Analytic Left Inversion of SISO Lotka-Volterra Models†

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† This research was supported by the BBVA Foundation Grant to Researchers, Innovators and Cultural Creators (Spain).

Overview

1. Introduction

- Given an operator F, describing the dynamics of a system, and a function y in its range, the left inversion problem (LIP) is to determine a unique input u such that $y = F[u]$.
- A sufficient condition (but not necessary) for solving this problem in a state space setting is to have well defined relative degree (Isidori, 1995).
- The solution of LIP does not require a state space realization.
- Fliess operators provide an explicit analytical solution.
- Introduced by M. Fliess in 1983, Fliess operators are analytic input-output systems described by coefficients and iterated integrals of the inputs.
- Fliess operators can be viewed as a functional generalization of a Taylor series. For example, any Volterra operator with analytic kernels has a Fliess operator representation.

2. Preliminaries

Population models:

- Here we apply the method to the population dynamical system:

$$\dot{z}_i = \beta_i z_i + \sum_{j=1}^{n} \alpha_{ij} z_i z_j, \quad i = 1, 2, \ldots, n, \qquad \text{(Lotka-Volterra model)}$$

  where $z_i \propto$ to the i-th population, β_i is the growth rate for the i-th population, and α_{ij} weights the effect of the j-th species on the i-th species.

- Input-output models are obtained by introducing time dependence on the $\beta_i(t)$'s or $\alpha_{ij}(t)$'s (inputs), and assuming $y = h(z)$ (outputs).

- For $n = 2$,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} \beta_1 z_1 - \alpha_{12} z_1 z_2 \\ -\beta_2 z_2 + \alpha_{21} z_1 z_2 \end{pmatrix} \qquad \text{(Predator-Prey model)}$$

  The vector fields are complete within the first quadrant giving concentric periodic trajectories about $z_e = (\beta_2/\alpha_{21}, \beta_1/\alpha_{12})$.

Fliess Operators:

- Let $X = \{x_0, x_1, \ldots, x_m\}$ be an alphabet and X^* the set of all words over X (including the empty word $\emptyset$).
- A formal power series is any mapping $c : X^* \to \mathbb{R}^\ell$. Typically, c is written as a formal sum $c = \sum_{\eta \in X^*} (c, \eta)\eta$, and the set of all such series is $\mathbb{R}^\ell\langle\langle X \rangle\rangle$.
- For a measurable function $u : [a, b] \to \mathbb{R}^m$ with finite L_1-norm, define $E_\eta : L_1^m[t_0, t_0 + T] \to \mathcal{C}[t_0, t_0 + T]$ by $E_\emptyset[u] = 1$, and

  $$E_{x_i \eta'}[u](t, t_0) = \int_{t_0}^{t} u_i(\tau) E_{\eta'}[u](\tau, t_0)\, d\tau,$$

  where $x_i \in X$, $\eta' \in X^*$ and $u_0 \triangleq 1$.
- Note that to each letter x_i has been assigned a function u_i.
- For each $c \in \mathbb{R}^\ell\langle\langle X \rangle\rangle \Rightarrow F_c[u](t) = \sum_{\eta \in X^*} (c, \eta)\, E_\eta[u](t, t_0)$, which is called a Fliess operator (Fliess, 1983).

Fliess Operator Inverses:

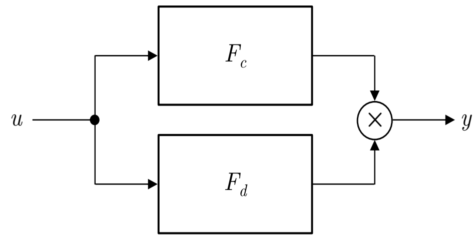

Fig. 2.1: Product connection.

$\Rightarrow F_c F_d = F_{c \sqcup\!\sqcup\, d}$, where $\sqcup\!\sqcup$ denotes the shuffle product.

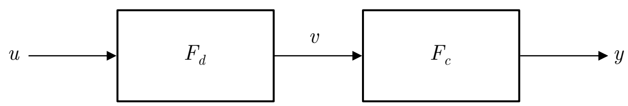

Fig. 2.2: Cascade connection.

$\Rightarrow F_c \circ F_d = F_{c \circ d}$, where $c \circ d$ denotes the composition product of $c \in \mathbb{R}^{\ell}\langle\langle X \rangle\rangle$ and $d \in \mathbb{R}^{m}\langle\langle X \rangle\rangle$ (Gray et al., 2014)

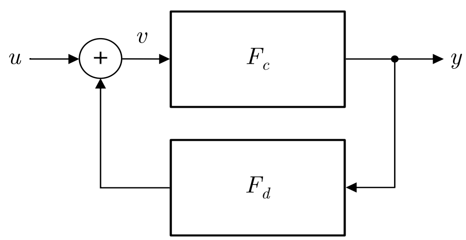

Fig. 2.3: Feedback connection.

Given $c, d \in \mathbb{R}^{m}\langle\langle X \rangle\rangle$, y satisfies

$$y = F_c[v] = F_c[u + F_d[y]].$$

If there exists e so that $y = F_e[u]$, then

$$F_e[u] = F_c[u + F_{d \circ e}[u]]. \quad \text{(contraction!)}$$

On the other hand, $\boxed{v = u + F_{d \circ c}[v] \quad \Rightarrow \quad (I + F_{-d \circ c})\,[v] = u.}$

Apply the compositional inverse to both sides of this equation:

$$v = (I + F_{-d \circ c})^{-1}[u] := \left(I + F_{(-d \circ c)^{\circ -1}}\right)[u].$$

In which case, $F_{c@d}[u] = F_c[v] = F_{c \circ (\delta - d \circ c)^{\circ -1}}[u]$, (explicit formula!)
or equivalently, $c@d = c \circ (\delta - d \circ c)^{\circ -1}$, where $F_\delta := I$.

The set of operators $\mathscr{F}_\delta = \{I + F_c \; : \; c \in \mathbb{R}\langle\langle X \rangle\rangle\}$, forms a group under composition, in particular, a Faà di Bruno Hopf algebra with antipode, α, satisfying

$$(\delta + c)^{\circ -1} := \delta + c^{\circ -1} = \delta + \sum_{\eta \in X^*} (\alpha\, a_\eta)(c)\, \eta,$$

where $c^{\circ -1}$ denotes the composition inverse of c and

$$a_\eta : \mathbb{R}\langle\langle X \rangle\rangle \to \mathbb{R} : c \mapsto (c, \eta).$$

Remark: The antipode has an explicit series representation
(Gray & Duffaut Espinosa, 2011, 2014).

Now observe that any $c \in \mathbb{R}\langle\langle X\rangle\rangle$ can be written as $c = c_N + c_F$, where $c_N := \sum_{k\geq 0}(c, x_0^k)x_0^k$ and $c_F := c - c_N$.

Definition 2.1: Given $c \in \mathbb{R}\langle\langle X\rangle\rangle$, let $r \geq 1$ be the largest integer such that $\mathrm{supp}(c_F) \subseteq x_0^{r-1}X^*$. Then c has relative degree r if the linear word $x_0^{r-1}x_1 \in \mathrm{supp}(c)$, otherwise it is not well defined.

Remark: This definition coincides with the usual definition of relative degree given in a state space setting. But this definition is independent of the state space setting.

Definition 2.2: Given $\xi \in X^*$, the corresponding left-shift operator is

$$\xi^{-1} : X^* \to \mathbb{R}\langle X\rangle : \eta \mapsto \begin{cases} \eta' & : \quad \eta = \xi\eta' \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Remark: The operation $F_c/F_d = F_{c/d}$ is given by $c/d := c \sqcup\!\sqcup\, d^{\sqcup\!\sqcup\, -1}$, where $c^{\sqcup\!\sqcup\, -1} := (c,\emptyset)^{-1}\sum_{k=0}^{\infty}(c')^{\sqcup\!\sqcup\, k}$, and $c' = 1 - c/(c,\emptyset)$ is proper.

$$\left.\begin{aligned} y &= F_c[u] \\ y^{(1)} &= F_{x_0^{-1}(c)}[u] \\ &\;\vdots \\ y^{(r-1)} &= F_{(x_0^{r-1})^{-1}(c)}[u] \\ y^{(r)} &= F_{(x_0^r)^{-1}(c)}[u] \\ &\quad + u\, F_{(x_0^{r-1}x_1)^{-1}(c)}[u]. \end{aligned}\right\} \quad \begin{aligned} u &= \frac{v - F_{(x_0^r)^{-1}(c)}[u]}{F_{(x_0^{r-1}x_1)^{-1}(c)}[u]} \qquad (y^{(r)} = v) \\ &= \frac{-F_{(x_0^r)^{-1}(c-c_y)}[u]}{F_{(x_0^{r-1}x_1)^{-1}(c)}[u]} = -F_d[u], \\ &\boxed{d = \frac{(x_0^r)^{-1}(c-c_y)}{(x_0^{r-1}x_1)^{-1}(c)}}. \end{aligned}$$

Theorem 2.3: Suppose $c \in \mathbb{R}\langle\langle X\rangle\rangle$ has relative degree r. Let y be analytic at $t=0$ with generating series $c_y \in \mathbb{R}_{LC}[[X_0]]$ satisfying $(c_y, x_0^k) = (c, x_0^k)$, $k = 0, \ldots, r-1$. (Here $X_0 := \{x_0\}$.) Then the input

$$u(t) = \sum_{k=0}^{\infty} (c_u, x_0^k)\frac{t^k}{k!},$$

where $c_u = ((x_0^r)^{-1}(c-c_y)/(x_0^{r-1}x_1)^{-1}(c))^{\circ -1}$, is the unique solution to $F_c[u] = y$ on $[0,T]$ for some $T > 0$.

Remark: The condition on c_y ensures that y is in the range of F_c.

3. Left System Inversion of LV Input-Output Systems

Four SISO predator-prey systems with output $y = z_1$ (prey):

I/0 map	state space realization	rel. degree
$F_c : \beta_1 \mapsto y$	$g_0(z) = \begin{bmatrix} -\alpha_{12} z_1 z_2 \\ -\beta_2 z_2 + \alpha_{21} z_1 z_2 \end{bmatrix}, \; g_1(z) = \begin{bmatrix} z_1 \\ 0 \end{bmatrix}$	1
$F_c : \alpha_{12} \mapsto y$	$g_0(z) = \begin{bmatrix} \beta_1 z_1 \\ -\beta_2 z_2 + \alpha_{21} z_1 z_2 \end{bmatrix}, \; g_1(z) = \begin{bmatrix} -z_1 z_2 \\ 0 \end{bmatrix}$	1
$F_c : \beta_2 \mapsto y$	$g_0(z) = \begin{bmatrix} \beta_1 z_1 - \alpha_{12} z_1 z_2 \\ \alpha_{21} z_1 z_2 \end{bmatrix}, \; g_1(z) = \begin{bmatrix} 0 \\ -z_2 \end{bmatrix}$	2
$F_c : \alpha_{22} \mapsto y$	$g_0(z) = \begin{bmatrix} \beta_1 z_1 - \alpha_{12} z_1 z_2 \\ -\beta_2 z_2 \end{bmatrix}, \; g_1(z) = \begin{bmatrix} 0 \\ z_1 z_2 \end{bmatrix}$	2

The population system under study:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} -\alpha_{12} z_1 z_2 \\ -\beta_2 z_2 + \alpha_{21} z_1 z_2 \end{pmatrix} + \begin{pmatrix} z_1 \\ 0 \end{pmatrix} u, \qquad y = z_1$$

with $z_1(0) = z_{1,0}$ and $z_2(0) = z_{2,0}$. Make $\alpha_{12} = \alpha_{21} = \beta_2 = 1$.

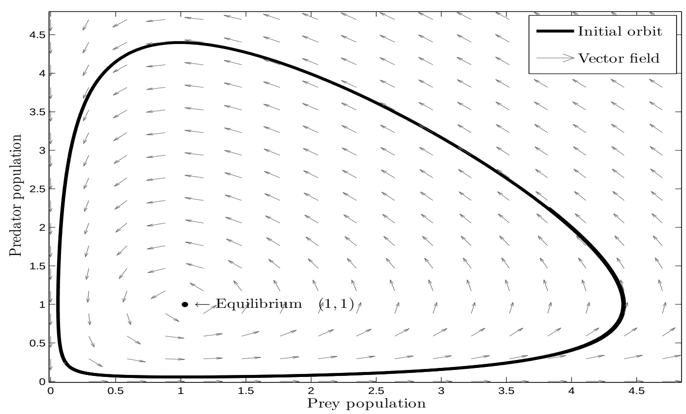

Fig. 3.1: $u = 1$.

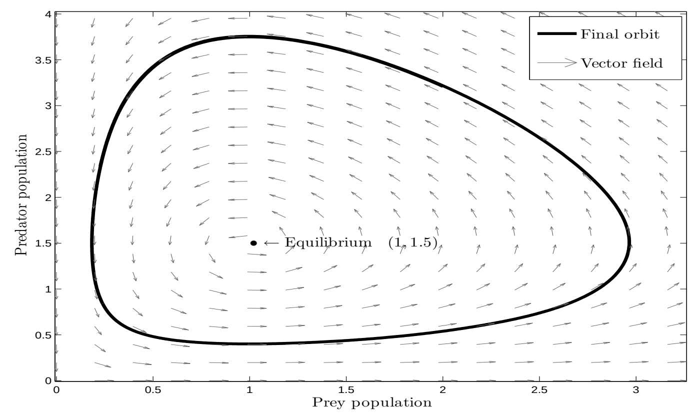

Fig. 3.2: $u = 1.5$.

$$c = z_{1,0} - \alpha_{12} z_{2,0} z_{1,0} x_0 + z_{1,0} \boxed{x_1} + \left({\alpha_{12}}^2 {z_{2,0}}^2 z_{1,0} - \alpha_{12}\alpha_{21} z_{2,0} {z_{1,0}}^2 \right.$$
$$\left. + \alpha_{12}\beta_2 z_{2,0} z_{1,0} \right) x_0^2 - \alpha_{12} z_{2,0} z_{1,0} x_0 x_1 - \alpha_{12} z_{2,0} z_{1,0} x_1 x_0 + z_{1,0} x_1^2 + \cdots .$$

$\boxed{\text{Relative degree } r = 1.}$

One must select an output function

$$y(t) = \sum_{k=0}^{\infty} (c_y, x_0^k) \frac{t^k}{k!},$$

where c_y is the generating series of y. It is sufficient to consider a polynomial of degree 3, so let $(c_y, \emptyset) = v_0$, $(c_y, x_0^i) = v_i$ for $i = 1, 2, 3$. Thus,

$$\begin{aligned} d := \frac{(x_0^r)^{-1}(c - c_y)}{(x_0^{r-1}x_1)^{-1}(c)} &= -\alpha_{12}z_{2,0} - \frac{v_1}{z_{1,0}} + \Bigg(\alpha_{12}\beta_2 z_{2,0} - \frac{v_2}{z_{1,0}} \\ &-\alpha_{12}\alpha_{21}z_{1,0}z_{2,0} - \frac{v_1\alpha_{12}z_{2,0}}{z_{1,0}}\Bigg) x_0 + \frac{v_1}{z_{1,0}} x_1 \\ &+ \Bigg(-\frac{v_1{\alpha_{12}}^2{z_{2,0}}^2}{z_{1,0}} + \frac{v_1\alpha_{12}\beta_2 z_{2,0}}{z_{1,0}} - v_1\alpha_{12}\alpha_{21}z_{2,0} \\ &-\frac{2v_2\alpha_{12}z_{2,0}}{z_{1,0}} + {\alpha_{12}}^2\alpha_{21}z_{1,0}{z_{2,0}}^2 + 2\alpha_{12}\beta_2\alpha_{21}z_{1,0}z_{2,0} \\ &-\frac{v_3}{z_{1,0}} - \alpha_{12}{\beta_2}^2 z_{2,0} - \alpha_{12}{\alpha_{21}}^2{z_{1,0}}^2 z_{2,0}\Bigg) x_0^2 + \cdots \end{aligned}$$

In which case,

$$c_u = \left(d^{\circ -1}\right)_N = \frac{v_1}{z_{1,0}} + \alpha_{12} z_{2,0} + \left(\frac{v_1 \left(-\frac{v_1}{z_{1,0}} - \alpha_{12} z_{2,0} \right)}{z_{1,0}} + \frac{v_1 \alpha_{12} z_{2,0}}{z_{1,0}} \right.$$
$$\left. + \frac{v_2}{z_{1,0}} - \alpha_{12} \beta_2 z_{2,0} + \alpha_{12} \alpha_{21} z_{1,0} z_{2,0} \right) x_0 + \cdots$$

Design example:

Given

- $[t_1, t_2] = [12.5, 12.7] \quad (\Delta t = 0.2)$,
- $u(t_1) = 1$, $u(t_2) = 1.5$,
- initial orbit exit point $[z_1(12.5), z_2(12.5)]^T = [3.82, 2.25]^T$,
- $y(t_2) = 2$,

find a smooth $u(t)$ for $t \in (t_1, t_2)$ so that all constraints are satisfied.

Solution:

- Select the output as

$$y(t) = \sum_{k=0}^{\infty} (c_y, x_0^k) \frac{t^k}{k!} = v_1 t + v_3 t^3/3!.$$

- From Theorem 2.3, $c_u = \left(d^{\circ -1}\right)_N$ is computed in terms of v_1 and v_3.
- Solving

$$\left.\begin{aligned} c_y(v_1, v_3)\Big|_{x_0^k \to (0.2)^k/k!, k>0} &= 2.0 \\ c_u(v_1, v_3)\Big|_{x_0^k \to (0.2)^k/k!, k>0} &= 1.5 \end{aligned}\right\} \leftarrow$$

system of nonlinear algebraic equations

gives the transfer input (up to order 6):

$$\boxed{\begin{aligned} u(t) = & -0.844733 - 3.22608(t-t_1) + 19.2847(t-t_1)^2 + 98.9718(t-t_1)^3 \\ & + 483.681(t-t_1)^4 + 1476.69(t-t_1)^5 + 2818.13(t-t_1)^6 \end{aligned}}$$

4. Numerical Simulation

Note that $u(t_2) = 1.5$ and $y(t_2) = 2.0$, and the error is

$$\boxed{y(t) - \hat{y}(t) = \; -153.04\mathbf{(t-t_1)^6} - 269.34(t-t_1)^7 - 1610.74(t-t_1)^8 + \cdots .}$$

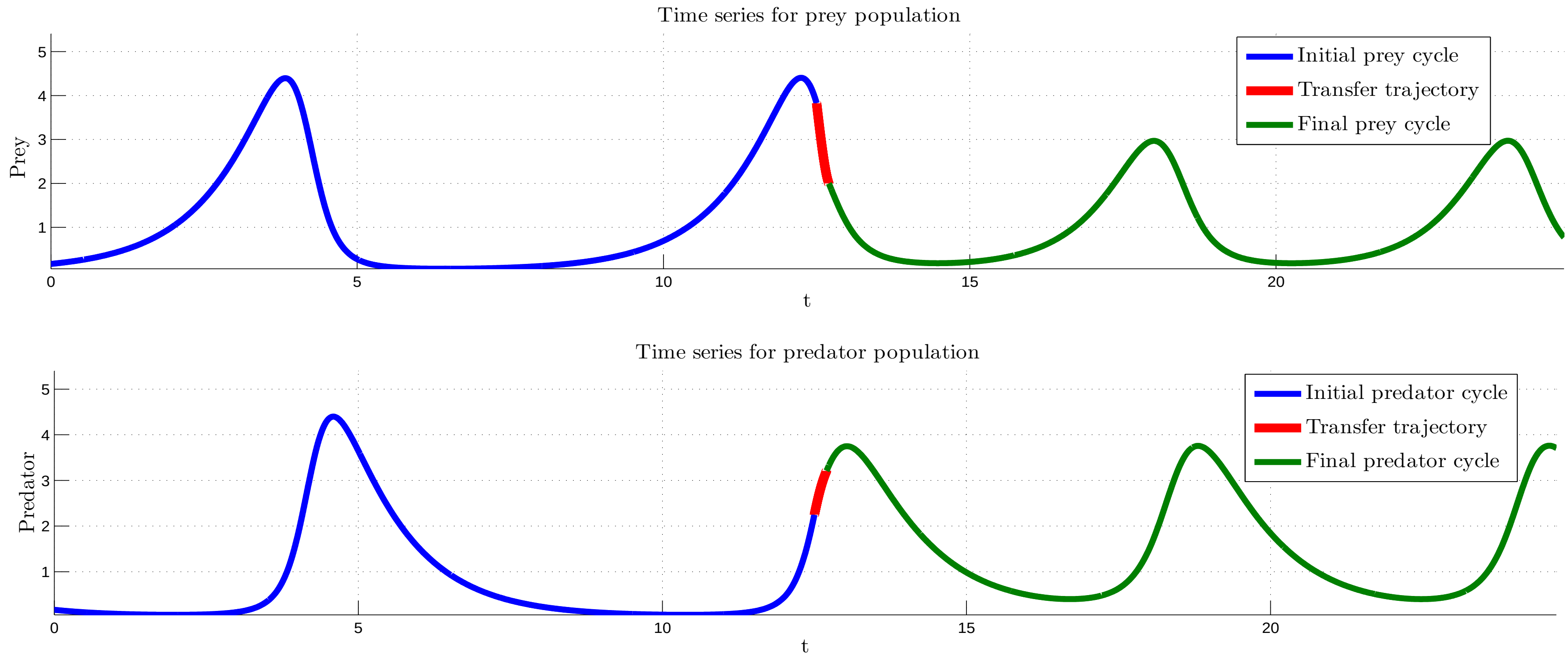

Fig. 4.1: Prey (top) and predator (bottom) populations.

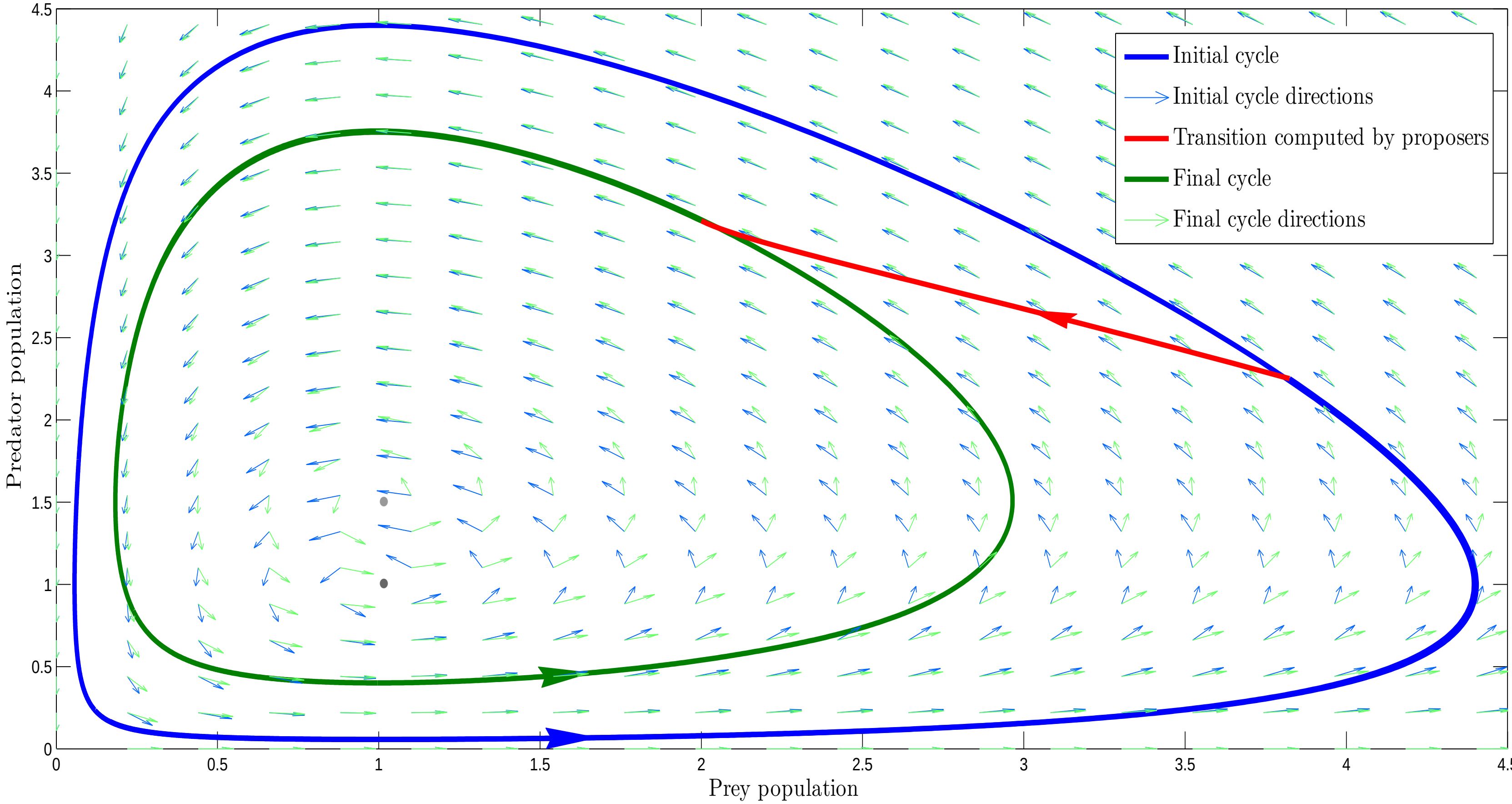

Fig. 4.2: Orbit transfer.

5. Conclusions and Future Work

- The LIP for SISO Lotka-Volterra systems was solved using Fliess operators having well defined relative degree.

- The method provides an exact, explicit and analytic formula for the LIP.

- The MIMO version of the Lotka-Volterra trajectory design problem is under review for the CDC 2015.

- Efficiency of the current software is currently being improved.