Combinatorial Hopf algebras for nonlinear system interconnections: A view towards discretization

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Overview

1. Introduction

2. Nonrecursive System Interconnections

3. Feedback Interconnection

4. Application in Control Theory: Output Tracking

5. Adapting the Method to the Discrete-Time Case

6. Concluding Remarks

1. Introduction

1.1 Fliess Operators

- Many systems encountered in control theory are in fact interconnections of more elementary subsystems.
- A natural class of nonlinear systems to consider for interconnection is the set of analytic input-output systems known as Fliess operators (Fliess, 1983).
- Such operators are described by convergent Chen-Fliess functional expansions indexed by words over a noncommutative alphabet.
- Their generating series are therefore specified in terms of noncommutative formal power series.

- Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the free monoid comprised of all words over X (including the empty word $\emptyset$) under the catenation product.
- A formal power series in X is any mapping of the form $c : X^* \to \mathbb{R}^\ell$, typically written as the formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$. The set of all such mappings is denoted by $\mathbb{R}^\ell \langle\langle X \rangle\rangle$.
- For each $c \in \mathbb{R}^\ell \langle\langle X \rangle\rangle$, one can associate an m-input, ℓ-output operator F_c in the following manner:
  - With $t_0, T \in \mathbb{R}$ fixed and $T > 0$, define recursively for each $\eta \in X^*$ the mapping $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by

$$E_{x_i \bar{\eta}}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\bar{\eta}}[u](\tau, t_0)\, d\tau,$$

    where $E_\emptyset = 1$, $x_i \in X$, $\bar{\eta} \in X^*$ and $u_0 := 1$.

- The input-output operator corresponding to c is the Fliess operator

$$y = F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0).$$

- If there exist real numbers $K_c, M_c > 0$ such that

$$|(c, \eta)| \leq K_c M_c^{|\eta|} |\eta|!, \quad \forall\, \eta \in X^*,$$

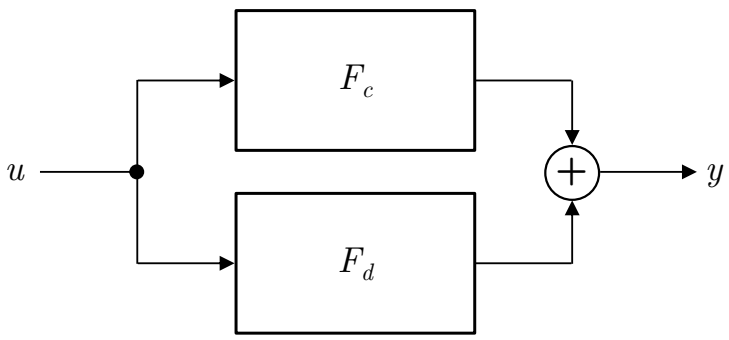
where $|\eta|$ denotes the number of symbols in η, then c is said to be locally convergent. The set of all such series is denoted by $\mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$.

- If $c \in \mathbb{R}^{\ell}_{LC}\langle\langle X \rangle\rangle$ then

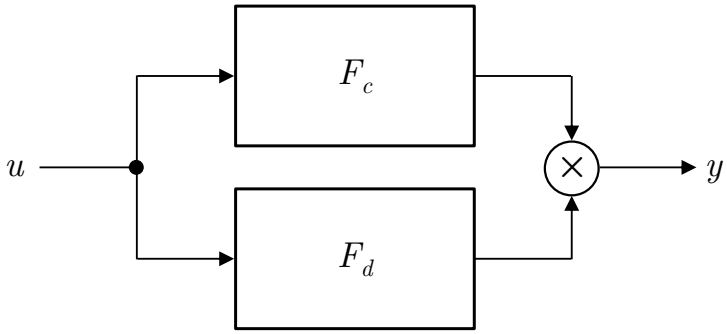
$$F_c : B^m_{\mathfrak{p}}(R)[t_0, t_0 + T] \to B^{\ell}_{\mathfrak{q}}(S)[t_0, t_0 + T]$$

for sufficiently small $R, T > 0$, where the numbers $\mathfrak{p}, \mathfrak{q} \in [1, \infty]$ are conjugate exponents, i.e., $1/\mathfrak{p} + 1/\mathfrak{q} = 1$ (Gray & Wang, 2002).

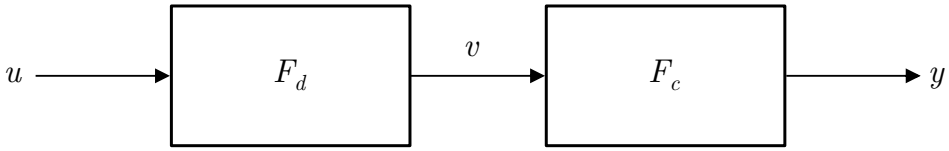
1.2 Interconnections of Fliess Operators



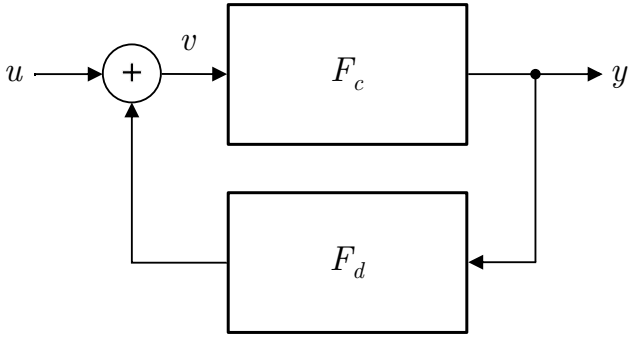
(a) parallel connection



(b) product connection



(c) cascade connection



(d) feedback connection

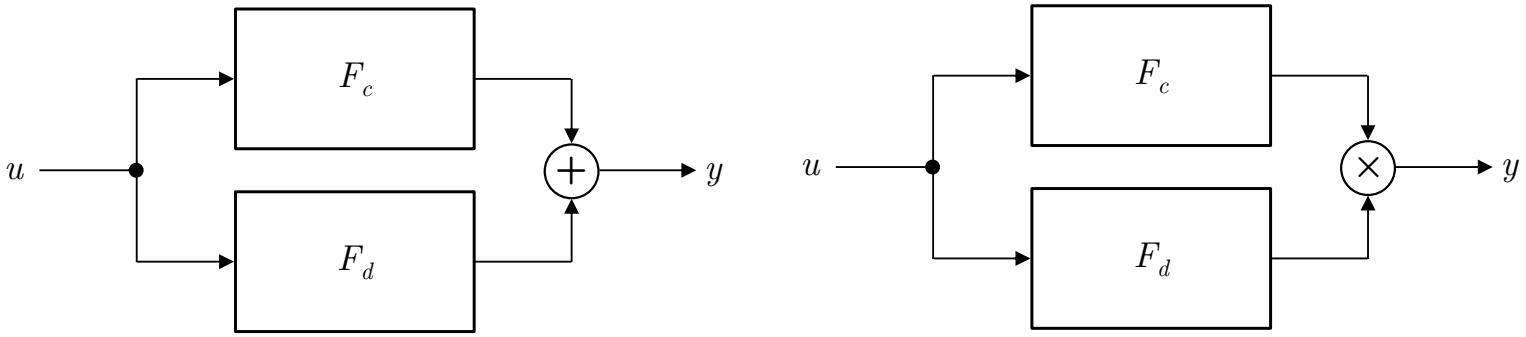
Fig. 1.1 Four elementary system interconnections

Goals of the talk:

- Introduce the algebraic structures that naturally arise when Fliess operators are interconnected.
- Describe some of the computational issues that arise when such theory is used in applications.
- Consider the specific control theoretic problem of output tracking via system inversion.
- Outline an approach to adapt the theory to the discrete-time setting.

2. Nonrecursive System Interconnections

2.1 Parallel and Product Interconnections



(a) parallel connection (b) product connection

Fig. 2.1 Parallel and product system interconnections

Theorem 2.1: (Fliess, 1981) Let $X = \{x_0, x_1, \ldots, x_m\}$. If $c, d \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$ then:

1. $F_c + F_d = F_{c+d}$
2. $F_c \cdot F_d = F_{c \sqcup\!\sqcup d}$.

Remarks:

- The shuffle product of two words is defined inductively by

$$(x_i\eta) \sqcup\!\sqcup (x_j\xi) = x_i(\eta \sqcup\!\sqcup (x_j\xi)) + x_j((x_i\eta) \sqcup\!\sqcup \xi),$$

  where $\eta \sqcup\!\sqcup \emptyset = \emptyset \sqcup\!\sqcup \eta = \eta$.

- A corresponding commutative bilinear product on $\mathbb{R}^{\ell}\langle\langle X\rangle\rangle$ is

$$c \sqcup\!\sqcup d = \sum_{\eta,\xi\in X^*} (c,\eta)(d,\xi)\, \eta \sqcup\!\sqcup \xi.$$

- The shuffle algebra on $\mathbb{R}\langle\langle X\rangle\rangle$ plays a fundamental role here. It describes the products of iterated integrals that appear in the product connection.

2.2 Cascade Interconnection

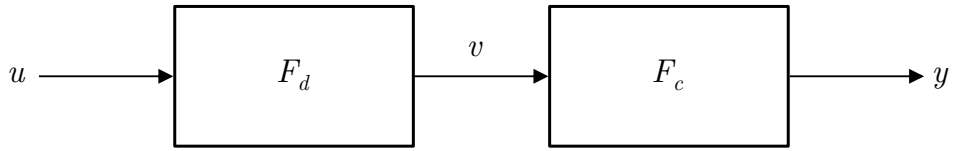


Fig. 2.2 Cascade interconnection

For any $x_i \in X$, $\eta \in X^*$ and $d \in \mathbb{R}^m\langle\langle X\rangle\rangle$ observe

$$E_{x_i\eta}[F_d[u]](t,t_0) = \int_{t_0}^{t} F_{d_i}[u](\tau)\underbrace{E_\eta[F_d[u]](\tau,t_0)}_{:=F_{\eta\circ d}[u](\tau)}\,d\tau = F_{\underbrace{x_0(d_i \sqcup\!\sqcup \eta\circ d)}_{:=\psi_d(x_i)\circ\psi_d(\eta)(1)}}[u](t).$$

Here ψ_d is the continuous algebra homomorphism from $\mathbb{R}\langle\langle X\rangle\rangle$ to $\mathrm{End}(\mathbb{R}\langle\langle X\rangle\rangle)$ uniquely specified by $\psi_d(x_ix_j) := \psi_d(x_i)\circ\psi_d(x_j)$ with

$$\psi_d(x_i)(e) = x_0(d_i \sqcup\!\sqcup e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X\rangle\rangle$, and where d_i is the i-th component series of d ($d_0 := 1$), and $\psi_d(\emptyset)$ is the identity map on $\mathbb{R}\langle\langle X\rangle\rangle$.

Example 2.1: Consider the case where $\eta = x_i x_j$:

$$(x_i x_j) \circ d = x_0(d_i \sqcup x_0(d_j \sqcup 1)) = \psi_d(x_i) \circ \psi_d(x_j)(1) = \psi_d(x_i x_j)(1).$$

Definition 2.1: The composition product of c and d is defined as

$$c \circ d = \sum_{\eta \in X^*} (c, \eta)\, \psi_d(\eta)(1).$$

Remark: This composition product:

- has no unit;
- is $\mathbb{R}$-linear in its left argument;
- is associative;
- distributes to the left over the shuffle product.

Theorem 2.2: (Ferfera, 1979) Let $X = \{x_0, x_1, \ldots, x_m\}$. If $c \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^{m}\langle\langle X \rangle\rangle$ then $F_c \circ F_d = F_{c \circ d}$.

Example 2.2: Consider two linear, time-invariant systems with kernel functions $h_c(t) = \sum_{i\geq 0}(c, x_0^i x_1)t^i/i!$ and $h_d(t) = \sum_{j\geq 0}(d, x_0^j x_1)t^j/j!$.

The kernel of the cascaded system is given by the convolution integral

$$\begin{aligned}
(h_c * h_d)(t) &= \int_0^t h_c(t-\tau)h_d(\tau)\,d\tau \\
&= \sum_{i,j=0}^{\infty}(c, x_0^i x_1)(d, x_0^j x_1)\int_0^t \frac{(t-\tau)^i}{i!}\frac{\tau^j}{j!}\,d\tau \\
&= \sum_{i,j=0}^{\infty}(c, x_0^i x_1)(d, x_0^j x_1)\frac{t^{i+j+1}}{(i+j+1)!} \\
&= \sum_{i=1}^{\infty}\underbrace{\sum_{j=0}^{i-1}(c, x_0^{i-j-1}x_1)(d, x_0^j x_1)}_{(c\circ d, x_0^i x_1)}\frac{t^i}{i!} \\
&= h_{c\circ d}(t).
\end{aligned}$$

3.0 Feedback Interconnection

3.1 Feedback Product Defined

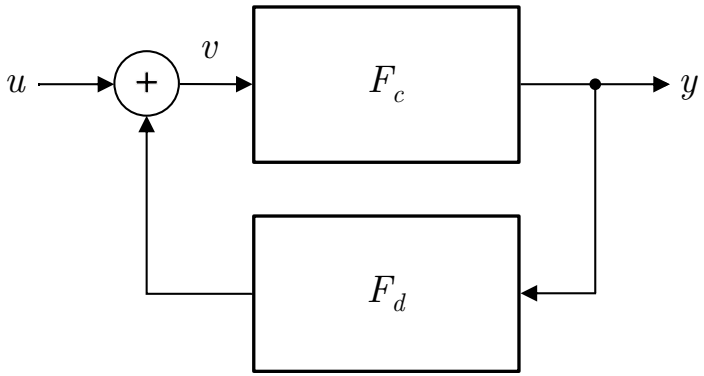


Fig. 3.1 Feedback connection of F_c and F_d

Given $c, d \in \mathbb{R}^m\langle\langle X\rangle\rangle$, the output y satisfies the feedback equation

$$y = F_c[v] = F_c[u + F_d[y]].$$

If there exists a generating series e so that $y = F_e[u]$, then

$$F_e[u] = F_c[u+F_{d\circ e}[u]] = F_{c\tilde{\circ}(d\circ e)}[u],$$

where $\tilde{\circ}$ denotes the modified composition product.

Specifically, the modified composition product is defined as

$$c \,\tilde{\circ}\, d = \sum_{\eta \in X^*} (c, \eta)\, \phi_d(\eta)(1),$$

where ϕ_d algebra homomorphism from $\mathbb{R}\langle\langle X \rangle\rangle$ to $\mathrm{End}(\mathbb{R}\langle\langle X \rangle\rangle)$ uniquely specified by

$$\phi_d(x_i)(e) = x_i e + x_0 (d_i \sqcup\!\sqcup e),$$

$i = 0, 1, \ldots, m$ for any $e \in \mathbb{R}\langle\langle X \rangle\rangle$, and where $d_0 := 0$.

Definition 3.1: The feedback product of c and d, namely $c@d$, is the unique fixed point of the (ultrametric) contractive map on $\mathbb{R}^m\langle\langle X \rangle\rangle$

$$\mathcal{T} : e_i \mapsto e_{i+1} = c\tilde{\circ}(d \circ e_i).$$

That is, $c@d = e$, where $e = c\tilde{\circ}(d \circ e)$.

Remarks:

- This approach yields no explicit way to compute $c@d$.
- So we turn to Hopf algebra methods to do the computation.

3.2 Faà di Bruno Hopf Algebra for Fliess Operators

Define the set of operators

$$\mathscr{F}_\delta = \{I + F_c \ : \ c \in \mathbb{R}^m\langle\langle X\rangle\rangle\}.$$

Define the Dirac symbol, δ, so that $F_\delta = I$ and $I + F_c = F_{\delta+c} = F_{c_\delta}$ with $c_\delta := \delta + c$. In which case, $c \,\tilde{\circ}\, d = c \circ (\delta + d)$.

The set of all such generating series for $\mathscr{F}_\delta$ will be denoted by $\mathbb{R}^m\langle\langle X_\delta\rangle\rangle$.

The composition of two elements in $\mathscr{F}_\delta$ is

$$\begin{aligned} F_{c_\delta} \circ F_{d_\delta} &= (I + F_c) \circ (I + F_d) = I + F_d + F_{c \,\tilde{\circ}\, d} \\ &= F_{\delta + \underbrace{d + c \,\tilde{\circ}\, d}_{:= c \circledcirc d}} =: F_{c_\delta \circ d_\delta}. \end{aligned}$$

Theorem 3.1: (Gray et al., 2014) The triple $(\mathscr{F}_\delta, \circ, I)$, or equivalently $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, forms a group.

The goal is to describe a Faà di Bruno type Hopf algebra associated with the group $(\mathbb{R}^m\langle\langle X_\delta\rangle\rangle, \circ, \delta)$, where the antipode, S, satisfies the identity

$$c_\delta^{-1} = \delta + c^{-1} = \delta + \sum_{\eta\in X^*} (Sa_\eta)(c)\,\eta,$$

with coordinate maps

$$a_\eta^i : \mathbb{R}^m\langle\langle X\rangle\rangle \to \mathbb{R} : c \mapsto (c_i, \eta).$$

Let V denote the $\mathbb{R}$-vector space spanned by these maps.

Consider the free unital commutative $\mathbb{R}$-algebra, H, with product

$$\mu : a_\eta^i \otimes a_\xi^j \mapsto a_\eta^i a_\xi^j.$$

The unit $\mathbf{1}$ maps every c to 1.

Three coproducts are considered. The first, Δ, is used to define a Hopf algebra on H, while the next two provide a recursive way to compute Δ.

Recalling that $c_\delta \circ d_\delta = \delta + c \circledcirc d$, define the first coproduct Δ for any $a^i_\eta \in V_+$ such that

$$\Delta a^i_\eta(c, d) = a^i_\eta(c \circledcirc d) = (c_i \circledcirc d, \eta).$$

The second coproduct $\Delta^j{}_{\sqcup\!\sqcup}\,(V_+) \subset V_+ \otimes V_+$, is a deshuffling isomorphism via the coordinate maps. That is,

$$\Delta^j{}_{\sqcup\!\sqcup}\, a^i_\emptyset = a^i_\emptyset \otimes a^j_\emptyset \tag{1a}$$

$$\Delta^j{}_{\sqcup\!\sqcup}\, \circ \theta_k = (\theta_k \otimes \mathrm{id} + \mathrm{id} \otimes \theta_k) \circ \Delta^j{}_{\sqcup\!\sqcup}\,, \tag{1b}$$

where θ_k denotes the endomorphism on V_+ specified by $\theta_k a^i_\eta = a^i_{x_k \eta}$. The first few terms of $\Delta^j{}_{\sqcup\!\sqcup}$ are:

$$\begin{aligned}
\Delta^j{}_{\sqcup\!\sqcup}\, a^i_\emptyset &= a^i_\emptyset \otimes a^j_\emptyset \\
\Delta^j{}_{\sqcup\!\sqcup}\, a^i_{x_{i_1}} &= a^i_{x_{i_1}} \otimes a^j_\emptyset + a^i_\emptyset \otimes a^j_{x_{i_1}} \\
\Delta^j{}_{\sqcup\!\sqcup}\, a^i_{x_{i_2} x_{i_1}} &= a^i_{x_{i_2} x_{i_1}} \otimes a^j_\emptyset + a^i_{x_{i_2}} \otimes a^j_{x_{i_1}} + a^i_{x_{i_1}} \otimes a^j_{x_{i_2}} + a^i_\emptyset \otimes a^j_{x_{i_2} x_{i_1}}.
\end{aligned}$$

The third coproduct is $\tilde{\Delta} a^i_\eta = \Delta a^i_\eta - \mathbf{1} \otimes a^i_\eta$ or, equivalently, the coproduct induced by the identity

$$\tilde{\Delta} a^i_\eta(c, d) = a^i_\eta(c \,\tilde{\circ}\, d) = (c_i \,\tilde{\circ}\, d, \eta).$$

Lemma 3.1: (Gray et al., 2014) The following identities hold:

(1) $\tilde{\Delta} a^i_\emptyset = a^i_\emptyset \otimes \mathbf{1}$

(2) $\tilde{\Delta} \circ \theta_i = (\theta_i \otimes \mathrm{id}) \circ \tilde{\Delta}$

(3) $\tilde{\Delta} \circ \theta_0 = (\theta_0 \otimes \mathrm{id}) \circ \tilde{\Delta} + (\theta_i \otimes \mu) \circ (\tilde{\Delta} \otimes \mathrm{id}) \circ \Delta^i{}_{\sqcup\!\sqcup}$,

$i = 1, 2, \ldots, m$, where id denotes the identity map on H.

Remark: The Einstein summation notation is used in item (3) and throughout to indicate summations from either 0 or 1 to m, e.g., $\sum_{i=1}^m a_i b^i = a_i b^i$.

Table 3.1: Products and coproducts induced by Fliess operator compositions

operator composition	generating series	coproduct	remarks
$F_c \circ F_d$	$c \circ d$	Δ'	associative
$F_c \circ (I + F_d)$	$c \,\tilde{\circ}\, d$	$\tilde{\Delta}$	nonassociative
$(I + F_c) \circ F_d$	$c \,\hat{\circ}\, d$	$\hat{\Delta}$	nonassociative
$(I + F_c) \circ (I + F_d)$	$c \circledcirc d$	Δ	group product

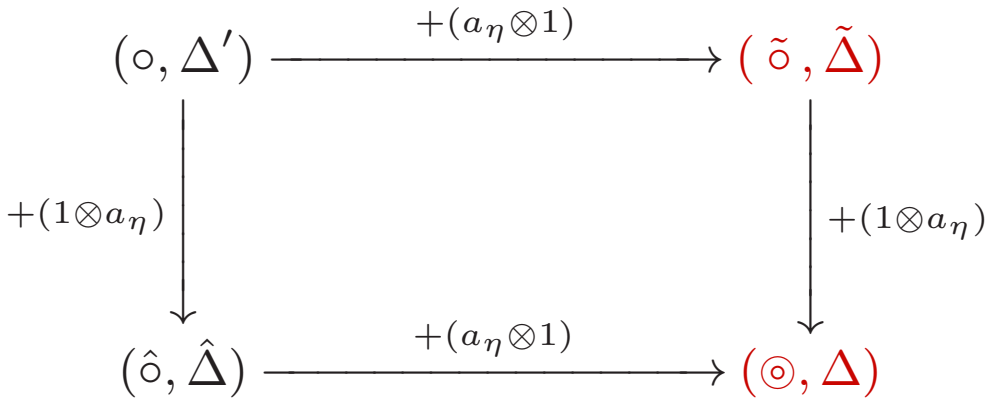


Fig. 3.2: Relationships between the products and coproducts induced by series composition

Let $|\eta|_{x_i}$ denote the number of times $x_i \in X$ appears in the word η.

The degree of a^i_η is $\deg(a^i_\eta) := 2\,|\eta|_{x_0} + \sum_{j=1}^m |\eta|_{x_j} + 1$. Let V_k be the subspace of degree k coordinate maps. Then V is a graded vector space.

This grading on V induces a connected grading on H since $H_0 = \mathbb{R}\mathbf{1}$.

Furthermore, $\tilde{\Delta} a^i_\eta \in (V_+ \otimes H)_n := \bigoplus_{\substack{j+k=n \\ j\geq 1, k\geq 0}} V_j \otimes H_k$. For example,

$$
\begin{aligned}
n = 1 &: \tilde{\Delta} a^i_\emptyset = a^i_\emptyset \otimes \mathbf{1} \\
n = 2 &: \tilde{\Delta} a^i_{x_j} = a^i_{x_j} \otimes \mathbf{1} \\
n = 3 &: \tilde{\Delta} a^i_{x_0} = a^i_{x_0} \otimes \mathbf{1} + a^i_{x_\ell} \otimes a^\ell_\emptyset \\
n = 3 &: \tilde{\Delta} a^i_{x_j x_k} = a^i_{x_j x_k} \otimes \mathbf{1} \\
n = 4 &: \tilde{\Delta} a^i_{x_0 x_j} = a^i_{x_0 x_j} \otimes \mathbf{1} + a^i_{x_\ell} \otimes a^\ell_{x_j} + a^i_{x_\ell x_j} \otimes a^\ell_\emptyset \\
n = 4 &: \tilde{\Delta} a^i_{x_j x_0} = a^i_{x_j x_0} \otimes \mathbf{1} + a^i_{x_j x_\ell} \otimes a^\ell_\emptyset \\
n = 4 &: \tilde{\Delta} a^i_{x_j x_k x_l} = a^i_{x_j x_k x_l} \otimes \mathbf{1}.
\end{aligned}
$$

Theorem 3.2: (Gray et al., 2014) (H, μ, Δ) is a connected graded commutative unital Hopf algebra.

Theorem 3.3: (Gray et al., 2014) The Hopf algebra (H, μ, Δ) has an antipode S satisfying $a_\eta^i(c^{-1}) = (Sa_\eta^i)(c)$ for all $\eta \in X^*$ and $c \in \mathbb{R}^m\langle\langle X\rangle\rangle$.

Theorem 3.4: (Figueroa and Gracia-Bondía, 2005) The antipode, S, of any graded connected Hopf algebra (H, μ, Δ) can be computed for $a \in H_k$, $k \geq 1$ by

$$Sa = -a - \sum (Sa'_{(1)})a'_{(2)} = -a - \sum a'_{(1)} Sa'_{(2)}, \tag{2}$$

where the reduced coproduct is defined as $\Delta' a = \Delta a - a \otimes \mathbf{1} - \mathbf{1} \otimes a$ $= \sum a'_{(1)} a'_{(2)}$.

Theorem 3.5: (Gray et al., 2014) The antipode, S, of any coordinate map $a_\eta^i \in V_k$, $k \geq 1$ in the Hopf algebra (H, μ, Δ) can be computed recursively by the following algorithm:

i. Recursively compute $\Delta^{\sqcup\!\sqcup}$ via (1).

ii. Recursively compute $\tilde{\Delta}$ via Lemma 3.1.

iii. Recursively compute S via Theorem 3.4 with $\Delta' a_\eta^i = \tilde{\Delta} a_\eta^i - a_\eta^i \otimes \mathbf{1}$.

Remark: While the left recursion $Sa = -a - \sum (Sa'_{(1)}) a'_{(2)}$ and the right recursion $Sa = -a - \sum a'_{(1)} Sa'_{(2)}$ are mathematically identical, they differ significantly from a computational point of view.

Applying the left recursion antipode formula gives:

$$H_1 : Sa^i_{\emptyset} = -a^i_{\emptyset}$$

$$H_2 : Sa^i_{x_j} = -a^i_{x_j}$$

$$H_3 : Sa^i_{x_0} = -a^i_{x_0} + a^i_{x_\ell} a^\ell_{\emptyset}$$

$$H_3 : Sa^i_{x_j x_k} = -a^i_{x_j x_k}$$

$$H_4 : Sa^i_{x_0 x_j} = -a^i_{x_0 x_j} + a^i_{x_\ell} a^\ell_{x_j} + a^i_{x_\ell x_j} a^\ell_{\emptyset}$$

$$H_4 : Sa^i_{x_j x_0} = -a^i_{x_j x_0} + a^i_{x_j x_\ell} a^\ell_{\emptyset}$$

$$H_4 : Sa^i_{x_j x_k x_l} = -a^i_{x_j x_k x_l}$$

$$\begin{aligned} H_5 : Sa^i_{x_0^2} &= -a^i_{x_0^2} - (Sa^i_{x_\ell}) a^\ell_{x_0} - (Sa^i_{x_\ell x_0}) a^\ell_{\emptyset} - (Sa^i_{x_0 x_\ell}) a^\ell_{\emptyset} - (Sa^i_{x_\ell x_\nu}) a^\ell_{\emptyset} a^\nu_{\emptyset} \\ &= -a^i_{x_0^2} - (-a^i_{x_\ell}) a^\ell_{x_0} - (-a^i_{x_\ell x_0} + \cancel{a^i_{x_\ell x_\nu} a^\nu_{\emptyset}}) a^\ell_{\emptyset} - \\ &\quad (-a^i_{x_0 x_\ell} + a^i_{x_\nu} a^\nu_{x_\ell} + a^i_{x_\nu x_\ell} a^\nu_{\emptyset}) a^\ell_{\emptyset} - \cancel{(-a^i_{x_\ell x_\nu}) a^\ell_{\emptyset} a^\nu_{\emptyset}} \\ &= -a^i_{x_0^2} + a^i_{x_\ell} a^\ell_{x_0} + a^i_{x_\ell x_0} a^\ell_{\emptyset} + a^i_{x_0 x_\ell} a^\ell_{\emptyset} - a^i_{x_\nu} a^\nu_{x_\ell} a^\ell_{\emptyset} - a^i_{x_\nu x_\ell} a^\nu_{\emptyset} a^\ell_{\emptyset}. \end{aligned}$$

Table 3.2: Total number of terms (counting multiplicities) and cancellations when the left recursion antipode formula is used

coordinate map degree	coordinate map antipode	total number of unique terms	total number of terms	number of cancelations
3	Sa_{x_0}	2	2	0
5	$Sa_{x_0^2}$	6	6	1
7	$Sa_{x_0^3}$	17	26	9
9	$Sa_{x_0^4}$	50	150	70
11	$Sa_{x_0^5}$	139	1082	427
13	$Sa_{x_0^6}$	390	9366	2417
15	$Sa_{x_0^7}$	1059	94,586	12,730

$$\Delta a^{i}_{x_j x_0} = a^{i}_{x_j x_0} \otimes \mathbf{1} + \mathbf{1} \otimes a^{i}_{x_j x_0} + a^{i}_{x_j x_\ell} \otimes a^{\ell}_{\emptyset}$$

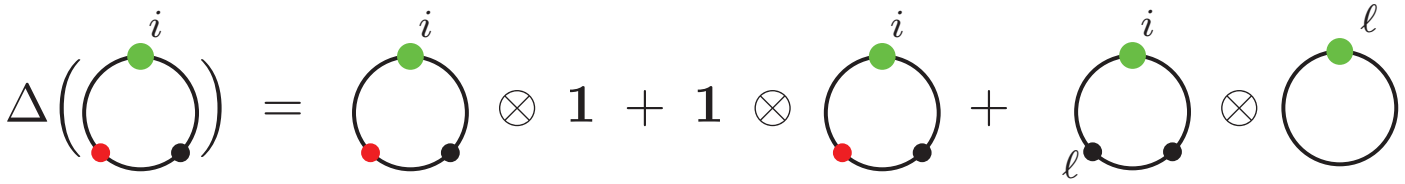


Fig. 3.3 Example of coproduct Δ in terms of rct's

- Each coordinate map can be associated with decorated rooted circle tree (rct).
- There exists a Hopf algebra of rct's which is isomorphic to H.
- Using extractions and decorated rooted trees to encode their hierarchical structure, the right recursion antipode formula can be shown to be cancellation-free (Duffaut Espinosa et al., 2014).
- Therefore, this antipode formula is a type of Zimmermann forest formula, which first appeared in the context of the renormalization problem in perturbative quantum field theory (Zimmermann, 1969).

3.3 Explicit Formula for the Feedback Product

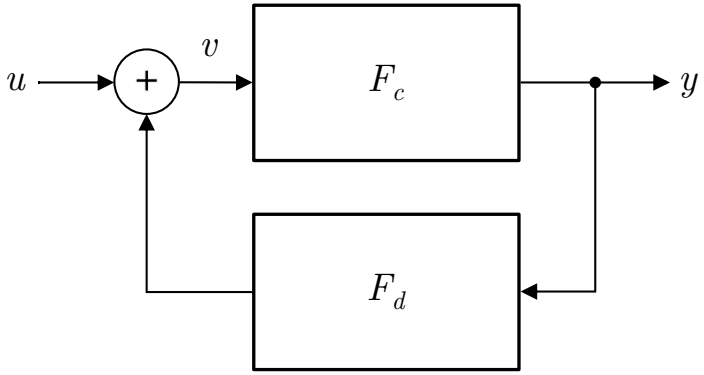


Fig. 3.4: Feedback connection of F_c and F_d

Observe that

$$v = u + F_{d \circ c}[v],$$

and therefore,

$$(I + F_{-d \circ c})\,[v] = u.$$

Simply apply the compositional inverse to both sides of this equation:

$$\begin{aligned} v &= (I + F_{(-d)\circ c})^{-1}[u] \\ &= \left(I + F_{(-d\circ c)^{-1}}\right)[u]. \end{aligned}$$

In which case,

$$F_{c@d}[u] = F_c[v] = F_{c\,\tilde{\circ}\,(-d\circ c)^{-1}}[u].$$

An explicit formula for the feedback product is

$$c@d = c\,\tilde{\circ}\,(-d\circ c)^{-1} = c\circ(\delta - d\circ c)^{-1}$$

$$(\delta - d\circ c)^{-1} = \delta + \textstyle\sum_{\eta\in X^*}(Sa_\eta)(-d\circ c)\,\eta.$$

4. Application in Control Theory: Output Tracking

- Given a single-input, single-output (SISO) system, F, and a function y in the range of F, the left inversion problem is to determine an input u such that $y = F[u]$.
- In control theory, finding a u to make the output $F[u]$ equivalent to a specific function y is known as the output tracking problem.
- Given a smooth input-affine state space realization of F with finite dimension, it is known that a well defined relative degree is a sufficient (but not necessary) condition for solving this problem via feedback linearization (Isidori, 1995).
- The goal here is to provide an explicit analytical solution to this problem in the case where F has a Fliess operator representation, and where y is a real analytic function (Gray et al., 2014).

First observe that any $c \in \mathbb{R}\langle\langle X \rangle\rangle$ can be written as $c = c_N + c_F$, where $c_N := \sum_{k \geq 0} (c, x_0^k) x_0^k$ and $c_F := c - c_N$.

Definition 4.1: Given $c \in \mathbb{R}\langle\langle X \rangle\rangle$, let $r \geq 1$ be the largest integer such that $\text{supp}(c_F) \subseteq x_0^{r-1} X^*$. Then c has relative degree r if the linear word $x_0^{r-1} x_1 \in \text{supp}(c)$, otherwise it is not well defined.

Remark: This definition coincides with the usual definition of relative degree given in a state space setting. But this definition does not require the existence of a state space realization.

Example 4.1: Consider Example 4.1.5 from Isidori's text:

$$\dot{z} = \begin{bmatrix} z_1 z_2 - z_1^3 \\ z_1 \\ -z_3 \\ z_1^2 + z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 + 2z_3 \\ 1 \\ 0 \end{bmatrix} u, \quad y = z_4.$$

Using the usual definition, the system has relative degree $r = 2$ at any point where $z_3(0) \neq -1$, and r is undefined otherwise.

If $z(0) = 0$ then the corresponding generating series is

$$c = 2x_0x_1 + 2x_0x_1^2 - 2x_0x_1x_0x_1 + 2x_0x_1x_0^2x_1 - \cdots$$

Here $r = 2$ since $\text{supp}(c_F) = \text{supp}(c) \subset x_0X^*$ and $x_0x_1 \in \text{supp}(c)$.

If $z(0) = [0, 0, -1, 0]^T$ then

$$c = 2x_0x_1x_0 + 2x_0x_1x_1 - 2x_0x_1x_0^2 - 2x_0x_1x_0x_1 + \cdots$$

So r is not defined since $\text{supp}(c_F) = \text{supp}(c) \subset x_0X^*$ but $x_0x_1 \notin \text{supp}(c)$.

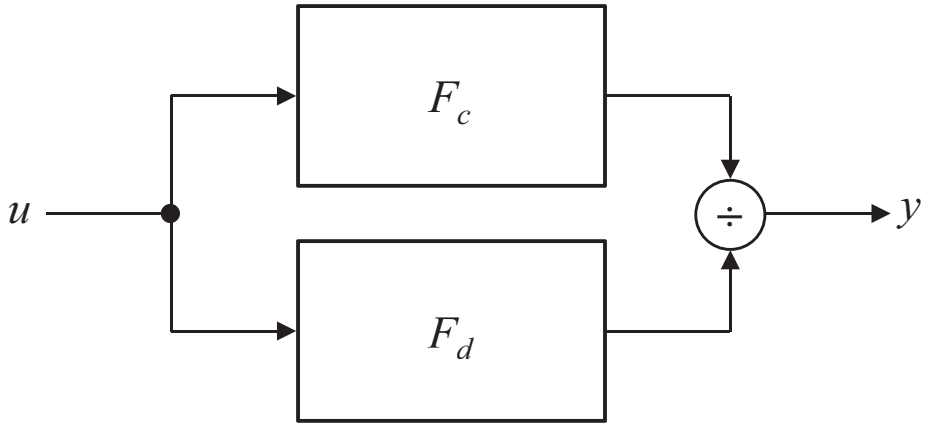


Fig. 4.1: Quotient connection

Theorem 4.1: The set of non proper series in $\mathbb{R}\langle\langle X\rangle\rangle$ is a group under the shuffle product, where

$$c^{\sqcup\!\sqcup -1} = ((c,\emptyset)(1-c'))^{\sqcup\!\sqcup -1} := (c,\emptyset)^{-1}\sum_{k=0}^{\infty}(c')^{\sqcup\!\sqcup k},$$

and $c' = 1 - c/(c,\emptyset)$ is proper.

Theorem 4.2: For $c, d \in \mathbb{R}\langle\langle X\rangle\rangle$, the quotient connection has a Fliess operator representation if and only if d is non proper. In particular, $F_c/F_d = F_{c/d}$, where $c/d := c \sqcup\!\sqcup d^{\sqcup\!\sqcup -1}$.

Definition 4.2: Given any $\xi \in X^*$, the corresponding left-shift operator is

$$\xi^{-1} : X^* \to \mathbb{R}\langle X \rangle : \eta \mapsto \begin{cases} \eta' & : \quad \eta = \xi\eta' \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Lemma 4.1: If $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has relative degree r then $(x_0^{r-1}x_1)^{-1}(c)$ is non proper.

Theorem 4.3: Suppose $c \in \mathbb{R}\langle\langle X \rangle\rangle$ has relative degree r. Let y be analytic at $t = 0$ with generating series $c_y \in \mathbb{R}_{LC}[[X_0]]$ satisfying $(c_y, x_0^k) = (c, x_0^k)$, $k = 0, \ldots, r-1$. (Here $X_0 := \{x_0\}$.) Then the input

$$u(t) = \sum_{k=0}^{\infty} (c_u, x_0^k) \frac{t^k}{k!},$$

where $c_u = (((x_0^r)^{-1}(c - c_y)/(x_0^{r-1}x_1)^{-1}(c))^{\circ -1})_N$, is the unique solution to $F_c[u] = y$ on $[0, T]$ for some $T > 0$.

Remark: The condition on c_y ensures that y is in the range of F_c.

Example 4.2: Reconsider the previous example with $z(0) = 0$. Since $r = 2$ and $c_N = 0$, the range of F_c contains all analytic outputs with generating series $c_y = \sum_{k\geq 2}(c_y, x_0^k)x_0^k$.

Suppose $y(t) = t^2/2$. Then $c_y = x_0^2$, $(x_0^{-2})(c - c_y) = -1$, and

$$(x_0x_1)^{-1}(c) = 2 + 2x_1 - 2x_0x_1 + 2x_0^2x_1 - 2x_0^3x_1 + \cdots .$$

With the assistance of Mathematica, it follows that

$$\frac{(x_0^{-2})(c - c_y)}{(x_0x_1)^{-1}(c)} = -\frac{1}{2} + \frac{1}{2}x_1 - \frac{1}{2}x_0x_1 - x_1^2 + \frac{1}{2}x_0^2x_1 + 2x_0x_1^2 + x_1x_0x_1 + 3x_1^3 - \cdots .$$

Computing the antipode of this series gives the corresponding input

$$u(t) = \frac{1}{2} - \frac{1}{4}t + \frac{5}{8}\frac{t^2}{2!} - \frac{35}{16}\frac{t^3}{3!} + \frac{307}{32}\frac{t^4}{4!} - \cdots .$$

If c_u is truncated to third order to give $\hat{c}_u$, then the error is

$$c_y - c \circ \hat{c}_u = \frac{331}{16}x_0^6 - \frac{1999}{32}x_0^7 + \frac{12855}{64}x_0^8 - \frac{83527}{128}x_0^9 + \cdots .$$

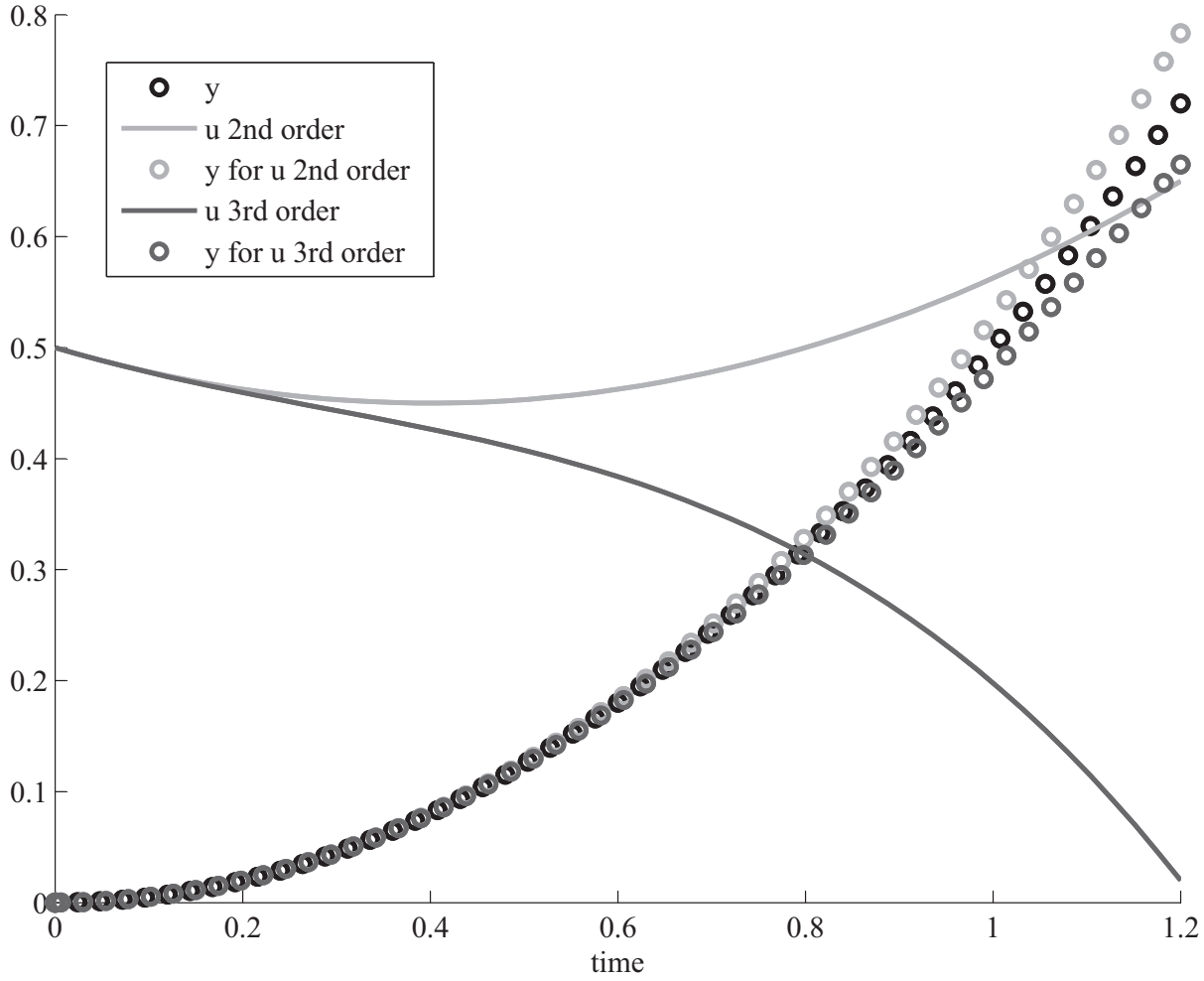


Fig 4.2: Output of F_c for Taylor series approximations of u up to third order and $y(t) = t^2/2$

5. Adapting the Method to the Discrete-time Case

One way to start is to construct a discrete-time approximation of a Fliess operator F_c via numerical integration.

Consider a fixed interval $[0, T]$ and select some $L \in \mathbb{N}^+$. With $\Delta := T/L$, define the sequence

$$\hat{u}_i(N) = \int_{(N-1)\Delta}^{N\Delta} u_i(t)\, dt, \quad i = 0, 1 \ldots, m$$

for all $N \in [1, L]$ so that

$$S_{x_i\eta}[\hat{u}](N) := \sum_{k=1}^{N} \hat{u}_i(k) S_\eta[\hat{u}](k)$$

with $S_\emptyset[\hat{u}](N) := 1$.

That is, iterated sums replace iterated integrals.

Theorem 5.1: (Gray et al., 2015) Over any interval $[0, T]$ with $T > 0$ and $\Delta = T/L$, $L \in \mathbb{N}^+$, it follows for $\eta \in X^k$ with $k \geq 2$ that

$$E_\eta[u](T, 0) = S_\eta[\hat{u}](L) + O\left(\frac{T^{|\eta|}}{(|\eta| - 2)!L}\right).$$

If $c \in \mathbb{R}^m_{LC}\langle\langle X \rangle\rangle$ with growth constants $K, M > 0$ then

$$F_c[u](T) = \sum_{\eta \in X^*} (c, \eta) S_\eta[\hat{u}](L) + O\left(\frac{1}{L} \frac{(MT(m+1))^2}{(1 - MT(m+1))^3}\right)$$

provided that $T < 1/M(m+1)$.

Remark: The condition $T < 1/M(m+1)$ ensures that F_c converges over $[0, T]$ for sufficiently small inputs (Duffaut Espinosa et al., 2009).

This motivates the following definition.

Definition 5.1: (version 1) For any $\mathrm{c} \in \mathbb{R}^m\langle\langle X\rangle\rangle$ the associated discrete-time Fliess operator is

$$\mathrm{F_c[u]}(N) = \sum_{\eta \in X^*} (\mathrm{c}, \eta) S_\eta[\mathrm{u}](N).$$

To pursue an interconnection theory for such operators, the shuffle algebra needs to be replaced with the quasi-shuffle algebra to recover the relation

$$S_\eta[\mathrm{u}] S_\xi[\mathrm{u}] = S_{\eta * \xi}[\mathrm{u}]$$

(Hoffman, 2000). Specifically,

$$(x_i\eta') * (x_j\xi') = x_i(\eta' * (x_j\xi')) + x_j((x_i\eta') * \xi') - [x_i, x_j](\eta' * \xi'),$$

where $[x_i, x_j]$ corresponds to the product $u_i u_j$.

Remarks:

- This scheme requires the product $[x_i, x_j]$ to be treated like a new letter.
- Consider the SISO case where $X = \{x_0, x_1\}$ corresponding to inputs $u_0 = 1$ and $u_1 := u$. Define the new alphabet $Q = \{q_0, q_1, q_2, \ldots\}$ corresponding to inputs $\{u_0, u_1, u_2, \ldots\} \sim \{u^0, u^1, u^2, \ldots\}$. This allows one to define the product $[q_i, q_j] = q_{i+j}$.
- In this context, it is natural to extend the notion of iterated sums so that

$$S_{q_i\eta}[\mathrm{u}](N) = \sum_{k=1}^{N} u^i(k) S_\eta[\mathrm{u}](k).$$

- Aside from the "index war" that this generalization creates in the multivariable case, there is also some concern it may cause well definedness problems downstream. This is an open problem.

Example 5.1: Consider the discrete-time Fliess operator $\mathrm{y} = \mathrm{F}_{\mathrm{c}}[\mathrm{u}]$ where $\mathrm{c} = \sum_{k \geq 0} q_1^k$.

In which case,

$$\mathrm{y}(N) = S_{\emptyset}[\mathrm{u}](N) + S_{q_1}[\mathrm{u}](N) + S_{q_1^2}[\mathrm{u}](N) + S_{q_1^3}[\mathrm{u}](N) + \cdots$$

so that

$$\begin{aligned}
\mathrm{y}(0) &= 1 \\
\mathrm{y}(1) &= 1 + \mathrm{u}(1) + \mathrm{u}^2(1) + \mathrm{u}^3(1) + \cdots \\
\mathrm{y}(2) &= 1 + (\mathrm{u}(1) + \mathrm{u}(2)) + (\mathrm{u}^2(1) + \mathrm{u}(2)\mathrm{u}(1) + \mathrm{u}^2(2)) + \\
&\qquad (\mathrm{u}^3(1) + \mathrm{u}^2(2)\mathrm{u}(1) + \mathrm{u}(2)\mathrm{u}^2(1) + \mathrm{u}^3(2)) + \cdots \\
&\vdots
\end{aligned}$$

These sums can be rearranged so that

$$\mathrm{y}(N) = \sum_{i_1,\ldots,i_N=0}^{\infty} \mathrm{u}^{i_N}(N)\mathrm{u}^{i_{N-1}}(N-1)\cdots\mathrm{u}^{i_1}(1) = \prod_{k=1}^{N}(1-\mathrm{u}(k))^{-1}.$$

Therefore, $\mathrm{F_c}$ has the rational-input state-affine realization

$$\mathrm{z}(N+1) = (1-\mathrm{u}(N+1))^{-1}\mathrm{z}(N),\ \mathrm{z}(0) = 1$$
$$\mathrm{y}(N) = \mathrm{z}(N),$$

provided $\|\mathrm{u}\|_\infty < 1$.

Remarks:

- The class of polynomial-input state-affine discrete-time systems has been widely studied by Sontag et al. (1979).
- Note that this state space realization is time-reversible.

The next logical step is to define a composition product on $\mathbb{R}\langle\langle Q\rangle\rangle$ to satisfy the identity $\mathrm{F_c} \circ \mathrm{F_d} = \mathrm{F_{c \circ d}}$.

In the special case where $c = q_i \eta \in Q^*$ observe

$$
\begin{aligned}
(\mathrm{F_{q_i \eta}} \circ \mathrm{F_d})(N) &= \sum_{j=0}^{N} (\mathrm{F_d[u]})^i(j) S_\eta[\mathrm{F_d}](j) \\
&= \sum_{j=0}^{N} u^0(j) \mathrm{F_{d^{*i}}[u]}(j) \mathrm{F_{\eta \circ d}[u]}(j) \\
&= \sum_{j=0}^{N} u^0(j)\, \mathrm{F_{d^{*i} * (\eta \circ d)}[u]}(j) \\
&= \mathrm{F_{q_0 (d^{*i} * (\eta \circ d))}[u]}(N) \\
&=: \mathrm{F_{\Psi_d(q_i) \circ \Psi_d(\eta)(1)}}(N) \\
&= \mathrm{F_{\Psi_d(q_i \eta)(1)}}(N).
\end{aligned}
$$

This analysis suggests the following definitions.

Definition 5.2: (version 2) For any $\mathrm{c} \in \mathbb{R}\langle\langle Q \rangle\rangle$ the associated discrete-time Fliess operator is

$$\mathrm{F_c[u]}(N) = \sum_{\eta \in Q^*} (\mathrm{c}, \eta) S_\eta[\mathrm{u}](N).$$

Definition 5.3: Given $c, d \in \mathbb{R}\langle\langle Q \rangle\rangle$, the discrete-time composition product is defined as

$$\mathrm{c} \circ \mathrm{d} = \sum_{\eta \in Q^*} (\mathrm{c}, \eta) \Psi_\mathrm{d}(\eta)(1),$$

where for each $q_i \in Q$

$$\Psi_\mathrm{d}(q_i) : \mathbb{R}\langle\langle Q \rangle\rangle \to \mathbb{R}\langle\langle Q \rangle\rangle : \mathrm{e} \mapsto q_0(\mathrm{d}^{*i} * \mathrm{e}),$$

and $\Psi_\mathrm{d}(q_i q_j) := \Psi_\mathrm{d}(q_i) \circ \Psi_\mathrm{d}(q_j)$.

Remarks:

- By construction $\mathrm{F_c} \circ \mathrm{F_d} = \mathrm{F_{cod}}$.
- Is this composition product always locally finite/summable?

Bigger Questions:

- What is the corresponding Faà di Bruno Hopf algebra in the discrete-time setting?
- Is this class of discrete-time Fliess operators closed under feedback?
- What are sufficient conditions for convergence of discrete-time Fliess operator, especially under interconnections.
- Does this notion of a discrete-time Fliess operator have any compatibility with state space methods, in particular, the foundational work of Sontag?

6. Concluding Remarks

- Interconnections of continuous-time Fliess operators are well characterized in terms of combinatorial Hopf algebras.
- It renders a theory which provides explicit and tractable algorithms for control applications.
- For example, the problem of left system inversion can be solved directly.
- One approach to adapting the method to the discrete-time setting is to consider a continuous-time approximation of F_c.
- The key idea here is to introduce iterated sums and use the quasi-shuffle to describe their products.
- Other approaches likely exist, in particular, ones that do not inherit the time-reversibility from the continuous-time setting.

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