Formal series for dynamical systems

J.M. Sanz-Serna
Universidad Carlos III de Madrid

March 30, 2015

In the last 40 years Butcher series (B-series) have played an important role in the analysis of integrators for systems of ordinary differential equations, in particular in the context of geometric integration [7].[1] In the talk I shall present a number of recent developments [1], [2], [3], [4], [6] where Butcher series and related formal series are applied to analyze the dynamical systems themselves rather than their discretizations.

The talk will begin with a brief review of the use of (possibly colored) B-series. These are relative to systems of the form

$$\frac{d}{dt}x = \sum_{a\in A} f_a(x),$$

and use the elementary differentials based on the vector fields f_a. There is one elementary differential associated with each 'colored' rooted tree (each element a of the set of indices A provides a 'color'). It is well known that B-series may be composed and this composition leads to the introduction of the Butcher group $\mathcal{G}_B$.

I shall then present two new tools: word series and extended word series [6], [7]. Word series are relative to systems of the form

$$\frac{d}{dt}x = \sum_{a\in A} \lambda_a(t) f_a(x), \tag{1}$$

where the λ_a take scalar values, and use the so-called *word basis functions*, rather than elementary differentials. There is one word basis function associated with each word constructed with the letters of the alphabet A. The concept of word series leads to the introduction of a group $\mathcal{G}$ that may be seen as a (small) subgroup of $\mathcal{G}_B$. While roughly speaking elements of $\mathcal{G}_B$ correspond to integrators (including Runge-Kutta schemes) that are equivariant with respect to affine changes of variables [5], elements of $\mathcal{G}$ correspond to integrators (such as splitting algorithms) that are equivariant with respect to arbitrary changes of variables. Extended word series are a variant of word series specially adapted to treat perturbations of integrable differential systems.

In the final part of the talk I shall show how B-series, word series and extended word series may provide new ways of performing advantageously some classical tasks,

[1] My publications may be downloaded from `http://www.sanzserna.org/`

including the *averaging* of periodic or quasiperiodic dynamical systems, the construction of *formal invariants* of motion and the reduction of dynamical systems to *normal forms*. The new approach simplifies the algebraic treatment of those issues and also the derivation of the associated error bounds.

References

[1] P. Chartier, A. Murua, and J.M. Sanz-Serna, *Higher-Order averaging, formal series and numerical integration I: B-series*, Found. Comput. Math. 10 (2010), pp. 695–727.

[2] P. Chartier, A. Murua, and J.M. Sanz-Serna, *Higher-Order averaging, formal series and numerical integration II: the quasi-periodic case*, Found. Comput. Math., 12 (2012), pp. 471-508.

[3] P. Chartier, A. Murua, and J.M. Sanz-Serna, *A formal series approach to averaging: exponentially small error estimates*, DCDS A, 32 (2012), pp. 3009-3027.

[4] P. Chartier, A. Murua, and J.M. Sanz-Serna, *Higher-Order averaging, formal series and numerical integration III*, Found. Comput. Math., 15 (2015), pp. 591-612.

[5] R. McLachlan, K. Modin, H. Munthe-Kaas and O. Verdier, *B-series methods are exactly the local, affine equivariant methods*, preprint.

[6] A. Murua and J.M. Sanz-Serna, J. M., *Word series for dynamical systems and their numerical integrators*, arXiv:1502.05528.

[7] J.M. Sanz-Serna and A. Murua, *Formal series and numerical integrators: some history and some new techniques*, to appear in the Proceedings of the ICIAM, Beijing, China, 2015; arXiv: 1503.06976.