. Atiyah, *Elliptic operators, discrete groups and von Neumann algebras*, Colloque "Analyse et Topologie" en l'Honneur de Henri Cartan (Orsay, 1974), Soc. Math. France, Paris, 1976, pp. 43–72. Astérisque, No. 32-33. MR0420729

[2] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982. MR658304

[3] D. Burghelea, L. Friedlander, T. Kappeler, and P. McDonald, *Analytic and Reidemeister torsion for representations in finite type Hilbert modules*, Geom. Funct. Anal. **6** (1996), no. 5, 751–859. MR1415762

[4] Jozef Dodziuk, *de Rham-Hodge theory for L^2-cohomology of infinite coverings*, Topology **16** (1977), no. 2, 157–165. MR0445560

[5] Józef Dodziuk, Peter Linnell, Varghese Mathai, Thomas Schick, and Stuart Yates, *Approximating L^2-invariants and the Atiyah conjecture*, Comm. Pure Appl. Math. **56** (2003), no. 7, 839–873. Dedicated to the memory of Jürgen K. Moser. MR1990479

[6] M. Gromov, *Kähler hyperbolicity and L_2-Hodge theory*, J. Differential Geom. **33** (1991), no. 1, 263–292. MR1085144

[7] Nigel Higson and John Roe, *Analytic K-homology*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. Oxford Science Publications. MR1817560

[8] Varghese Mathai, *L^2-analytic torsion*, J. Funct. Anal. **107** (1992), no. 2, 369–386. MR1172031

[9] Martin Olbrich, *L^2-invariants of locally symmetric spaces*, Doc. Math. **7** (2002), 219–237. MR1938121

[10] Peter A. Linnell, *Division rings and group von Neumann algebras*, Forum Math. **5** (1993), no. 6, 561–576. MR1242889

[11] John Lott, *Heat kernels on covering spaces and topological invariants*, J. Differential Geom. **35** (1992), no. 2, 471–510. MR1158345

[12] Wolfgang Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR1926649

[13] D. V. Efremov and M. A. Shubin, *Spectrum distribution function and variational principle for automorphic operators on hyperbolic space*, Séminaire sur les Équations aux Dérivées Partielles, 1988–1989, École Polytech., Palaiseau, 1989, pp. Exp. No. VIII, 19. MR1032284

[14] Thomas Schick, *L^2-determinant class and approximation of L^2-Betti numbers*, Trans. Amer. Math. Soc. **353** (2001), no. 8, 3247–3265. MR1828605

[15] ______, *Integrality of L^2-Betti numbers*, Math. Ann. **317** (2000), no. 4, 727–750. MR1777117

[16] Hassler Whitney, *Geometric integration theory*, Princeton University Press, Princeton, N. J., 1957. MR0087148

amsbib entries; as from mathscinet