Geometry, invariants, and linearization of mechanical control systems

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Aim

To discuss three structural problems

- When is a control system mechanical?
- To analyze compatibility of two structures of control systems: mechanical structure and linear structure
- To describe equivariants of mechanical control systems

Outline

- Problem description
- Mechanical control systems
- Linearization preserving the mechanical structure
- Control systems that admit a mechanical structure
- Linearization of Mechanizable Control Systems
- Lagrangian linear control systems
- When a control system is a nonholonomic mechanical system
- Equivariants of mechanical control systems

Problem statement

- Assume that a control system Σ is equivalent to a mechanical control system $(\mathcal{MS})$

$$\Sigma \longleftrightarrow (\mathcal{MS})$$

- Assume that Σ is equivalent to a linear control system Λ

$$\Sigma \longleftrightarrow \Lambda$$

- Question: Are the linear and mechanical structures of Σ compatible, i.e., is Σ equivalent to a linear mechanical control system $(\mathcal{LMS})$?

$$\Sigma \longleftrightarrow (\mathcal{LMS})$$

- Two variants of our problem: we may wish $(\mathcal{MS})$ and $(\mathcal{LMS})$ to have equivalent mechanical structures or we may allow for non equivalent ones (the latter possibility being, obviously, related with the problem of (non)uniqueness of mechanical structures that a control system may admit).

- To make the problem precise: define the class of systems Σ, linear systems Λ, mechanical control systems $(\mathcal{MS})$, linear mechanical control system $(\mathcal{LMS})$, and the equivalence.

Notions

- We will consider smooth control-affine systems of the form

$$\Sigma : \quad \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z), \ z \in M$$

- Σ and $\tilde{\Sigma} : \ \dot{\tilde{z}} = \tilde{F}(\tilde{z}) + \sum_{r=1}^{m} u_r \tilde{G}_r(\tilde{z})$ on $\tilde{M}$ are (locally) state-space equivalent, shortly (locally) S-equivalent, if there exists a (local) diffeomorphism $\Psi : \ M \to \ \widetilde{M}$ such that

$$D\Psi(z) \cdot F(z) = \tilde{F}(\tilde{z}) \ \text{ and } \ D\Psi(z) \cdot G_r(z) = \tilde{G}_r(\tilde{z}), \quad 1 \leq r \leq m.$$

- Ψ preserves trajectories.
- Σ is S-linearizable if it is S-equivalent to a linear system of the form

$$\Lambda : \quad \dot{\tilde{z}} = A\tilde{z} + \sum_{r=1}^{m} u_r B_r.$$

Mechanical Control Systems

A *mechanical control system* ($\mathcal{MS}$) as a 4-tuple $(Q, \nabla, \mathfrak{g}_0, d)$, in which

(i) Q is an n-dimensional manifold, called *configuration manifold*;

(ii) ∇ is a symmetric affine connection on Q;

(iii) $\mathfrak{g}_0 = (e, g_1, \dots, g_m)$ is an $(m+1)$-tuple of vector fields on Q;

(iv) $d : TQ \to TQ$ is a map preserving each fiber and linear on fibers.

defining the system that, in local coordinates (x, y) of TQ, reads

$$\begin{aligned} \dot{x}^i &= y^i \\ \dot{y}^i &= -\Gamma^i_{jk}(x) y^j y^k + d^i_j(x) y^j + e^i(x) + \sum_{r=1}^{m} u_r g^i_r(x). \end{aligned}$$

- Γ^i_{jk} are the Christoffel symbols of ∇ (Coriolis and centrifugal forces)
- the terms $d^i_j(x) y^j$ correspond to dissipative-type (or gyroscopic-type) forces acting on the system,
- e represents an uncontrolled force (which can be potential or not)
- $g_1, \dots, g_m$ represent controlled forces.

Examples: planar rigid body

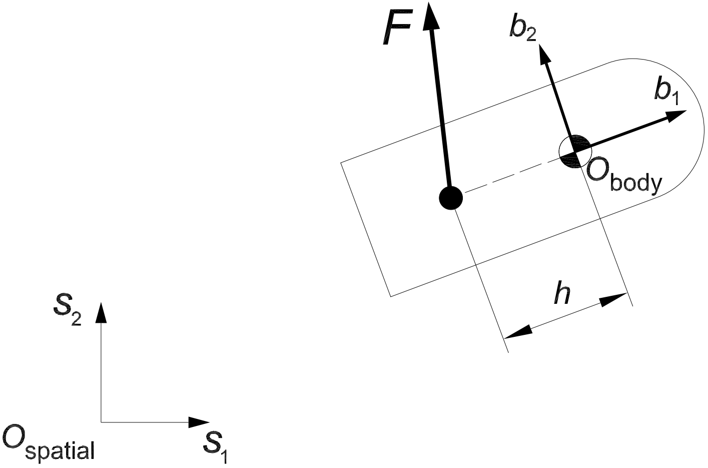

Figure: The planar rigid body

Examples: planar rigid body

- Configuration: $q = (\theta, x_1, x_2) \in \mathbb{S}^1 \times \mathbb{R}^2$, where

$$\begin{aligned} \theta &= \text{relative orientation of } \Sigma_{\text{body}} \text{ w.r.t. } \Sigma_{\text{spatial}} \\ (x_1, x_2) &= \text{position of the center of mass} \end{aligned}$$

- Equations of motion:

$$\begin{aligned} \ddot{\theta} &= -u_2 \frac{h}{J} \\ \ddot{x}_1 &= u_1 \frac{\cos\theta}{m} - u_2 \frac{\sin\theta}{m} \\ \ddot{x}_2 &= u_1 \frac{\sin\theta}{m} + u_2 \frac{\cos\theta}{m} \end{aligned}$$

- no d-forces
- The Christoffel symbols Γ^i_{jk} of the Euclidean metric $J\mathrm{d}\theta \otimes \mathrm{d}\theta + m(\mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2)$ vanish

Examples: robotic leg

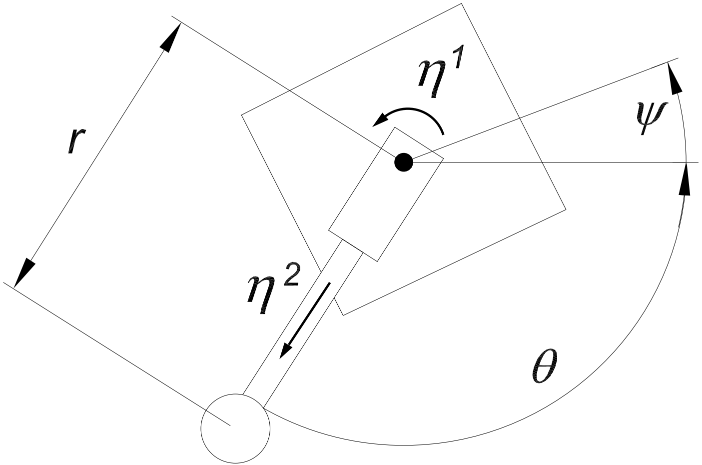

Figure: Robotic leg

Examples: robotic leg

- Configuration: $q = (r, \theta, \psi) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1$, where

$$\begin{aligned} r &= \text{extension of the leg} \\ \theta &= \text{angle of the leg from an inertial reference frame} \\ \psi &= \text{angle of the body} \end{aligned}$$

- Equations of motion:

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 + \frac{1}{m}u_2 \\ \ddot{\theta} &= -\frac{2}{r}\dot{r}\dot{\theta} + \frac{1}{mr^2}u_1 \\ \ddot{\psi} &= -\frac{1}{J}u_1. \end{aligned}$$

- no d-forces
- The Christoffel symbols of the Riemannian metric $m\mathrm{d}r \otimes \mathrm{d}r + mr^2\mathrm{d}\theta \otimes \mathrm{d}\theta + J\mathrm{d}\psi \otimes \mathrm{d}\psi$ are $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$.

Vertical distribution and mechanical MS-equivalence

- Any mechanical control system $(\mathcal{MS})$ evolves on TQ and thus defines the *vertical distribution* $\mathfrak{V}$, of rank n, that is tangent to fibers T_qQ. In (x, y)-coordinates it is given by

$$\mathfrak{V} = \text{span} \left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\}.$$

  Clearly, $\mathfrak{V}$ contains all control vector fields $g_r^i(x)\frac{\partial}{\partial y^i}$ of $(\mathcal{MS})$.

- Two mechanical systems $(\mathcal{MS})$ and $(\widetilde{\mathcal{MS}})$ are MS-equivalent if there exists a diffeomorphism φ between their configuration manifolds Q and $\tilde{Q}$ such that the corresponding control systems on the tangent bundles TQ and $T\tilde{Q}$ are S-equivalent via the extended point diffeomorphism $\Phi = (\varphi, \mathrm{D}\varphi \cdot y)^T$.

- The diffeomorphism Φ, establishing the MS-equivalence, maps the vertical distribution into the vertical distribution.

Linear Mechanical Control Systems

Systems that are simultaneously linear and mechanical form the class of Linear Mechanical Control Systems

$$(\mathcal{LMS}) \qquad \begin{array}{rcl} \dot{\tilde{x}} & = & \tilde{y}, \\ \dot{\tilde{y}} & = & D\tilde{y} + E\tilde{x} + \sum_{r=1}^{m} u_r b_r, \end{array}$$

where D and E are matrices of appropriate sizes.

Example

The mechanical system

$$(\mathcal{MS})_1 : \begin{array}{ll} \dot{x}^1 = y^1, & \dot{y}^1 = u, \\ \dot{x}^2 = y^2, & \dot{y}^2 = x^1(1+x^1) + \frac{y^1\ y^2}{1+x^1} \end{array}$$

on TQ, where $Q = \{(x^1, x^2) \in \mathbb{R}^2 \ : \ x^1 > -1\}$. is transformed via the diffeomorphism Ψ

$$\begin{array}{rclcrcl} \tilde{x}^1 & = & x^1, & & \tilde{y}^1 & = & y^1, \\ \tilde{x}^2 & = & x^2 - \frac{1}{2}\left(\frac{y^2}{1+x^1}\right)^2, & & \tilde{y}^2 & = & \frac{y^2}{1+x^1}, \end{array}$$

into the linear control system

$$(\mathcal{LMS})_1 : \quad \begin{array}{rclcrcl} \dot{\tilde{x}}^1 & = & \tilde{y}^1, & & \dot{\tilde{y}}^1 & = & u, \\ \dot{\tilde{x}}^2 & = & \tilde{y}^2, & & \dot{\tilde{y}}^2 & = & \tilde{x}^1. \end{array}$$

Notice that $(\mathcal{LMS})_1$ is a linear mechanical system but its mechanical structure is not MS-equivalent to that of $(\mathcal{MS})_1$. Indeed, Ψ does not map the vertical distribution $\mathfrak{V} = \text{span}\,\{\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}\}$ of $(\mathcal{MS})_1$ onto the vertical distribution $\tilde{\mathfrak{V}} = \text{span}\,\{\frac{\partial}{\partial \tilde{y}^1}, \frac{\partial}{\partial \tilde{y}^2}\}$ of $(\mathcal{LMS})_1$. The question is thus whether we can bring $(\mathcal{MS})_1$ into a linear system that would be mechanically equivalent to $(\mathcal{MS})_1$? ◁

Linearization preserving the mechanical structure: main result

Theorem

The mechanical system $(\mathcal{MS})$ *is, locally around* $(x_0, y_0) \in TQ$, *MS-equivalent to a linear controllable mechanical system* $(\mathcal{LMS})$ *if and only if it satisfies, in a neighborhood of* (x_0, y_0), *the following conditions*

(LM1) $\dim \operatorname{span}\{\operatorname{ad}_F^q G_r,\ 0 \le q \le 2n-1, 1 \le r \le m\}(x, y) = 2n$,

(LM2) $\left[\operatorname{ad}_F^p G_r, \operatorname{ad}_F^q G_s\right] = 0$, *for* $1 \le r, s \le m,\ 0 \le p, q \le 2n$,

(LM3) *there exist* $d_{iq}^r \in \mathbb{R}$, *where* $1 \le i \le n$, $1 \le r \le m$, $0 \le q \le 2n-1$, *such that the vector fields*

$$V_i = \sum_{r,q} d_{iq}^r \operatorname{ad}_F^q G_r$$

span the vertical distribution $\mathfrak{V}$.

(LM3) is a compatibility condition

- It is well known that the conditions (LM1) and (LM2) are necessary and sufficient for a nonlinear control system of the form Σ : $\dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z)$ to be, locally, S-equivalent to a linear controllable system.
- In linearizing coordinates the vector fields $\mathrm{ad}_F^q G_r$ are constant
- The condition (LM3) is thus, clearly, a compatibility condition that assures that the mechanical and linear structure are conform: it implies that well chosen $\mathbb{R}$-linear combinations of the vector fields $\mathrm{ad}_F^q G_r$ span the vertical distribution $\mathfrak{V}$ that defines the tangent bundle structure of the mechanical system.

Example - cont.

For the system $(\mathcal{MS})_1$ of Example, we have

$$\mathfrak{V} = \text{span}\ \left\{ \frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2} \right\}.$$

Simple Lie bracket calculations yield

$$\begin{array}{rcl} \mathrm{ad}_F\, G & = & -\frac{\partial}{\partial x^1} - \frac{y^2}{1+x^1}\frac{\partial}{\partial y^2}, \\ \mathrm{ad}_F^2\, G & = & \frac{y^2}{1+x^1}\frac{\partial}{\partial x^2} + (1+x^1)\frac{\partial}{\partial y^2}, \\ \mathrm{ad}_F^3\, G & = & -\frac{\partial}{\partial x^2}, \qquad \mathrm{ad}_F^4\, G = 0. \end{array}$$

We take $V_1 = G = \frac{\partial}{\partial y^1}$, that is, $d_{10} = 1$ and $d_{11} = d_{12} = d_{13} = 0$. In order to have $\mathfrak{V} = \text{span}\ \{V_1, V_2\}$, where $V_2 = d_{21}\,\mathrm{ad}_F\, G + d_{22}\,\mathrm{ad}_F^2\, G + d_{23}\,\mathrm{ad}_F^3\, G$, we need $d_{21} = 0$ and $d_{23} = \frac{y^2}{1+x^1} d_{22}$ so d_{22} and d_{23} cannot be taken as real constants, thus violating the condition (LM3) of Theorem 4. It follows that although the system $(\mathcal{MS})_1$ of Example 1 is S-equivalent to a linear mechanical system, it is not MS-equivalent to a linear mechanical system, that is, it cannot be linearized with simultaneous preservation of its mechanical structure. $\lhd$

Interpretation of linearizability conditions

- The linearizing diffeomorphism φ simultaneously rectifies the control vector fields, annihilates the Christoffel symbols, transforms the fiber-linear map $d(x)y$ into a linear one, and the vector field $e(x)$ into a linear vector field. Conditions that guarantee that all those normalizations take place and, moreover, that they can be effectuated simultaneously must be somehow encoded in the conditions (LM1)-(LM3). How?

- By (LM3), there exist $V_i = \sum_{r,q} d_{iq}^r \operatorname{ad}_F^q G_r,\ 1 \leq i \leq\ n$, that span the vertical distribution $\mathfrak{V}$ and are vertical lifts of vector fields v_i on Q. The commutativity conditions

$$0 = [\operatorname{ad}_F V_i, \operatorname{ad}_F V_j] = [v_i, v_j] \bmod \mathfrak{V}, \quad 1 \leq i, j \leq n, \tag{1}$$

imply that there exists a local diffeomorphism $\tilde{x} = \varphi(x)$ rectifying simultaneously all v_i, that is, $\varphi_* v_i = \frac{\partial}{\partial \tilde{x}_i}$. The extended point transformation $(\tilde{x}, \tilde{y})^T = \Phi(x, y) = (\varphi(x), \mathrm{D}\varphi \cdot y)^T$ maps V_i into $\tilde{V}_i = \Phi_* V_i = \frac{\partial}{\partial \tilde{y}_i}$.

- Now calculating in the $(\tilde{x}, \tilde{y})$-coordinates the commutativity relations

$$0 = [\tilde{V}_i, \mathrm{ad}_{\tilde{F}} \tilde{V}_j] = \Gamma^k_{ij} \frac{\partial}{\partial y^k}, \tag{2}$$

we conclude that all Christoffel symbols vanish implying that the connection ∇ defining the mechanical system is locally Euclidean (its Riemannian tensor R vanishes) and that the local $\tilde{x}$-coordinates are flat and, simultaneously, rectifying coordinates for the v_i's.

- Finally, calculating the commutativity relations

$$0 = [\mathrm{ad}_F \tilde{V}_i, \mathrm{ad}^2_F \tilde{V}_j], \tag{3}$$

we conclude that in the $\tilde{x}$-coordinates, the $(1,1)$-tensor d is constant and the vector field $e(x)$ is linear.

- All those informations are encoded in the commutativity conditions (LM2) but they are mixed up. Passing to the vector fields $V_i = \sum_{r,q} d^r_{iq} \mathrm{ad}^q_F G_r$ and using the conditions (LM2) in the form $0 = [\mathrm{ad}^p_F V_i, \mathrm{ad}^q_F V_j]$, for $0 \leq p, q \leq 2$ (equivalent to (1)-(3)), instead of applying directly to $\mathrm{ad}^q_F G_r$, allows to clearly identify the conditions responsable for the required form of, respectively, g_r's, the connection ∇, $d(x)$, and $e(x)$.

Linearization of Mechanizable Control Systems

- So far: is the linear structure compatible with a given mechanical structure?
- Now: we discuss general control-affine systems that admit both: a mechanical and a linear structure.
- If a system admits a unique mechanical structure, then the situation is that of the previous theorem
- When does a control system admit a mechanical structure and when is it unique?

Equivalence problem

When is the control system

$$\Sigma \ : \ \dot{z} = F(z) + \sum_{r=1}^{m} u_r Gr(z), \quad z \in M^{2n}, \ u \in \mathbb{R}^m,$$

mechanical? That is, when does there exist a (local) diffemorphism $\Phi : M \to TQ$ transforming Σ into a mechanical system $(\mathcal{MS})$?

In other words, a diffeomorphism $\Phi : M \to TQ$ such that

$$\begin{aligned} \Phi_* F &= y^i \frac{\partial}{\partial x^i} + \left(-\Gamma^i_{jk}(x) y^j y^k + d^i_j(x) y^j + g^i_0(x) \right) \frac{\partial}{\partial y^i} \\ \Phi_* G_r &= g^i_r(x) \frac{\partial}{\partial y^i}, \end{aligned}$$

Links with the inverse problem

When for the control system (differential equation)

$$\Sigma \ : \ \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z), \quad z \in M, \ u \in \mathbb{R}^m,$$

does there exist a (local) diffeomorphism $\Phi : M \to TQ$ such that

$$\begin{aligned} \Phi_* F &= y^i \frac{\partial}{\partial x^i} + \left(-\Gamma^i_{jk}(x) y^j y^k + d^i_j(x) y^j + g^i_0(x) \right) \frac{\partial}{\partial y^i} \\ \Phi_* G_r &= g^i_r(x) \frac{\partial}{\partial y^i}, \end{aligned}$$

- our problem is more specific: the right hand side is quadratic in velocities
- our problem is more general:
  - no á priori tangent bundle structure TQ
  - non potential forces g_0 are allowed
  - dissipative forces are allowed
- The vector fields G_r provide additional information encoded in the Lie algebra generated by them and F.

Symmetric product

- An affine connection ∇ defines the symmetric product:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X \qquad X, Y \in \mathfrak{X}(Q).$$

- In coordinates given by

$$\langle X : Y \rangle = \left(\frac{\partial X^i}{\partial x^j} Y^j + \frac{\partial Y^i}{\partial x^j} X^j + \Gamma^i_{jk} X^j Y^k + \Gamma^i_{jk} Y^j X^k \right) \frac{\partial}{\partial x^i}.$$

- A distribution $\mathcal{D}$ on Q is called *geodesically invariant* with respect to an affine connection ∇ if every geodesic $\gamma : I \to Q$, such that $\gamma'(t_0) \in \mathcal{D}(\gamma(t_0))$ for some $t_0 \in I$, satisfies $\gamma'(t) \in \mathcal{D}(\gamma(t))$ for all $t \in I$.
- Geometric interpretation of the symmetric product (A. Lewis): a distribution $\mathcal{D}$ on a manifold Q, equipped with an affine connection ∇, is geodesically invariant if and only if

$$\langle X : Y \rangle \in \mathcal{D}, \quad \text{for every} \quad X, Y \in \mathcal{D}.$$

- So the symmetric products plays the same role for the geodesic invariance as the Lie brackets for integrability.

Geodesic accessibility

Consider the mechanical control system $(\mathcal{MS}) = (Q, \nabla, \mathfrak{g}_0, d)$. Let $\mathcal{SYM}(g_1, \ldots, g_m)$ be the smallest distribution on Q containing the input vector fields $g_1, \ldots, g_m$ and such that it is closed under the symmetric product defined by the connection ∇.

Definition

The system $(\mathcal{MS})$ is called geodesically accessible at $x_0 \in Q$ if

$$\mathcal{SYM}(g_1, \ldots, g_m)(x_0) = T_{x_0}Q,$$

and geodesically accessible if the above equality holds for all $x_0 \in Q$. A geodesically accessible mechanical system will be denoted by $(\mathcal{GAMS})$.

- For geodesically accessibile mechanical control systems, the smallest geodesically invariant distribution containing the control vector fields $g_1, \cdots, g_m$ is TQ.
- The planar rigid body is geodesically accessible but the robotic leg is NOT geodesically accessible (although accessible).

The basic object

- We will call a *zero-velocity point* for the mechanical control system $(\mathcal{MS})$ any point of the form $(x_0, \dot{x}_0) = (x_0, 0)$, that is, any point of the zero section of the tangent bundle TQ.
- For the control system

$$\Sigma \ : \ \dot{z} = F(z) + \sum_{r=1}^{m} u_r Gr(z),$$

let $\mathcal{V}$ denote the smallest vector space, over $\mathbb{R}$, containing the vector fields $G_1, \ldots, G_m$ and satisfying

$$[\mathcal{V}, \operatorname{ad}_F \mathcal{V}] \subset \mathcal{V},$$

where $[\mathcal{V}, \operatorname{ad}_F \mathcal{V}] = \{[V_i, \operatorname{ad}_F V_j] \mid V_i, V_j \in \mathcal{V}\}$.

Characterization of mechanical control systems

Theorem (Respondek-Ricardo)

Let M be a smooth $2n$-dimensional manifold. A system Σ is locally, at $z_0 \in M$, S-equivalent to a mechanical system $(\mathcal{MS})$ around a zero-velocity point $(x_0, 0)$ if (and only if $(\mathcal{MS})$ is geodesically accessible)

(MS0) $F(z_0) \in \mathcal{V}(z_0)$,

(MS1) $\dim \mathcal{V}(z) = n$ *and* $\dim \big(\mathcal{V} + [F, \mathcal{V}]\big)(z) = 2n$,

(MS2) $[\mathcal{V}, \mathcal{V}](z) = 0$,

for any z in a neighborhood of z_0.

Moreover, under (MS0)-(MS2), the mechanical structure is unique.

- The condition (MS0) implies that the diffeomorphism establishing the S-equivalence (if it exists) will map z_0 into a zero-velocity point.
- A mechanical system (more generally, a control system that is S-equivalent to a mechanical system $(\mathcal{MS})$) is geodesically accessible around a zero-velocity point if and only if it satisfies (MS0) and (MS1).

- The condition (MS2) $[\mathcal{V}, \mathcal{V}] = 0$, is always necessary for S-equivalence to a mechanical system $(\mathcal{MS})$ and sufficient provided that (MS1) and (MS2) hold.
- It states that the Lie algebra $\mathcal{L} = \{F, G_1, \ldots, G_m\}_{LA}$ contains an abelian subalgebra $\mathcal{V}$ (that spans a distribution of rank n) which is the structural condition reflecting the existence of a mechanical structure.
- The conditions (MS0)-(MS2) are veryfiable: define

$$\begin{aligned} \mathcal{V}_1 &= \{G_r \mid 1 \le r \le m\} \\ \mathcal{V}_2 &= \{[G_r, \mathrm{ad}_F\, G_s] \mid 1 \le r, s \le m\} \end{aligned}$$

and, inductively,

$$\mathcal{V}_i = \bigcup_{p+l=i} \left[\mathcal{V}_p, \mathrm{ad}_F\, \mathcal{V}_l\right]. \quad \text{Put} \quad \mathcal{V} := \mathrm{Vect}_{\mathbb{R}} \bigcup_{i=1}^{\infty} \mathcal{V}_i.$$

- Control systems that admit a unique mechanical structure are S-equivalent to a geodesically accessible mechanical systems (Respondek-Ricardo). But linear mechanical control systems are never geodesically accessible (unless the number of controls m equals n, the dimension of the configuration manifold Q) so a new approach to the problem is needed.

Theorem

The following conditions are equivalent for a nonlinear control system of the form $\Sigma : F(z) + \sum_{r=1}^{m} u_r G_r(z)$ *on a* $2n$*-dimensional manifold:*

(i) the system Σ *is S-equivalent, locally at* z_0*, to a controllable linear mechanical system* $(\mathcal{LMS})$*;*

(ii) Σ *satisfies, in a neighborhood of* z_0*, the following conditions*

(LM1) $\dim \operatorname{span} \{\operatorname{ad}_F^q G_r,\ 1 \le r \le m, 0 \le q \le 2n-1\}(z) = 2n,$

(LM2) $\left[\operatorname{ad}_F^p G_r, \operatorname{ad}_F^q G_s\right] = 0,$ *for* $1 \le r, s \le m,\ 0 \le p, q \le 2n,$

(LM3)' *there exist* $d_{iq}^r \in \mathbb{R}$*, where* $1 \le i \le n$*,* $1 \le r \le m$*,* $0 \le q \le 2n-1$*, such that the distribution*

$$\mathfrak{V} = \operatorname{span} \{\sum_{r,q} d_{iq}^r \operatorname{ad}_F^q G_r,\ 1 \le i \le n\}$$

is of rank n*, contains* G_r*, for* $1 \le r \le m$*, and satisfies*

$$\mathfrak{V} + [F, \mathfrak{V}] = TM.$$

(iii) Σ *satisfies (LM1), (LM2) and*

(LM3)" $\dim \operatorname{span} \{G_r, \operatorname{ad}_F G_r,\ 1 \le r \le m\}(z) = 2m.$

Interpretation of the conditions

- The difference between the condition (LM3) and (LM3)' (or (LM3)") explains very clearly the difference between the problems considered in this and the previous theorem.

- If a mechanical system is given (the case of the former theorem), then n vector fields of the form $V_i = \sum_{r,q} d^r_{iq} \operatorname{ad}^q_F G_r$ have to span its vertical distribution $\mathfrak{V}$.

- If a mechanical structure is not given (the case of the last theorem), it is the distribution $\mathfrak{V} = \operatorname{span}\{V_1, \ldots, V_n\}$ which will be the vertical distribution of the mechanical structure to be constructed, provided that $\mathfrak{V}$ satisfies (LM3)' (or, equivalently, (LM3)").

Example-cont.

Clearly, the system $(\mathcal{MS})$ of Example, satisfies the conditions (LM1) and (LM2) (actually, we have given a linearizing diffeomorphism Ψ explicitly). To analyze the condition (LM3)', we take $V_1 = G = \frac{\partial}{\partial y^1}$, that is, $d_{10} = 1$ and $d_{11} = d_{12} = d_{13} = 0$, and $V_2 = d_{21} \operatorname{ad}_F G + d_{22} \operatorname{ad}_F^2 G + d_{23} \operatorname{ad}_F^3 G$. We look for reals d_{21}, d_{22}, d_{23} such that the distribution $\mathfrak{V} = \operatorname{span}\{V_1, V_2\}$ satisfies $\mathfrak{V} + [F, \mathfrak{V}] = TM$. A direct calculation shows that this is the case if and only if

$$d_{21}d_{23} - d_{22}^2 \neq 0.$$

Therefore the system $(\mathcal{MS})_1$ of Example admits infinitely many non-equivalent linear mechanical structures whose vertical distribution can be any distribution $\operatorname{span}\{G, d_{21} \operatorname{ad}_F G + d_{22} \operatorname{ad}_F^2 G + d_{23} \operatorname{ad}_F^3 G\}$, where the real coefficients d_{2q} satisfy the above condition. $\triangleleft$

Reducing the problem to the case of linear systems

- The conditions (LM1) and (LM2) are necessary and sufficient for S-equivalence of Σ to a linear controllable system.
- Therefore the problem becomes that of when a linear control system admits a linear mechanical structure.
- Therefore the last Theorem reduces actually to the following one, which is of independent interest.

Proposition

Consider a linear controllable system of the form $\Lambda: \ \dot{z} = Az + \sum_{r=1}^{m} u_r b_r$, *where* $z \in \mathbb{R}^{2n}$. *The following conditions are equivalent:*

(i) the system Λ *is S-equivalent, via a linear transformation, to a linear mechanical system* $(\mathcal{LMS})$;

(ii) there exists an n*-dimensional linear subspace* $\mathsf{V} \subset \mathbb{R}^{2n}$ *containing the vectors* b_r, *for* $1 \leq r \leq m$, *and satisfying*

$$\mathsf{V} + A\mathsf{V} = \mathbb{R}^{2n}.$$

(iii) all controllability indices of Λ *equal at least two.*

The above proposition explains that all linear controllable systems (excepts for those possessing a controllability index equal to one) admit a linear mechanical structure. Moreover, such a structure is, in general, highly non unique: any n-dimensional linear subspace V satisfying (ii) of the above proposition leads to such a structure.

- Our linear mechanical control systems are general

$$(\mathcal{LMS}) \qquad \begin{array}{rcl} \dot{x} & = & y, \\ \dot{y} & = & Dy + Ex + Bu, \end{array}$$

given by any positional force Ex and any force Dy depending on velocities (any linear controlled SODE)

- And if we want the drift of the system to be Lagrangian? (only potential positional forces)

When is a linear system Lagrangian?

Given

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= Ex \ (+Bu), \end{aligned}$$

when does there exist a quadratic Lagrangian

$$L = \frac{1}{2} \underbrace{y^T M y}_{kinetic} - \frac{1}{2} \underbrace{x^T P x}_{potent.}$$

such that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= M\ddot{x} + Px = M(\ddot{x} - M^{-1}Px) \\ &= M(\ddot{x} - Ex) = 0, \end{aligned}$$

where $M = M^T$, $P = P^T$ and M-invertible. We conclude $P = ME$ and the question is:

- Can we represent a given matrix E as a product of symmetric matrices $E = M^{-1}P$?

Linear inverse problem

Theorem (Helmholtz, Douglas, Sarlet)

(i) *SODE* $\ddot{x} = f(x, \dot{x})$ *is Lagrangian if and only if satisfies Helmholz conditions (algebraic and differential);*

(ii) *for the linear SODE* $\ddot{x} = Ex$*, Helmholz conditions are purely algebraic and read: there exists* $M = M^T$ *such that* $ME = E^T M$ *(clearly, if and only if there exists* $P = P^T$ *such that* $P = ME$*).*

So when is E a product of two symmetric matrices?

All linear mechanical systems, with positional forces, are Lagrangian

Theorem (Frobenius, 1910)

Any real (complex) square matrix can be written as a product of two real (complex) symmetric matrices.

Proposition

Any SODE $\ddot{x} = Ex$ *(control system* $\ddot{x} = Ex + Bu$ *is a Lagrangian system* $M\ddot{x} + Px = 0$ *(control system* $M\ddot{x} + Px = Bu$*), given by Lagrangian* $L = \frac{1}{2}y^T My - \frac{1}{2}x^T Px$ *(by the controlled Lagrangian* $L = \frac{1}{2}y^T My - \frac{1}{2}x^T Px + x^T Bu$*).*

Nonholonomic constraints

- When is the control system

$$\Sigma : \quad \dot{z} = F(z) + \sum_{r=1}^{m} u_r G_r(z), \; z \in M$$

equivalent to a mechanical control systems in the presence of nonholonomic constraints?

- Constraints: $\dot{x} \in \mathcal{C} = \operatorname{span}\{c_1, \ldots, c_k\}$ - constraint distribution
- Constraint forces $\omega = \sum_{i=1}^{n-k} \lambda_i \omega_i$, where $\operatorname{span}\{\omega_1, \ldots, \omega_{n-k}\} = \operatorname{ann} \mathcal{C}$.
- Passsing to vecotor fields constraint forces we get

$$(\mathcal{NHS}) \qquad \begin{array}{rcl} \dot{x} & = & y \\ \dot{y} & = & -y^T \Gamma(x) y + d(x) y + e(x) + \sum\limits_{i=1}^{m} u_i g_i(x) + \sum\limits_{i=1}^{n-k} \lambda_i r_i(x). \end{array}$$

- Eliminating the Lagrange multipliers and taking the constraints into account, we get

Poincaré representation

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{k} c_i(x) v^i & \dim x = n \\
\dot{v} &= -v^T \tilde{\Gamma}(x) v + \tilde{d}(x) v + \tilde{e}(x) + \sum_{i=1}^{m} u_i \tilde{g}_i(x), & \dim v = k.
\end{aligned}$$

- The system evolves on the manifold $\mathcal{C} \subset \mathrm{T}M$, equipped with the coordinates $(x, v) = (x^1, \ldots, x^n, v^1, \ldots, v^k)$
- Analogous definition of the geodesic acessibilty: the smallest distribution containing the $\tilde{g}_i$'s and closed under the symmetric product defined by $\tilde{\nabla}$ is $\mathrm{T}Q$
- $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of the connection $\tilde{\nabla}$, which (in general) is not symmetric and not metrizable (although comming via projection onto $\mathcal{C}$ from the metric connection connection ∇)

When is a control system nonholonomic?

Theorem

Let M be a smooth d-dimensional manifold. A system Σ is locally, at $z_0 \in M$, S-equivalent to a completely nonholonomic system $(\mathcal{NHS})$ around a zero-velocity point $(x_0, 0)$ if (and only if $(\mathcal{MS})$ is geodesically accessible)

(MS0) $F(z_0) \in \mathcal{V}(z_0)$,

(MS1) $\dim \mathcal{V}(z) = k$, *where* $d \geq 2k$ *and* $\dim \left(\mathcal{V} + \overline{[F, \mathcal{V}]}\right)(z) = d$,

(MS2) $[\mathcal{V}, \mathcal{V}](z) = 0$,

for any z in a neighborhood of z_0.

- $\overline{[F, \mathcal{V}]}$ stands for the involutive closure of the distribution $[F, \mathcal{V}]$.
- The only difference for the unconstrained case is $k = n$ and $d = 2n$
- There are no new structural conditions for the constrained case

When a control system admits a mechanical structure?

Why is that question interesting?

- If system admits a mechanical structure, we can apply to it the whole machinery of the mechanical control theory
- If we reduce or constrain a mechanical system, we want to know whether the reduced (constraint) system is still mechanical
- For observed dynamics we define dummy input vector fields and the properties of the virtual control system determine properties of the observer

Affine connection control systems

Mechanical control systems subject

- neither to dissipative-type (or gyroscopic-type) forces, i.e., $d = 0$
- nor to uncontrolled forces, i.e., $g_0 = 0$

are called affine connection control systems and are thus defined as a 3-tuple $(\mathcal{ACS}) = (Q, \nabla, \mathfrak{g})$, with Q and ∇ as before and $\mathfrak{g} = (g_1, \ldots, g_m)$ an m-tuple of input vector fields on Q. For an $(\mathcal{ACS})$, we have

$$\begin{aligned} \dot{x}^i &= y^i, \\ \dot{y}^i &= -\Gamma^i_{jk}(x) y^j y^k + \sum_{r=1}^{m} u_r g_r^i(x), \end{aligned}$$

- Let $\mathrm{Sym}(\mathfrak{g})$ denote the smallest family of vector fields on Q containing $g_1, \ldots, g_m$ and closed under the symmetric product defined by the connection ∇. Elements of $\mathrm{Sym}(\mathfrak{g})$ are thus iterative symmetric products of vector fields $g_1, \ldots, g_m$.

- Let Let $\mathcal{SYM}(\mathfrak{g})$ be the distribution on Q spanned by $\mathrm{Sym}(\mathfrak{g})$.

- Recall that the system $(\mathcal{MS})$ is called geodesically accessible at $x_0 \in Q$ if

$$\mathcal{SYM}(\mathfrak{g})(x_0) = T_{x_0}Q,$$

  and geodesically accessible if the above equality holds for all $x_0 \in Q$.

- Geodesically accessible mechanical control systems are denoted by $(\mathcal{GAMS})$. If additionally, the system is affine connection then it will be called geodesically accessible affine connection system and it will be denoted shortly by $(\mathcal{GACS})$.

Conform frames

- The geodesic accessibility property guarantees the existence of n independent vector fields $v_1, \ldots, v_n \in \mathrm{Sym}(\mathfrak{g})$ and $\tilde{v}_1, \ldots, \tilde{v}_n \in \mathrm{Sym}(\tilde{\mathfrak{g}})$.
- Two frames $(v_1, \ldots, v_n)$ and $(\tilde{v}_1, \ldots, \tilde{v}_n)$, for two systems, are conform if each $\tilde{v}_j,\ 1 \leq j \leq n$, is constructed as an analogous iterative symmetric product as that defining v_j

Fundamental relations

- Fix a frame $(v_1, \ldots, v_n)$ and consider the fundamental equalities

$$\text{(LAR)} \qquad \left[v_{i_q}, \ldots, \left[v_{i_3}, [v_{i_2}, v_{i_1}]\right] \ldots\right] = \alpha^s_{i_1 \ldots i_q}\, v_s, \text{ and}$$

$$\text{(SAR)} \qquad \langle v_{i_q} : \ldots \langle v_{i_3} : \langle v_{i_2} : v_{i_1} \rangle\rangle \ldots \rangle = \beta^s_{i_1 \ldots i_q}\, v_s,$$

  defining the *structure functions* $\alpha^s_{i_1 \ldots i_q}$ and $\beta^s_{i_1 \ldots i_q}$, where $q \geq 2$ and $1 \leq i_1, \ldots, i_q \leq n$.

- Equalities (LAR) and (SAR) give, respectively, information about the Lie algebraic relations and the symmetric algebraic relations of the system.

- Analogously, we can derive the structure functions $\tilde{\alpha}^s_{i_1 \ldots i_q}$ and $\tilde{\beta}^s_{i_1 \ldots i_q}$ for $(\widetilde{\mathcal{GACS}})$. We consider the families of structure functions

$$\begin{aligned} \mathfrak{s} &= \{\alpha^s_{i_1 \ldots i_q},\ \beta^s_{i_1 \ldots i_q} \mid q \geq 2\} \quad \text{and} \\ \tilde{\mathfrak{s}} &= \{\tilde{\alpha}^s_{i_1 \ldots i_q},\ \tilde{\beta}^s_{i_1 \ldots i_q} \mid q \geq 2\} \end{aligned}$$

  defined by the Lie algebraic relations (LAR) and the symmetric algebraic relations (SAR).

Rank and order of a family of functions

- A family of smooth functions $\{\gamma^s_{i_1\dots i_q} \mid q \geq 2\}$ is of a constant rank r, in an open neighborhood U of $x_0 \in Q$, if $\left\{d\gamma^s_{i_1\dots i_q}(x) \mid q \geq 2\right\}$ span an r-dimensional space at any $x \in U$.
- We call the order of a family of constant rank r to be the minimal number ρ such that

$$\dim \operatorname{span} \left\{d\gamma^s_{i_1\dots i_q} \mid 2 \leq q \leq \rho\right\}(x_0) = r.$$

Eqivariants of mechanical control systems

Theorem

Two geodesically accessible affine connection systems $(\mathcal{GACS}) = (Q, \nabla, \mathfrak{g})$ *and* $(\widetilde{\mathcal{GACS}}) = (\tilde{Q}, \tilde{\nabla}, \tilde{\mathfrak{g}})$*, whose families of structure functions* $\mathfrak{s}$ *and* $\tilde{\mathfrak{s}}$ *are of constant rank in neighborhoods of* $x_0 \in Q$ *and* $\tilde{x}_0 \in \tilde{Q}$*, are MS-equivalent around* x_0 *and* $\tilde{x}_0$*, respectively, if and only if there exists a diffeomorphism* $\varphi : W_{x_0} \to \tilde{W}_{\tilde{x}_0}$*, where* W_{x_0} *and* $\tilde{W}_{\tilde{x}_0}$ *are neighborhoods of* x_0 *and* $\tilde{x}_0$ *in* Q *and* $\tilde{Q}$*, respectively, such that*

$$\begin{array}{rrcl} (\mathrm{LAC}) & \alpha^s_{i_1 \ldots i_q} & = & \tilde{\alpha}^s_{i_1 \ldots i_q} \circ \varphi, \\ (\mathrm{SAC}) & \beta^s_{i_1 \ldots i_q} & = & \tilde{\beta}^s_{i_1 \ldots i_q} \circ \varphi, \end{array}$$

for $q \leq \rho + 1$*, with* ρ *being the common order of families* $\mathfrak{s}$ *and* $\tilde{\mathfrak{s}}$*.*

- (LAC) says that the Lie modules, generated by the symmetric vector fields $\mathrm{Sym}(g_1, \ldots, g_m)$ of $(\mathcal{GACS})$ and and $\mathrm{Sym}(\tilde{g}_1, \ldots, \tilde{g}_m)$ of $(\widetilde{\mathcal{GACS}})$, coincide (up to the conjugation by a diffeomorphism of the configuration manifolds Q and $\tilde{Q}$); (SAC) states that the symmetric modules, generated by all symmetric vector fields of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$, coincide (up to the conjugation by the same diffeomorphism).

- If a diffeomorphism ϕ establishing the equivalence of $(\mathcal{GACS})$ and $(\widetilde{\mathcal{GACS}})$ exists then it is unique (since it transforms the frame $(v_1, \ldots, v_n)$ onto the frame $(\tilde{v}_1, \ldots, \tilde{v}_n)$ and $\phi(x_0) = \tilde{x}_0$). On the other hand, the diffeomorphism φ conjugating the structure functions may or may not be unique: we can distinguish three cases:

  (i) If $r = n$, that is, the families $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ are of maximal possible rank, then the diffeomorphism φ conjugating them is unique and φ and ϕ coincide;

  (ii) If $r = 0$, which correspond to $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$ consisting of constant functions only (homogenous case), then (LAC) and (SAC) imply that the structure functions have to be the same and, if this is the case, any diffeomorphism φ conjugates them;

  (iii) If $0 < r < n$, then only a "part" of the diffeomorphism ϕ is determined by the diffeomorphism φ.

Conclusions

- We described mechanical systems that admit a mechanical structure (both holonomic and nonholonomic)
- We discussed linearization of mechanic and mechanizable control systems
- Equivariants of mechanical control systems