An introduction to Lie group integrators

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Summary

The significance of the geometry of differential equations was well understood already in the nineteenth century, and in the last few decades such aspects have played an increasing role in numerical methods for differential equations. Nowadays, there is a rich selection of integrators which preserve properties like symplecticity, reversibility, phase volume and first integrals, either exactly or approximately over long times [6]. Differential equations are inherently connected to Lie groups, and in fact one often sees applications in which the phase space is a Lie group or a manifold with a Lie group action. In the early nineties, two important papers appeared which used the Lie group structure directly as a building block in the numerical methods. Crouch and Grossman [4] suggested to advance the numerical solution by computing flows of vector fields in some Lie algebra. Lewis and Simo [8] wrote an influential paper on Lie group based integrators for Hamiltonian problems, considering the preservation of symplecticity, momentum and energy. These ideas were developed in a systematic way throughout the nineties by several authors. In a series of three papers, Munthe-Kaas [11, 12, 13] presented what are now known as the Runge–Kutta–Munthe-Kaas methods. By the turn of the millennium, a survey paper [7] summarised most of what was known by then about Lie group integrators. More recently a survey paper on structure preservation appeared with part of it dedicated to Lie group integrators [3].

W shall begin by giving an elementary, geometric introduction to the ideas behind Lie group integrators. Some examples, both illustrative and real applications will be shown. There are many possible examples to choose from, and we give here only a few teasers.

Symplectic Lie group integrators have been known for some time, derived by Marsden and coauthors [10] by means of variational principles. We consider a group structure on the cotangent bundle of a Lie group and derive symplectic Lie group integrators using the model for vector fields on manifolds defined by Munthe-Kaas in [13].

Finally, we extend the notion of discrete gradient methods as proposed by Gonzalez [5] to Lie groups, and thereby we obtain a general method for preserving first integrals in differential equations on Lie groups. We also offer extensions to more general manifolds which can be furnished with a retraction map.

We would like to briefly mention some of the issues we are *not* pursuing in this talk. One is the important family of Lie group integrators for problems of linear type, including methods based on the Magnus and Fer expansions. An excellent review of the history, theory and applications of such integrators can be found in [1]. We will also skip all discussions of order analysis of Lie group integrators. This is a large area by itself which involves technical tools and mathematical theory which we do not wish to include in this relatively elementary exposition. There have been several new developments in this area recently, in particular by Lundervold and Munthe-Kaas, see e.g. [9]. The presentation is for a large part based on the recent survey paper [2].

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