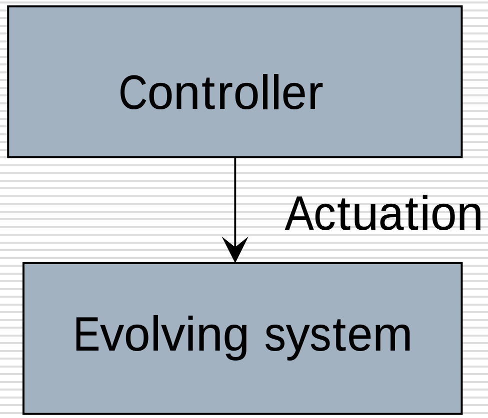
Thermodynamics of feedback controlled systems

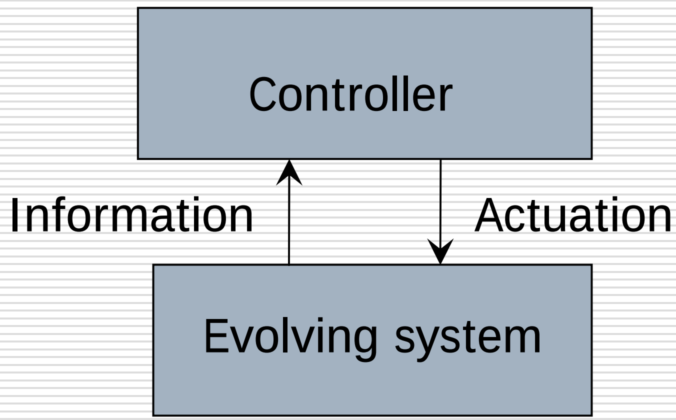
Francisco J. Cao

Open-loop and closed-loop control

- **Open-loop control**: the controller actuates on the system **independently** of the the system state.



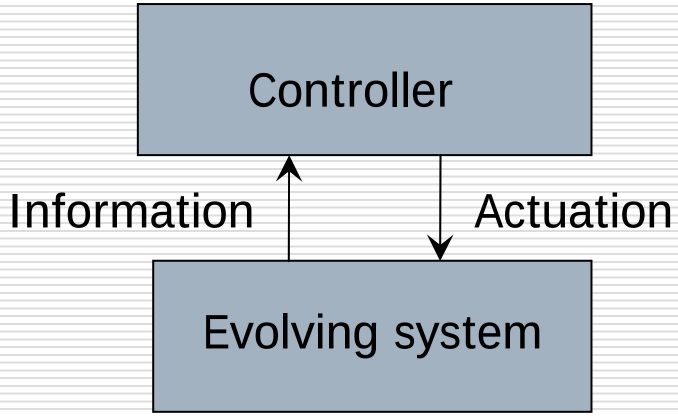
- **Closed-loop or feedback control**: the controller actuates on the system using **information** of the system state.



J. Bechhoefer, Rev. Mod. Phys. 77, 783 (2005)

Information and feedback control

- The information about the state of the system allows the external agent to optimize its actuation on the system, in order to improve the system performance.
- **Thermodynamics** of feedback control is **incomplete**: the role of **information** in feedback controlled systems is still not completely understood. In particular, its implications for the **entropy** of the system.
- The understanding of feedback systems and their limitations is very important form the technological point of view.



Overview

1. Entropy in Thermodynamics
2. Entropy in Statistical Physics and Information
3. Entropy and Thermodynamics of feedback controlled systems
4. Conclusions

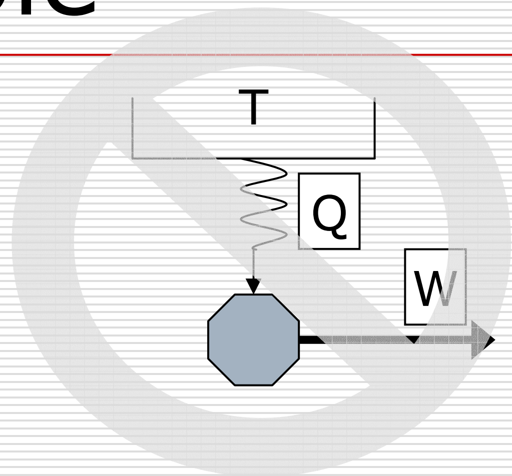
1. Entropy in Thermodynamics

Second law and entropy

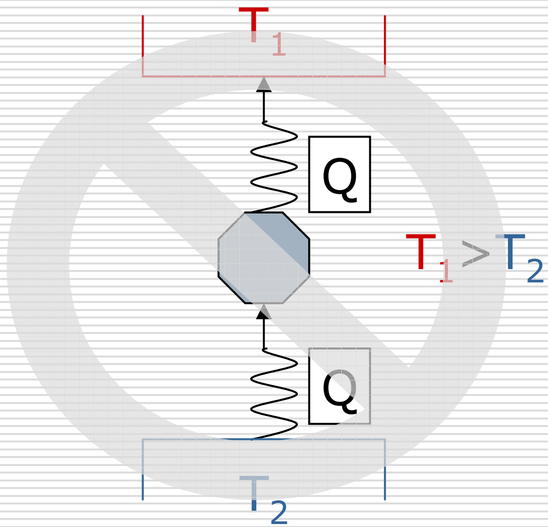
intimately linked

1.1. Second principle

- Kelvin-Planck statement:
  "It is not possible to find any **spontaneous** process whose **only result** is to convert a given amount of **heat to** an equal amount of **work** through the exchange of heat with **only one heat source**".



- Clausius statement:
  "It is not possible to find an **spontaneous** process which its **only result** is to pass **heat** from a **system to another system with greater temperature**".



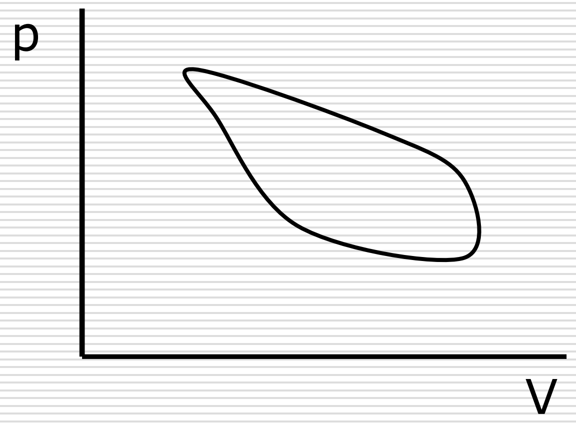
1.2. Clausius Theorem

- For a system that follows a cyclic process we have for each cycle

$$\oint \frac{\delta Q}{T_{TB}} \leq 0$$

with δQ the infinitesimal amount of work interchanged with the thermal bath at temperature T_{TB}.

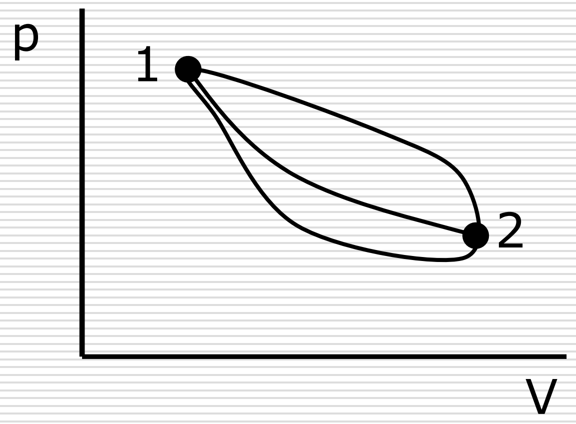
- The equality holds if the process is reversible (in this case also $T_{system}=T_{TB}$)



1.3. Thermodynamic definition of entropy

- The application of the Clausius theorem for reversible cycles tell us that there exist a state function, named entropy, defined by

$$S_2 - S_1 = \int_1^2 \frac{\delta Q}{T}\bigg|_{REVERSIBLE}$$



- As a consequence in any cycle the change in entropy of the system is zero.

1.4. Second principle in terms of entropy

- The entropy of an isolated system either increases or remains constant

$$\Delta S_{ISOLATED} \geq 0$$

- Thus, in an isolated systems only processes that increase or keep constant the entropy will spontaneously occur.
- The increase of the entropy of an isolated system indicates its evolution towards the equilibrium state, which has the maximum entropy.

2. Entropy in Statistical Physics and Information

Microstate and Macrostate

+

Entropy expression in Statistical Physics

+

Basic concepts in Information Theory

=

Fruitful and clear interpretation of entropy

2.1. Microstate and macrostate

- Microstate:

  Complete description of the state of the system, where all the microscopic variables are specified.

- Macrostate:

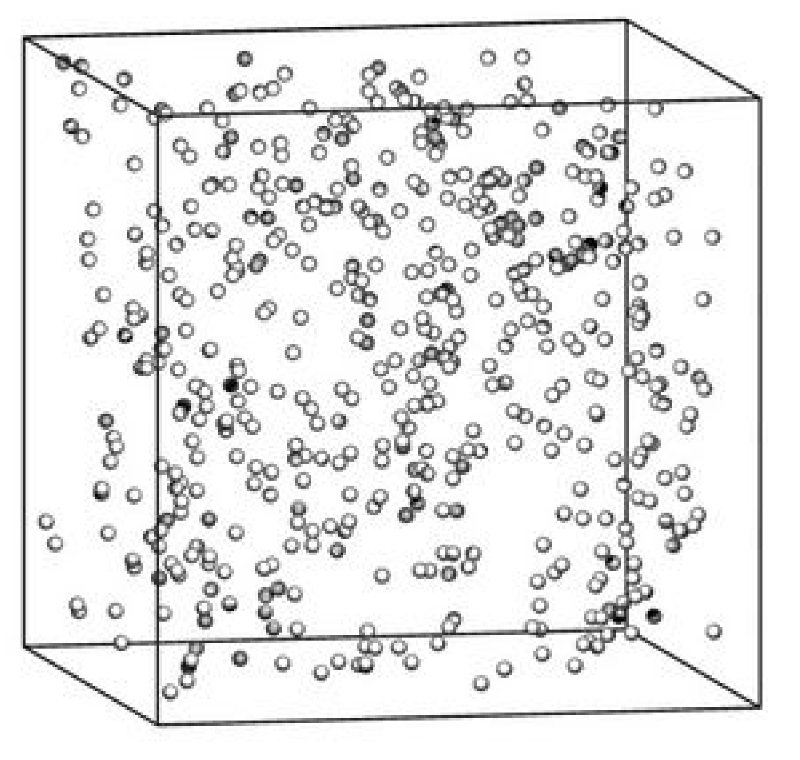
  Partial description of the state of the system, where only some macroscopic variables are specified.

2.1. Microstate and macrostate

- Example: gas of a great number of point particles

  Microstate: position and velocity of each particle at a time t.

  Macrostate: E, V and N; o p, V and T.
- In general, for systems with a great number of constituents experimentally it is only possible to determine the macrostate.

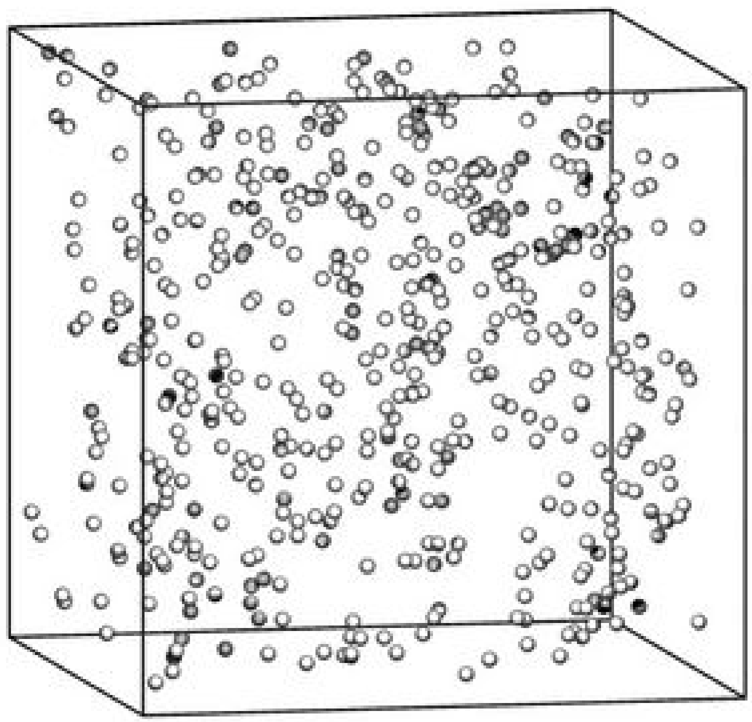


2.2. Entropy in the microcanonical ensemble

- **Isolated** system in an equilibrium state defined by E, V and N.
- Macrostate E, V and N has Ω equiprobable compatible microstates
- Entropy

$$S = k \ln \Omega$$

k=1,38 10^{-23} J/K
Boltzmann constant

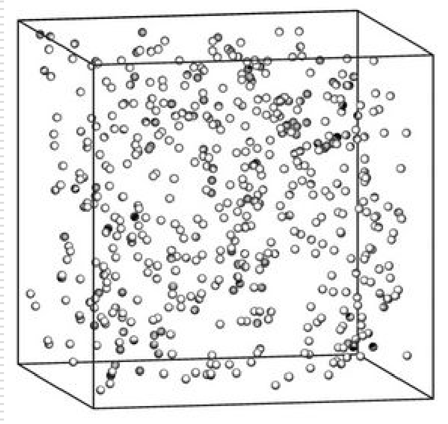


2.3. Boltzmann entropy

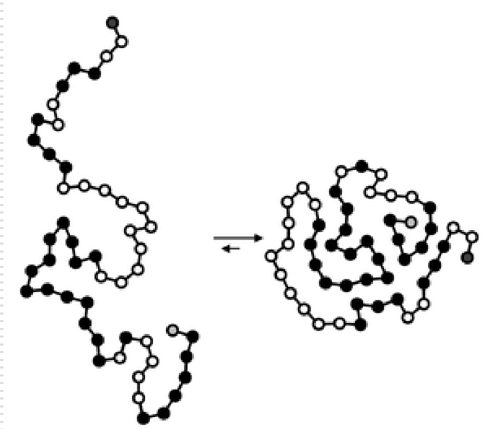
- Entropy of a macrostate

$$S = -k \sum_{i=1}^{n} p_i \ln p_i$$

p_i probability of microstate i
n number of microstates compatible with the macrostate



- Example with equal probability: isolated system in equilibrium → microcanonical ensemble $p_i = 1/\Omega$
- Example with different probabilities:
  - system in equilibrium with a thermal bath (particle gas) → canonical ensemble.
  - Proteins.



2.5. Entropy and information

- Shannon defined the quantity

$$H = -\sum_{i=1}^{n} p_i \log_2 p_i$$

(Shannon "entropy")

- It is a measure of the average uncertainty of a random variable that takes n values each with probability p_i.
- It is the number of bit needed in average to describe the random variable.

C.E. Shannon, The Bell System Tech. J. 27, 379 (1948).

2.5. Entropy and information

- If the values are equiprobable, the number of bits needed in average to describe the random variable, is simply $\log_2 n$.

- But when the values are not equiprobable, the average number of bits can be reduced, using a shorter description for the more probable cases.

Example with four values:

values	p_i	*chain*	l_i	$p_i \cdot l_i$
a	1/2	0	1	1/2
b	1/4	10	2	1/2
c	1/8	110	3	3/8
d	1/8	111	3	3/8

With this codification the average number of bits needed is

$\Sigma p_i l_i = 7/4 = 1.75$ bits

Which coincides with the Shannon "entropy"

$H = \Sigma p_i \log_2 p_i = 7/4 = 1.75$ bits

While it they were equiprobable it would be $\log_2 4 = 2$ bits

2.5. Entropy and information

- Recall that the Boltzmann entropy of a macrostate and the Shannon "entropy" are

$$S = -k\sum_{i=1}^{n} p_i \ln p_i \qquad H = -\sum_{i=1}^{n} p_i \log_2 p_i$$

p_i probability or the i microstate

n number of microstates compatible with the macrostate

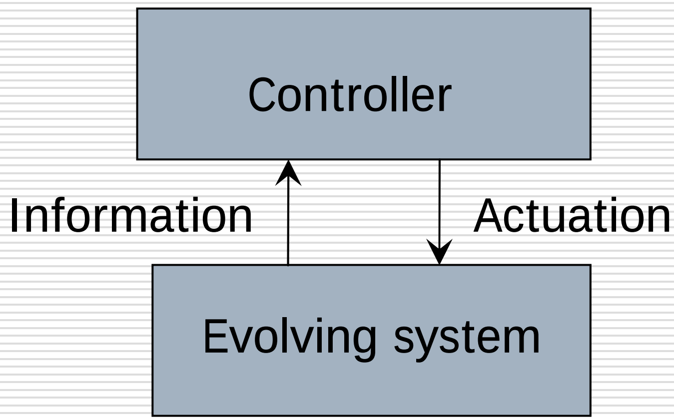
- Botzmann entropy of a macrostate: average amount of information needed to specify the microstate

$$S = k\ln(2)H$$

[The ln(2) factor comes from the change of base.]

3. Entropy and thermodynamics of feedback systems

- Feedback controlled system: system that is coupled to an external agent that uses information of the system to actuate on it.
- **Thermodynamics** of feedback control is **incomplete**: the role of **information** in feedback controlled systems is still not completely understood. In particular, its implications for the **entropy** of the system.
- Much of the progress has come from the study of the Maxwell's demon, and mainly from a computation theory point of view.

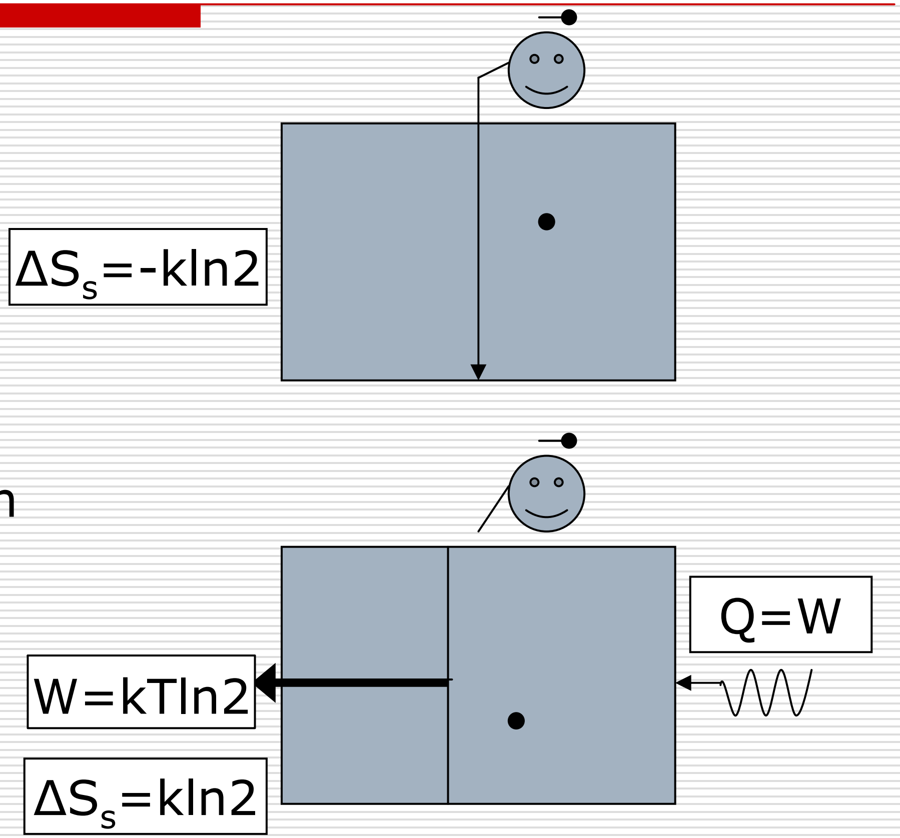


3.1. Maxwell demon: Szilard engine

- The demon puts a wall in the middle, and observes where is the particle.
- Once the demon knows in which side is the particle, it attaches a piston in the correct side of the wall to extract a work W. Meanwhile the system is connected to a thermal bath of temperature T extracting from it a heat Q=W.
- Apparently the efficiency is ¿¿¿$\eta = W/Q = 1$???

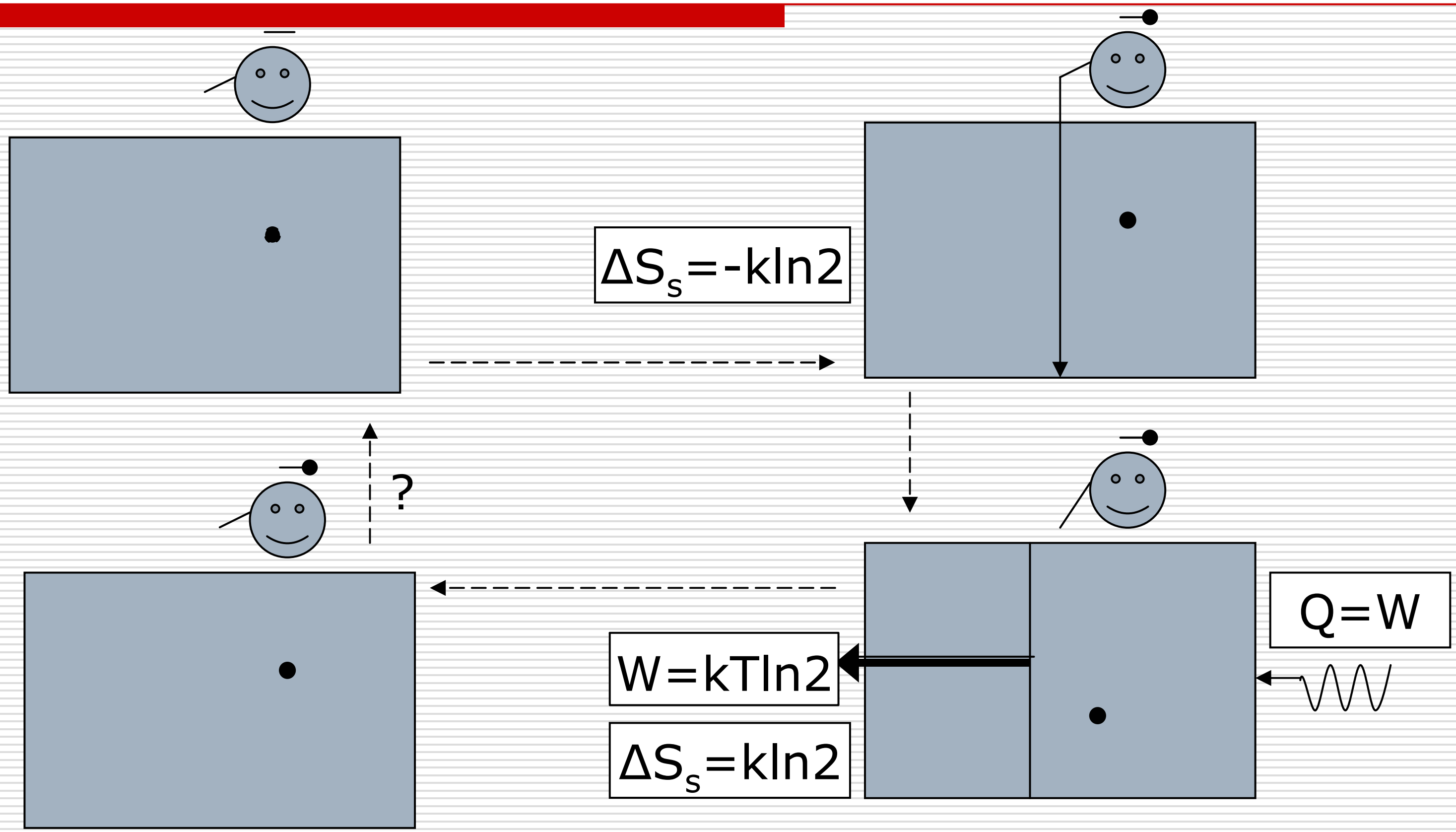
  and with only one thermal bath

  (¡¡¡2nd principle!!!)



J.C.Maxwell, Theory of heat (1871)
L.Szilard, Z. Phys. 53, 840 (1929)

3.1. Maxwell demon: Szilard engine



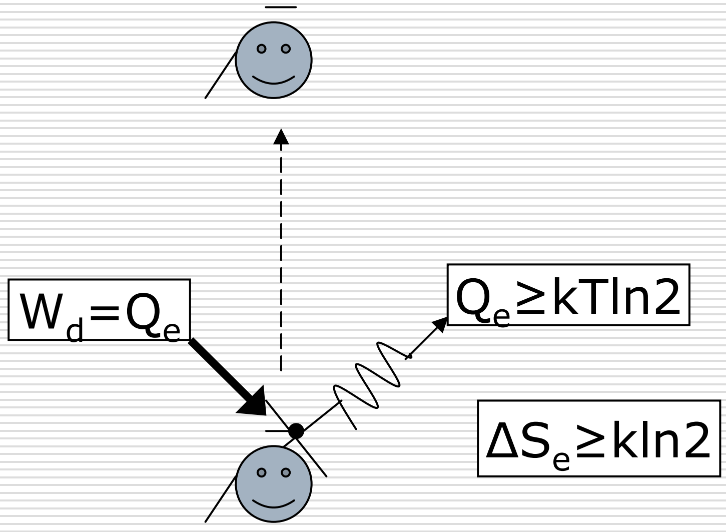
J.C.Maxwell, Theory of heat (1871)
L.Szilard, Z. Phys. 53, 840 (1929)

3.2. Landauer principle

- It can be obtained from the second law, therefore it is not a principle.

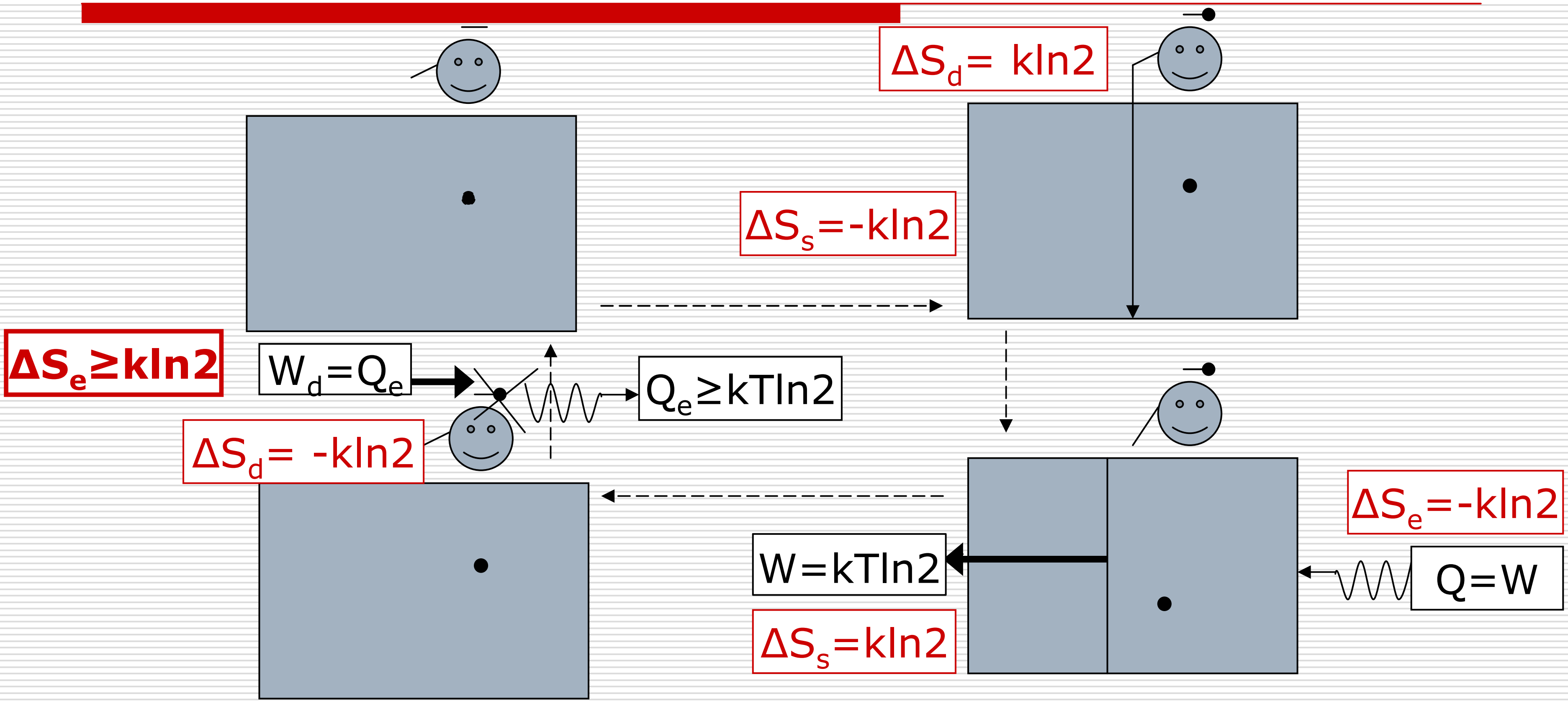
- **The erasure of one bit of information produces a growth in the entropy of the enviroment of $\Delta S_e \geq k \ln 2$**

  (Szilard engine: one bit is enough to store the information, for example:
  0 left, 1 right)



R. Landauer, IBM J. Res. Dev. 5, 183 (1961)

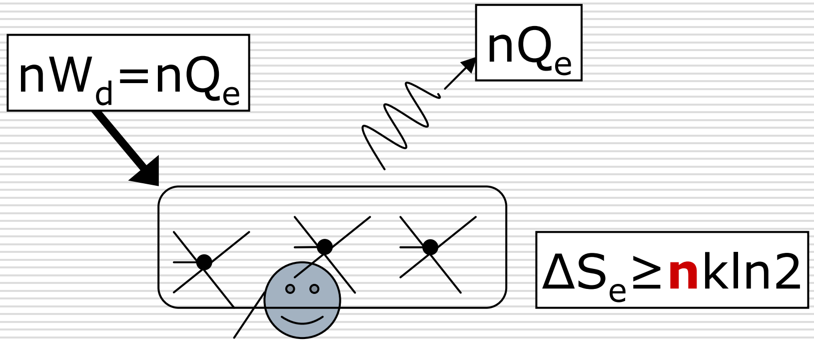
3.3. Maxwell demon “solution” (system + demon perspective)



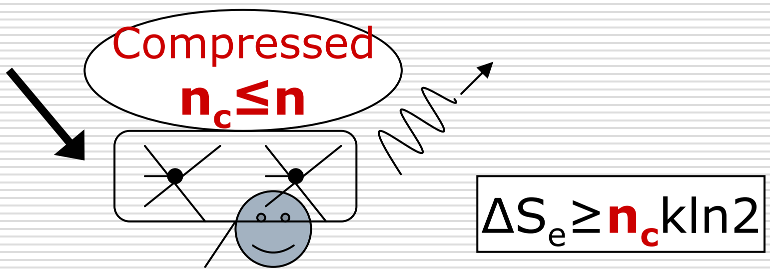
$\Delta S_s + \Delta S_d + \Delta S_e \geq 0$

3.4. Many measurements (demon + system perspective)

- Zurek shows how to minimize the erasure cost, using an algorithmic complexity approach



- The clever demons compress the information (less bits = lower erasure cost)



W.H.Zurek, Phys.Rev.A 40, 4731 (1989)

3.5. Open questions

There are already many open questions in the physics of feedback controlled systems.

- From the point of view of system + controller the understanding is advanced, but it uses concepts like algorithmic complexity (Zurek) which do not have a clear physical meaning, and which it is neither clear how to compute them in real cases.
- The understanding from the point of view of the system (without entering in the controller details) is still incomplete.
- The thermodynamics of the feedback controlled systems is still incomplete.

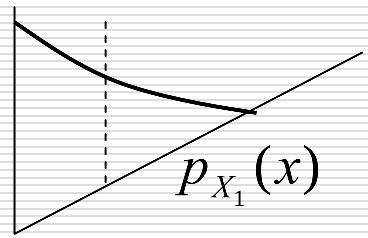
3.6. Entropy reduction due to information

System perspective: For the controller we only need the (deterministic or not) correspondence between the states of the system and the actions of the controller.

left → off

right → on

The entropy of the system *before* being measured by the system for the first time



$$S_1^b = -k \sum_{x \in X} p_{X_1}(x) \ln p_{X_1}(x) = k \ln(2) H(X_1)$$

F. J. Cao, M. Feito, Phys. Rev. E 79, 041118 (2009)

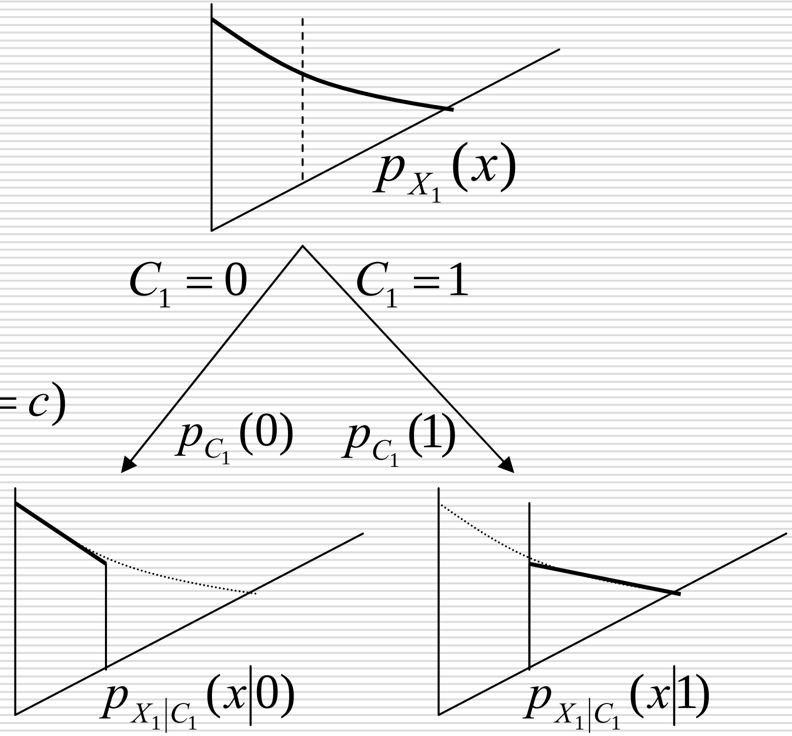
3.6. Entropy reduction due to information

If the first measurement implies that the first action of the controller C_1 is c, the entropy decreases to

$$-k\sum_{x\in X} p_{X_1|C_1}(x|c)\ln p_{X_1|C_1}(x|c) =: k\ln(2)H(X_1|C_1=c)$$

Therefore the average entropy *after* the measurement is

$$S_1^a = k\ln(2)\sum_{c\in C} p_{C_1}(c)H(X_1|C_1=c) =: k\ln(2)H(X_1|C_1)$$



F. J. Cao, M. Feito, Phys. Rev. E 79, 041118 (2009)

3.7. Derivation of the Landauer "principle"

The average change in a measurement is

$$\Delta S_1 = S_1^d - S_1^a = k \ln(2)[H(X_1 | C_1) - H(X_1)]$$

where its appears the mutual information

$$I(X_1; C_1) := H(X_1) - H(X_1 | C_1) = \sum_{x \in X, c \in C} p_{X_1 C_1}(x,c) \ln \frac{p_{X_1 C_1}(x,c)}{p_{X_1}(x) p_{C_1}(c)}$$

which is a measurement of the ((dependency)) between two random variables.

Thus, we obtain the Landauer "principle" as a consequence

$$\Delta S_1 = -k \ln(2) I(X_1; C_1)$$

F. J. Cao, M. Feito, Phys. Rev. E 79, 041118 (2009)

3.8. Many measurements (system perspective)

For systems with **deterministic control** after **M measurements** we obtain

$$\Delta S_{info} = -k\ln(2)H(C_M,\ldots,C_1)$$
$$= k \sum_{c_M,\ldots,c_1\in C} p_{C_M\cdots C_1}(c_M,\ldots,c_1)\ln p_{C_M\cdots C_1}(c_M,\ldots,c_1)$$

$H(C_M,\ldots,C_1)$ is the average amount of information needed to especify the M actions of the controller on the system

This result indicates that only **nonredundant information** is useful to reduce the entropy of the system (in correspondence with the Zurek idea of compressing the information)

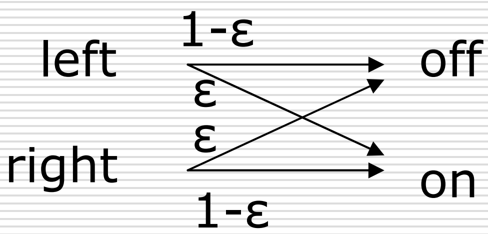
3.8. Many measurements (system perspective)

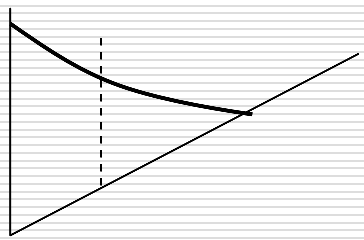
For system with **NONdeterministic control** after **M measurements** we have

$$\Delta S_{info} = -k\ln(2)\left[H(C_M,\ldots,C_1) - \sum_{k=1}^{M} H(C_k \mid C_{k-1},\ldots,C_1,X_k) \right]$$

where the additional term is nonzero if the present state of the system and the previous history of the controller does not completely determine the action of the controller.

Example:





The entropy reduction in the system due to information after M measurements is

$$\Delta S_{info} = -k\ln(2)\left[H(C_M,\ldots,C_1) - MH_b(\varepsilon) \right]$$

with $H_b(\varepsilon) = -\varepsilon\ln\varepsilon - (1-\varepsilon)\ln(1-\varepsilon)$

3.9. Application and example

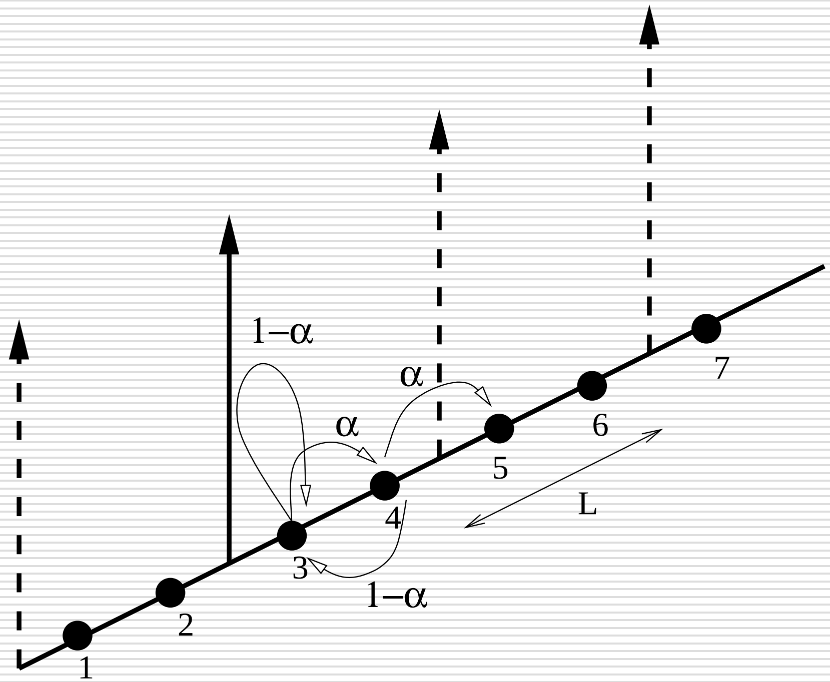
- **Isothermal feedback systems**: its efficiency can be defined as

$$\eta = \frac{W}{-\Delta F_{cont}}$$

If the controller does not transfer heat to the system, the maximum efficiency is

$$\eta = \frac{W}{-\Delta U_{cont} - T\Delta S_{info}}$$

- **Markovian particle pump**: we have computed the rate of reduction of entropy, the work, and the efficiency, both in the quasistatic and in a nonquasistatic regime.



F. J. Cao, M. Feito, Phys. Rev. E 79, 041118 (2009)

4. Conclusions

- Entropy of a macrostate can be interpreted as the average amount of information needed to specify the microstate

$$S = k \ln(2) H$$

- This approach allows us to establish the thermodynamics of feedback controlled classical systems, even for nonquasistatic cases (where measurements are correlated) and also for nondeterministic controllers.
- Open questions: continuous time limit, thermodynamics of feedback controlled quantum systems, applications to particular systems of relevance.