What is integrability of partial difference equations?

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In this talk I will review the basic notions of integrability of partial difference equations which I consider as a guiding principle for construction of integrable discretizations of integrable PDEs.

In case of hyperbolic type PDEs $f(u, u_x, u_y, u_{xy}) = 0$, we consider discretizations of the type
$$F(u_{m,n}, u_{m+1,n}, u_{m,n+1}, u_{m+1,n+1}) = 0, \quad (m,n) \in \mathbb{Z}^2$$
(quad-equations). A typical integrable example – the sine-Gordon equation $u_{xy} = \sin u$ and its Hirota's discretization
$$\sin \frac{1}{4}(u_{m+1,n+1} - u_{m+1,n} - u_{m,n+1} + u_{m,n}) = \frac{\epsilon^2}{4} \sin \frac{1}{4}(u_{m+1,n+1} + u_{m+1,n} + u_{m,n+1} + u_{m,n}).$$
I will argue that an adequate definition of integrability of such equations is the *multi-dimensional consistency*, i.e., the possibility to impose the equation on any two-dimensional sublattice of $\mathbb{Z}^m$ for any $m \geq 3$. This definition is constructive, yields the more classical integrability attributes (such as zero-curvature representations, permutable Bäcklund-type transformations, conserved quantities etc.) in an algorithmic way, and allows for a complete classification of such equations [1, 2].

Then, I will turn to an extension of the notion of multi-dimensional consistency to the case of variational (Lagrangian) equations coming from a least action principle. In this case, the relevant notion of integrability is that of *pluri-Lagrangian systems*. Let $L[u]$ be a discrete 2-form on $\mathbb{Z}^m$, depending on a field $u : \mathbb{Z}^m \to \mathbb{R}$.

- To an arbitrary quad-surface $\Sigma \subset \mathbb{Z}^m$ with boundary, there corresponds the *action functional*, which assigns to $u|_\Sigma$ the number
$$S_\Sigma[u] = \int_\Sigma L[u].$$
- We say that the field $u : \Sigma \to \mathbb{R}$ is a *critical point* of S_Σ, if for any interior vertex $n \in \Sigma$,
$$\frac{\partial S_\Sigma[u]}{\partial u(n)} = 0.$$
- We say that the field $u : \mathbb{Z}^m \to \mathbb{R}$ solves the *pluri-Lagrangian problem* for the Lagrangian 2-form L if, *for any oriented quad-surface* $\Sigma \subset \mathbb{Z}^m$, the restriction $u|_\Sigma$ is a critical point of the corresponding action S_Σ.

I will discuss why this is the relevant notion of integrability of discrete variational systems, what is its continuous counterpart, and what are the open problems in this approach [4, 3].

References

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