# Combinatorial Hopf algebras for nonlinear system interconnections: A view towards discretization

W. Steven Gray*
Old Dominion University
Norfolk, Virginia USA

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**Extended Abstract**

A central problem in control theory is understanding how dynamical systems behave when they are interconnected. In a typical design problem, for example, one is given a fixed system, and the objective is to determine a feedback system which will produce a desired closed-loop transient response or track a specified output trajectory. When the component systems are nonlinear, these problems are difficult to address by purely analytical means. The prevailing methodologies are geometric in nature and entirely based on state space models [17]. A complementary approach, however, has begun to emerge where the input-output map of each component system is represented in terms of a Chen-Fliess functional expansion. In this setting, concepts from combinatorial algebra are then employed to produce an explicit description of the interconnected system. The geneses of this method can be found in the work of Fliess [4, 5] and Ferfera [2, 3] who described key elements of these underlying algebras in terms of noncommutative formal power series. In particular, they identified the shuffle algebra's central role in this theory. The method was further developed by Gray et al. [14–16, 18] and Wang [19] who addressed basic analytical questions such as what inputs guarantee convergence of the component series, when are the interconnections well defined, and what is the nature of the output functions? A fundamental open problem on the algebraic side until 2011 was how to explicitly compute the generating series of two feedback connected systems. Largely inspired by the interactions at the 2010 Trimester in Combinatorics and Control (COCO2010) in Madrid, the problem was eventually solved in four steps by identifying a new combinatorial Hopf algebra (CHA) underlying the calculation. The single-input, single-output case was first addressed by Gray and Duffaut Espinosa in [8] via a certain graded CHA. Foissy then introduced a new grading in [7] which rendered a graded and connected version of this algebra. This provided a fully recursive formula for the antipode, which is central to the feedback calculation [10]. The multivariable case was then treated in [11]. Finally, a full combinatorial treatment, including a Zimmermann type forest formula for the antipode, was presented in [1]. This last result, based on an equivalent Hopf algebra of decorated rooted circle trees, greatly reduces the number of computations involved by eliminating the inter-term cancelations that are intrinsic in the antipode calculation. Practical problems would be largely intractable without this innovation. Control applications ranging from mechanics and chemical engineering to systems biology can be found in [9, 11–13].

The goal of this presentation is two-fold. First an introduction to the method of CHA's in feedback control theory will be given in its most up-to-date form. The full solution to the problem of computing the generating series for a multivariable continuous-time dynamic output feedback system will be given and connections to classical geometric methods such as feedback linearization will be described. The origins of the forest formula related to this calculation will also be outlined. Then the focus is shifted to open problems in this area. In particular, the issue of discretization will be explored. It can be approached from two points of view. The first viewpoint is motivated by the need to simulate the interconnection of continuous-time systems numerically, i.e., in discrete-time. In which case, the iterated integrals in the Chen-Fliess functional expansion give rise to iterated sums, and the role of the shuffle product is now replaced by that of a quasi-shuffle product. The second viewpoint is to start the theoretical development directly in the discrete-time setting and see what kind of algebraic structures naturally emerge in the context of system

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interconnection. Given the well known dichotomy between continuous-time and discrete-time systems in the nonlinear setting [6], there are likely to be a number of different paths that this type of analysis could take depending on the underlying assumptions. The current state-of-the-art will be described.

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