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- Feedback equivalence and linearization
  - 4 Orbital feedback equivalence and linearization
- 5 Linearization via dynamic feedback and flatness
  - 6 4 Definitions of flatness
  - 7 Flat systems of minimal differential weight
  - B) Conclusions

Introduction

### Summary

### Introduction

- 2) State-space equivalence and linearization
- 3 Feedback equivalence and linearization
- 4) Orbital feedback equivalence and linearization
- 5 Linearization via dynamic feedback and flatness
- 6 4 Definitions of flatness
- Flat systems of minimal differential weight

### 8 Conclusions

#### Introduction

Class of control systems

- finite-dimensional
- smooth
- time-continous

We will consider

$$\Xi$$
 :  $\dot{x} = F(x, u)$ 

- $x \in X$ , state space, an open subset of  $\mathbb{R}^n$
- $u \in U$ , set of control values, a subset of  $\mathbb{R}^m$
- *F* is smooth ( $C^k$  or  $C^\infty$ ) with respect to (x, u)
- a control system is an underdetermined system of differential equations: n equations for n + m variables

#### Introduction

Very often: control-affine systems

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x), \quad x \in X \subset \mathbb{R}^n, u \in \mathbb{R}^m$$

- f and  $g_1, \ldots, g_m$  are smooth vector fields on X
- state-dependent nonlinearities
- common in applications

Introduction

### Linearization problem

When is  $\Xi$  or  $\Sigma$  equivalent (transformable) to a linear control system?

- define equivalence (or the class of transformations)
- find conditions for linearization
- construct linearizing transformations

State-space equivalence and linearization

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### Introduction

- State-space equivalence and linearization
- 3 Feedback equivalence and linearization
- 4) Orbital feedback equivalence and linearization
- 5 Linearization via dynamic feedback and flatness
- 6 4 Definitions of flatness
- Flat systems of minimal differential weight

### 8 Conclusions

#### The system

$$\Xi$$
 :  $\dot{x} = F(x, u)$ ,  $x \in X$ ,  $u \in U$  and

 $\tilde{\Xi}$  :  $\dot{z} = \tilde{F}(z, u), \ z \in Z, \ u \in U$  (the same control)

are state-space equivalent, shortly S-equivalent, if there exists a diffeomorphism  $z = \Phi(x)$  such that

$$\frac{\partial \Phi}{\partial x} \cdot F(x, u) = \tilde{F}(\Phi(x), u) \text{ i.e., } \Phi_*F = \tilde{F}.$$

- the Jacobian matrix of Φ (the derivative of Φ, i.e, the tangent map of Φ) maps the dynamics F of Ξ into F̃ of Ξ̃
- A diffeomorphism is a map  $\Phi$  such that
  - Φ is bijective
  - $\Phi$  and  $\Phi^{-1}$  are  $C^k$  ( $C^{\infty}$ )
- A (local) diffeomorphism defines a (local) nonlinear change of coordinates z = Φ(x)

### S-equivalence preserves trajectories



The image under  $\Phi$  of a trajectory of  $\Xi$  is a trajectory of  $\tilde{\Xi}$  corresponding to the same control.

### S-linearization

Problem 1 When is  $\Sigma$  S-equivalent to a linear system, i.e., when does there exist  $z = \Phi(x)$  transforming  $\Sigma$  into a linear system of the form

$$\dot{z} = Az + \sum_{i=1}^{m} u_i b_i, \quad x \in \mathbb{R}^n$$

that is, for  $1 \le i \le m$ ,

$$\frac{\partial \Phi}{\partial x}(x) \cdot f = \Phi_* f = Az$$
 and  $\frac{\partial \Phi}{\partial x}(x) \cdot g_i(x) = \Phi_* g_i = b_i$ 

We want the same diffeomorphism Φ to transform *f* into Az (a linear vector field) and g<sub>i</sub> into b<sub>i</sub>, for 1 ≤ i ≤ m (constant vector fields)

Why is S-linearization interesting?

- If we want to solve a control problem for Σ and Σ is S-equivalent to a linear system Λ, then
- transform  $\Sigma$  into  $\Lambda$
- solve the problem for the linear system  $\Lambda$
- transform the solution (via the inverse  $\Phi^{-1}$  of  $\Phi$ )
- we identify intrinsic nonlinearities

### A little bit of geometry: Lie bracket

• Given two vector fields f and g on X, we define their Lie bracket as

$$[f,g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x)$$

It is a new vector field on X.

• It is a geometric (invariant) object

$$\Phi_*[f,g] = [\Phi_*f,\Phi_*g].$$

• It measures to what extent the flows of f and g do not commute

State-space equivalence and linearization

#### Define

$$ad_f^0g = g$$
  
 $ad_fg = [f,g]$   
and, inductively,  $ad_f^kg = [f,ad_f^{k-1}g] = [f,\ldots,[f,g],]$ 

For the single-input system

$$\dot{x} = f(x) + ug(x)$$

the Lie bracket  $ad_fg = [f, g] = [f, f+g]$  measures to what extent the trajectories of *f* (corresponding to  $u \equiv 0$ ) do not commute with those of f + g (corresponding to  $u \equiv 1$ ).

### Theorem

 $\Sigma$  is, locally around  $x_0,$  S-equivalent to a controllable linear system  $\Lambda$  if and only if

(SL1) span 
$$\{ad_f^q g_i(x_0): 1 \leq i \leq m, 0 \leq q \leq n-1\} = \mathbb{R}^n$$

(SL2)  $[ad_{f}^{q}g_{i}, ad_{f}^{r}g_{j}] = 0$ , for  $1 \le i, j \le m, 0 \le q, r \le n$ 

- Interpretation
  - (SL1) guarantees controllability of  $\Lambda$
  - (SL2) implies that all iterative Lie brackets containing at least two g<sub>i</sub>'s vanish, i.e., [L<sub>0</sub>, L<sub>0</sub>] = 0, where L<sub>0</sub> is the strong accessibility Lie algebra.
- Verification
  - (SL1) and (SL2) are verifiable in terms of *f* and *g<sub>i</sub>*'s using differentiation and algebraic operations only (no need to solve PDE's)

State-space equivalence and linearization

#### Theorem

 $\Sigma$  on X is globally S-equivalent to a controllable linear system  $\Lambda$  on  $\mathbb{R}^n$  if and only if

(SL1) span {  $ad_{f}^{q}g_{i}(x_{0}): 1 \leq i \leq m, 0 \leq q \leq n-1$  } =  $\mathbb{R}^{n}$ 

 $(SL2) [ad_{f}^{q}g_{i}, ad_{f}^{r}g_{j}] = 0, \text{ for } 1 \leq i, j \leq m, 0 \leq q, r \leq n$ 

(SL3) the vector fields  $f, g_1, \ldots, g_m$  are complete (equivalently,  $ad_f^q g_i$ ,  $1 \le i \le m$ ,  $0 \le q \le n-1$  are complete).

(SL4) X is simply connected

 If we drop (SL4), then Σ is globally S-equivalent to a controllable linear system Λ on T<sup>k</sup> × R<sup>n-k</sup>.

# Constructing linearizing coordinates

Assume, for simplicity, the scalar-input case m = 1. In order to find the linearizing diffeomorphism  $z = \Phi(x)$  solve the system of *n* 1st order PDE's:

$$(S) \frac{\partial \Phi}{\partial x} A(x) = Id,$$

where  $A(x) = (A_1(x), ..., A_n(x))$  and  $A_q(x) = ad_f^{q-1}g(x)$ , for  $1 \le q \le n$ . (SL2) form the integrability conditions for (S) and assure the existence of solutions.

- Do not confuse S-linearization with linear approximation
- Assume  $F(x_0, u_0) = 0$ . The linear approximation of  $\dot{x} = F(x, u)$  is

$$\dot{z} = Az + Bv + higher order terms$$
  
 $\dot{z} = Az + Bv$ ,

where 
$$A = \frac{\partial F}{\partial x}(x_0, u_0)$$
 and  $B = \frac{\partial F}{\partial u}(x_0, u_0)$ 

- So we neglect (erase) higher order terms
- In S-linearization higher order terms are compensated via the diffeomorphism  $\Phi$  (no terms are neglected)

#### Consider the pendulum



The states are  $(x_1, x_2) = (\theta, \dot{\theta})$  and the control is the torque *u* 

### The equations are

$$\dot{x}_1 = x_2$$
  
 $\dot{x}_2 = -\frac{g}{7} \sin x_1 + \frac{1}{m^2} u.$ 

We have

$$f = \begin{pmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{pmatrix}$$
,  $g = \begin{pmatrix} 0 \\ \frac{1}{ml^2} \end{pmatrix}$ ,  $ad_f g = -\begin{pmatrix} \frac{1}{ml^2} \\ 0 \end{pmatrix}$ 

yielding

$$[g, \textit{ad}_{\it f}g] = 0$$
 but  $[\textit{ad}_{\it f}g, \textit{ad}_{\it f}^2g] 
eq 0$ 

which implies that the pendulum is not S-linearizable

• But put 
$$u = ml^2(\frac{g}{l}\sin x_1 + v)$$

• we get the linear controllable system (in the Brunovsky form)

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& v_. \end{array}$$

• therefore there are systems that become linear after applying a (nonlinear) transformation in the control space

Feedback equivalence and linearization

### Summary

### 1 Introduction

- 2 State-space equivalence and linearization
- 3 Feedback equivalence and linearization
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### 8 Conclusions

#### The systems

$$\Xi$$
 :  $\dot{x} = F(x, u)$ ,  $x \in X$ ,  $u \in U$  and

$$\tilde{\Xi}$$
 :  $\dot{z} = \tilde{F}(z, v)$ ,  $z \in Z$ ,  $v \in V$  not the same control

are feedback equivalent, shortly F-equivalent, if there exists

• a diffeomorphism 
$$z = \Phi(x)$$
 and

• a control transformation  $v = \Psi(x, u)$ , invertible with respect to u

such that

$$\frac{\partial \Phi}{\partial x} \cdot F(x, u) = \tilde{F}(\Phi(x), \Psi(x, u)).$$

Why is F-equivalence interesting?

#### Does F-equivalence preserve trajectories?



Is the image of a trajectory, via the diffeomorphism  $z = \Phi(x)$ , a trajectory?

Yes, the image of a trajectory of  $\Xi$ , for a control u(t), is a trajectory of  $\tilde{\Xi}$  corresponding to

 $\mathbf{v}(t) = \Psi(\mathbf{x}(t), \mathbf{u}(t))$ 



Therefore, F-equivalence preserves the set of all trajectories (the totality of trajectories)



F-equivalence is thus interesting for all problems that depend on the set of all trajectories (and not on a particular parametrization with respect to control). Examples of such problems are: point-to-point controllability, trajectory tracking, stabilization.

Problem 2 When is  $\Xi$  F-equivalent to a linear system, i.e., when do there exist  $z = \Phi(x)$  and  $\Psi(x, u)$  transforming  $\Xi$  into a linear system of the form

$$\dot{z} = Az + \sum_{i=1}^{m} u_i b_i, \quad x \in \mathbb{R}^n$$
?

For control affine systems

$$\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x), \quad x \in X$$

we apply  $z = \Phi(x)$  and control-affine feedback transformation

$$u = \alpha(x) + \beta(x)v$$
,

where the matrix  $\beta$  is invertible.



- Let  $\mathcal{D} = \text{span} \{f_1, \dots, f_k\}$  be a distribution spanned by vector fields
- $\mathcal{D}$  is involutive if  $[f_i, f_j] \in \mathcal{D}$ , for any  $1 \le i, j \le k$

• Put 
$$\mathcal{D}^j = \operatorname{span} \{ ad_f^q g_i; \ 1 \le i \le m, \ 0 \le q \le j-1 \}$$

#### Theorem

 $\Sigma$  is, locally around  $x_0$ , F-equivalent to a controllable linear system  $\Lambda$  if and only if

```
(FL1) dim \mathcal{D}^{j}(x)=const.
```

(FL2) dim  $\mathcal{D}^n(x) = n$ 

(FL3)  $\mathcal{D}^{j}$  are involutive, for  $0 \leq j \leq n$ 

- (FL2) guarantees controllability of  $\Lambda$
- (FL1)-(FL3) are verifiable in terms of *f* and *g<sub>i</sub>*'s using differentiation and algebraic operations only (no need to solve PDE's)
- Geometry:  $\mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{n-1} \subset \mathcal{D}^n = TX$ .

Assume, for simplicity m = 1. Involutivity of  $\mathcal{D}^{n-1}$  (of dimension n-1 at any x) is equivalent to the existence of a family of hypersurfaces  $H_c = \{x \in X : h(x) = c\}$  tangent to  $\mathcal{D}^{n-1}$ 



# Constructing linearizing transformations

• The normal vector to the hypersurface  $H_c$  has to be annihilated by  $g, \ldots, ad_f^{n-2}g$  spanning  $\mathcal{D}^{n-1}$ . So solve

(S) 
$$\frac{\partial h}{\partial x}A(x) = 0$$
, where  $A(x) = (g(x), \dots, ad_f^{n-2}g(x))$ 

• any solution h,  $dh \neq 0$  of (S) gives linearizing coordinates

$$z_i = L_f^{i-1}h$$
, for  $1 \le i \le n$ 

• and linearizing feedback

$$v = L_f^n h + u L_g L_f^{n-1} h$$

Orbital feedback equivalence and linearization

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### 1 Introduction

- 2) State-space equivalence and linearization
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### 8 Conclusions

Orbital feedback equivalence and linearization

#### For the system

$$\Xi$$
:  $\frac{dx}{dt} = \dot{x} = F(x, u), x \in X, u \in U$ 

define a new time scale  $\tau$  such that

$$\frac{dt}{d\tau} = \gamma(\mathbf{x}(t)),$$

where  $\gamma$  is a nonvanishing function on X. With respect to the new time scale  $\tau$ 

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = \gamma(x)F(x, u).$$

We change the velocity along the trajectories.

The systems

$$\Xi$$
 :  $\dot{x} = F(x, u)$ ,  $x \in X$ ,  $u \in U$  and

 $\tilde{\Xi}$  :  $\dot{z} = \tilde{F}(z, v)$ ,  $z \in Z$ ,  $v \in V$  not the same control

are orbitally feedback equivalent, shortly OF-equivalent, if there exists

- a diffeomorphism  $z = \Phi(x)$  and
- a control transformation  $v = \Psi(x, u)$ , invertible with respect to u
- a nonvanishing function  $\gamma$  on X

such that

$$\frac{\partial \Phi}{\partial x} \cdot \gamma(x) F(x, u) = \tilde{F}(\Phi(x), \Psi(x, u)).$$

Why is OF-equivalence interesting?

### Does OF-equivalence preserve trajectories?



Is the image of a trajectory, via the diffeomorphism  $z = \Phi(x)$ , a trajectory?

Yes, the image of a trajectory of  $\Xi$ , for a control u(t), is a trajectory of  $\tilde{\Xi}$  corresponding to

$$\mathbf{v}(t) = \Psi(\mathbf{x}(t), \mathbf{u}(t))$$

and parameterized by the new time  $\tau = \int_0^t \frac{ds}{\gamma(x(s))}$ 



Therefore, OF-equivalence preserves the set of all trajectories (the totality of trajectories) as unparameterized curves



OF-equivalence is thus interesting for all problems that depend on the set of all trajectories and not on a particular parametrization with respect to control and time.
Orbital feedback equivalence and linearization

For 
$$\Sigma$$
 :  $\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)$ , define

the distributions

$$\begin{array}{lll} \mathcal{G} &=& \mathrm{span}\,\{g_1,\ldots,g_m\},\\ \mathcal{G}_f^j &=& \mathrm{span}\,\{f,g_i,\mathrm{ad}_fg_i,\ldots,\mathrm{ad}_f^{j-1}g_i, \quad 1\leq i\leq m\}, \ \text{ for } 1\leq j\leq n \end{array}$$

the differential forms

$$\omega^{j}(h) = 0, \text{ for any } h \in \mathcal{G}_{f}^{n},$$
  
 $\omega^{j}(\mathrm{ad}_{f}^{n}g_{j}) = \delta_{i}^{j}$ 

and the functions:

$$\mathbf{T}_{i,j}^{k,l} = \omega^k([\mathrm{ad}_f^{n-1}g_i, \mathrm{ad}_f^lg_j])$$

Attach to Σ the distribution

$$\mathcal{D} = \operatorname{span}\{f, g_1, \ldots, g_m\} = \operatorname{span}\{f\} + \mathcal{G}$$

Orbital feedback equivalence and linearization

## Theorem (ShunJie Li-Respondek)

The following conditions are equivalent:

- Σ is, locally around x<sub>0</sub>, OF-equivalent to a controllable linear system Λ
- $\Sigma$  satisfies (OFL1) dim  $\mathcal{G}_{f}^{n+1}(x) = (n+1)m+1$ ; (OFL2)  $[\mathcal{G}_{f}^{j}, \mathcal{G}_{f}^{j}] \subset \mathcal{G}_{f}^{j+1}$ , for  $1 \leq j \leq n$ ; (OFL3)  $[\mathcal{G}, \mathcal{G}_{f}^{2}] \subset \mathcal{G}_{f}^{2}$ ; (OFL4) The functions  $T_{i,j}^{k,l}$  equal zero or one.

### Σ satisfies

(OFL1)'  $C(D^{(1)}) = G$ , where  $C = C(D^{(1)})$  is the characteristic distribution of  $D^{(1)} = [D, D]$ , i.e.,  $[C, D^{(1)}] \subset D^{(1)}$ . (OFL2)'  $D_{\Sigma}$  is locally equivalent to the  $J^n(1, m)$  contact system. Orbital feedback equivalence and linearization

- (OFL1)-(OFL4) are generalizations of involutivity conditions for feedback linearization
- (OFL1)-(OFL4) are verifiable in terms of *f* and *g<sub>i</sub>*'s using differentiation and algebraic operations only (no need to solve PDE's)

Linearization via dynamic feedback and flatness

# Summary

## 1) Introduction

- 2) State-space equivalence and linearization
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## 8 Conclusions

Linearization via dynamic feedback and flatness

# Example: Unicycle

The unicycle on the plane subject to a nonholonomic constraint: the wheel is not allowed to slide.

- (x<sub>1</sub>, x<sub>2</sub>) ∈ ℝ<sup>2</sup>: the position of the mid-point of the unicycle;
- θ ∈ ℝ: the angle between the wheel and x<sub>1</sub>-axis;
- (u<sub>1</sub>, u<sub>2</sub>) ∈ ℝ<sup>2</sup>: controls allowing to move (forward and backward) the unicycle and to turn.



Figure: The unicycle system

Linearization via dynamic feedback and flatness

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \end{pmatrix} = u_1 \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = u_1 g_1 + u g_2.$$

We have

$$[g_1, g_2] = Dg_2 \cdot g_1 - Dg_1 \cdot g_2 = \begin{pmatrix} \sin \theta \\ -\cos \theta \\ 0 \end{pmatrix} \notin \operatorname{span} \{g_1, g_2\}$$

The distribution spanned by the control vector fields

$$\mathcal{D} = \operatorname{span} \{g_1, g_2\}$$

is not involutive. Thus the unicycle is not static feedback linearizable, i.e., not *F*-equivalent to the controllable linear system  $\dot{z} = Az + Bv$  (even, locally on  $X \subset \mathbb{R}^3$ ). But...

Linearization via dynamic feedback and flatness

Consider the control system (precompensator)

$$\dot{y} = v_1$$

and link it to the unicycle via

$$u_1 = y$$
$$u_2 = v_2$$



Figure: The precompensated unicycle

We control the derivative  $\dot{u}_1 = v_1$  of the first control (the second control  $u_2 = v_2$  remaining the same).  $\Rightarrow$  Dynamic precompensation (preintegration)

The precomompensated unicycle becomes F-linearizable (the linearizability distributions are involutive

Linearization via dynamic feedback and flatness

The precomompensated unicycle becomes

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{\theta} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \cos \theta \\ y \sin \theta \\ v_2 \\ v_1 \end{pmatrix}$$

applying the coordinates change

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ y \cos \theta \\ y \sin \theta \end{pmatrix}$$

and the control transformation

$$\left(\begin{array}{c} \tilde{v}_1\\ \tilde{v}_2 \end{array}\right) = \left(\begin{array}{c} \cos\theta & -y\sin\theta\\ \sin\theta & y\cos\theta \end{array}\right) \left(\begin{array}{c} v_1\\ v_2 \end{array}\right)$$

we get the linear controllable system

$$\dot{z}_1 = z_3$$
  $\dot{z}_2 = z_4$   
 $\dot{z}_3 = \tilde{v}_1$   $\dot{z}_4 = \tilde{v}_2$ ,

Linearization via dynamic feedback and flatness

## Remarks:

- It is a dynamic feedback since  $v_1 = \dot{u}_1$ .
- The feedback law is invertible for  $y = u_1 \neq 0$ .
- The unicycle has the same trajectories (for  $u_1 \neq 0$ ) as a linear system.
- Knowing  $z_1(t) = x_1(t)$  and  $z_2(t) = x_2(t)$  we can calculate all states and control via differentiation only.
- The dimension of the state space is not preserved.

Questions:

- Dynamic feedback is involved: what is a dynamic invertible feedback?
- How to formalize equivalence via such a transformation?

4 Definitions of flatness

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## 8 Conclusion

4 Definitions of flatness

# Infinite preintegrations

Consider also its infinite prolongation

$$\Xi^{\infty}: \begin{cases} \dot{x} = F(x, u^{0}) \\ \dot{u}^{0} = u^{1} \\ \vdots \\ \dot{u}^{\prime} = u^{\prime+1} \\ \vdots \end{cases}$$

- The system Ξ<sup>∞</sup> is a dynamical system (differential equation and not a control system) evolving on X × U<sup>∞</sup> = X × U × ℝ<sup>m</sup> × ℝ<sup>m</sup> × · · · .
- A function on X × U<sup>∞</sup> is C<sup>∞</sup>-smooth if, locally, it depends on a finite number of variables (it is actually a function on X × U<sup>l</sup>) and is C<sup>∞</sup>-smooth with respect to those variables.
- Denote the right hand side of  $\Xi^{\infty}$  by  $F^{\infty}$  and put  $u^{\infty} = (u^0, u^1, u^2, \cdots)$ .

4 Definitions of flatness

# Dynamic equivalence

### Definition

Two control systems  $\Xi$  and  $\tilde{\Xi}$  are  $D^\infty$  -equivalent if there exists a diffeomorphism

$$\chi: X imes U^{\infty} o ilde{X} imes U^{\infty}$$

such that

$$D\chi(x, u^{\infty}) \cdot F^{\infty}(x, u^{\infty}) = \tilde{F}^{\infty}(\chi(x, u^{\infty})).$$

### Definition (Flatness, first version)

A nonlinear control system  $\Xi$  is flat if it is  $D^{\infty}$ -equivalent to a linear controllable system  $\Lambda$ .

• Analogous to S-equivalence of ODE's (dynamical systems):

$$D\phi(\mathbf{x}) \cdot f(\mathbf{x}) = \tilde{f}(\phi(\mathbf{x})).$$

•  $D^{\infty}$ -equiv. is elegant and compact but involves infinite prolongations

Two systems  $\Xi$  and  $\tilde{\Xi}$  are dynamically equivalent, shortly D-equivalent, if there exist maps  $\Phi$  and  $\Psi$  mapping trajectories onto trajectories and mutually inverse on trajectories. How to formalize?



## Two control systems

$$\Xi$$
 :  $\dot{x} = F(x, u)$ ,  $x \in X \subset \mathbb{R}^n$ ,  $u \in U \subset \mathbb{R}^m$ 

and

$$ilde{\Xi}$$
 :  $\dot{ extsf{x}} = ilde{F}( ilde{ extsf{x}}, ilde{ extsf{u}}), \ ilde{ extsf{x}} \in ilde{ extsf{X}} \subset \mathbb{R}^{ ilde{ extsf{n}}}, \ u \in ilde{ extsf{U}} \subset \mathbb{R}^{ extsf{m}}$ 

are *D*-equivalent if there exist two integers *I* and  $\tilde{I}$  and two pairs of maps

$$\tilde{\mathbf{x}} = \phi(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \dots, \mathbf{u}^{(l)})$$
$$\tilde{\mathbf{u}} = \psi(\mathbf{x}, \mathbf{u}, \dot{\mathbf{u}}, \dots, \mathbf{u}^{(l)})$$

and (with a different numbers of derivatives)

$$\begin{aligned} \mathbf{x} &= \tilde{\phi}(\tilde{\mathbf{x}}, \tilde{u}, \dot{\tilde{u}}, \dots, \tilde{u}^{(l)}) \\ \mathbf{u} &= \tilde{\psi}(\tilde{\mathbf{x}}, \tilde{u}, \dot{\tilde{u}}, \dots, \tilde{u}^{(l)}) \end{aligned}$$

that map trajectories into trajectories and are mutually inverse on trajectories. The new states and controls depend on the old states, old controls and their derivatives and vice-versa.

4 Definitions of flatness

# Dynamic precompensation

Consider the control system

$$\Xi$$
 :  $\dot{x} = F(x, u), x \in X \subset \mathbb{R}^n, u \in U \subset \mathbb{R}^m$ 

together with the precompensation

$$\Pi : \begin{cases} \dot{y} = G(x, y, v), & y \in Y \subset \mathbb{R}^{l}, & v \in V \subset \mathbb{R}^{m} \\ u = \psi(x, y, v) \end{cases}$$

The precompensated system becomes

$$\Xi \circ \Pi : \begin{cases} \dot{x} = F(x, \psi(x, y, v)) \\ \dot{y} = G(x, y, v). \end{cases}$$

# Dynamic endogeneous invertible feedback

The dynamic feedback defining the precompensation is endogeneous if

$$\mathbf{y}=\mu(\mathbf{x},\mathbf{u},\ldots,\mathbf{u}^{(l)}),$$

for a smooth function  $\mu$ , that is, the state of the precompensator is a function of the original state, original control and its derivatives. The dynamic feedback is said invertible if the precompensated system, together with the output

$$u = \psi(x, y, v)$$

is input-output invertible, that is, if we can express

$$\mathbf{v}=\bar{\mathbf{v}}(\mathbf{x},\mathbf{y},\mathbf{u},\ldots,\mathbf{u}^{(l)}),$$

which, in the case of an endogenous feedback, yields

$$\mathbf{v}=\mathbf{v}(\mathbf{x},\mathbf{u},\ldots,\mathbf{u}^{(l)}).$$

4 Definitions of flatness

## Theorem (FLMR, Jakubczyk, Pomet)

Given two control systems  $\Xi$  and  $\tilde{\Xi}$ , the following conditions are equivalent:

- (i) The systems are  $D^{\infty}$ -equivalent;
- (ii) The systems are D-equivalent;
- (iii) There exist two endogeneous and invertible precompensators  $\Pi$  for  $\Xi$  and  $\tilde{\Pi}$  for  $\tilde{\Xi}$  such that the precomposed systems  $\Xi \circ \Pi$  and  $\tilde{\Xi} \circ \tilde{\Pi}$  are *S*-equivalent.

## Definition (Flatness, second version)

A nonlinear control system  $\Xi$  is flat if it is D-equivalent to a linear controllable system  $\Lambda$ .

4 Definitions of flatness

# Example

*D*-equivalence does not preserve the dimension of the state space. Two control systems

$$\dot{x}_1 = x_2$$
 and  $\dot{\tilde{x}}_1 = \tilde{u}$   
 $\dot{x}_2 = u$ 

are D-equivalent. Indeed, the transformations

$$\begin{aligned} \tilde{x}_1 &= x_1 & x_1 &= \tilde{x}_1 \\ \tilde{u} &= x_2 & \text{and} & x_2 &= \tilde{u} \\ u &= \dot{\tilde{u}} \end{aligned}$$

map trajectories into trajectories and are are mutually inverse on trajectories.

Any linear controllable system is *F*-equivalent to the Brunovsky canonical form:

Ż <sub>11</sub>	=	<i>Z</i> <sub>12</sub>	ż <sub>m1</sub>	=	Z <sub>m2</sub>
	÷			÷	
$\dot{z}_{1\rho_{1}-1}$	=	$Z_{1\rho_1}$	$\dot{z}_{m ho_m-1}$	=	$Z_{m\rho_m}$
$\dot{z}_{1 ho_1}$	=	<i>V</i> 1	ż <sub>mρm</sub>	=	<b>v</b> <sub>m</sub> ,

and thus *D*-equivalent to the trivial system consisting of *m* functions  $z_{11}, \ldots, z_{m1}$  with no dynamics. The trajectories of that system are arbitrary evolutions of  $z_{11}(t), \ldots, z_{m1}(t)$  subject to no constraints, so the variables are completely free.

## Definition (Flatness, third version)

A nonlinear control system  $\Xi$  is flat if it is D-equivalent to a trivial system with no dynamics.

φ

4 Definitions of flatness

# Solving trajectory generation problem via flatness

Using flatness we easily solve the constructive controllability problem: given  $x_0$  and  $x_T$ , find a trajectory joining them. Assume that  $\Xi$  is *D*-equivalent to a controllable linear system  $\Lambda$  (single-input, for simplicity), which is in Brunovsky canonical form

$$\dot{z}_1 = z_2$$

$$\vdots$$

$$\dot{z}_{n-1} = z_{n-2}$$

$$\dot{z}_n = v$$

In order to go from  $z_0 = \phi(x_0)$  into  $z_T = \phi(x_T)$ , choose a  $C^n$ -function  $\varphi(t), t \in [0, T]$ , such that

$\begin{aligned} \varphi(T) &= z_{1T} \\ \dot{\varphi}(T) &= z_{2T} \end{aligned}$	$   \phi(0) = z_{10} $ $   \dot{\phi}(0) = z_{20} $
:	÷
$\varphi^{(n-1)}(T) = z_{nT}$	$^{(n-1)}(0) = z_{n0}$

Then the control  $v(t) = \varphi^{(n)}(t)$  steers the system from  $z_0$  into  $z_T$  and the control u(t) for the original system  $\Xi$  can be computed with the help of v(t) and its derivatives (an invertible transformation!)

# Flatness: the most popular definition

If  $\tilde{\Xi}$  is a system with no dynamics and *m* free variables are denoted  $\tilde{u}_1, \ldots, \tilde{u}_m$ , then a direct application of the third definition (*D*-equivalence of  $\Xi$  and  $\tilde{\Xi}$ ), requires the existence of a map

$$\tilde{u} = \psi(x, u, \ldots, u^{(l)})$$

such that

$$\begin{aligned} \mathbf{x} &= \tilde{\phi}(\tilde{u}, \dots, \tilde{u}^{(\tilde{l})}) \\ u &= \tilde{\psi}(\tilde{u}, \dots, \tilde{u}^{(\tilde{l})}) \end{aligned}$$

since  $\tilde{x}$  is not present. Renaming the variables  $\tilde{u}_i$  by  $\varphi_i$  as well as  $\tilde{\phi}$  and  $\tilde{\psi}$  by  $\gamma$  and  $\delta$ , respectively, gives

#### Flatness: the most popular definition

$$\Xi$$
:  $\dot{x} = F(x, u), x \in X \subset \mathbb{R}^n, u \in U \subset \mathbb{R}^m$ 

is flat at  $(x_0, u_0, \dot{u}_0, \dots, u_0^{(p)}) \in X \times U \times \mathbb{R}^{mp}$ , for  $p \ge -1$ , if there exists *m* smooth functions  $\varphi_i = \varphi_i(x, u, \dot{u}, \dots, u^{(p)})$ , called flat outputs, such that

$$\begin{aligned} \mathbf{x} &= \gamma(\varphi, \dot{\varphi}, \dots, \varphi^{(\mathbf{s})}) \\ \mathbf{u} &= \delta(\varphi, \dot{\varphi}, \dots, \varphi^{(\mathbf{s})}) \end{aligned}$$

where  $\varphi = (\varphi_1, \ldots, \varphi_m)$ .

• Remark: If  $\varphi_i = \varphi_i(x)$ , for  $1 \le i \le m$ ,  $\Xi$  is *x*-flat.

# To memorize flatness

Consider the mechanical control system

$$\dot{q} = v$$
  
 $\dot{v} = rac{u}{m}$ 

 To know all trajectories (q(t), v(t)) (configurations and velocities), we apply all control forces u(t) and integrate

$$u(t) \Rightarrow \frac{1}{m} \int u(t) dt = v(t) \Rightarrow \int v(t) dt = q(t)$$

• But we can look all configuration trajectories q(t) and differentiate

$$q(t) \Rightarrow \dot{q}(t) = v(t) \Rightarrow \dot{v}(t) = \ddot{q}(t) = u(t)$$

so q is a flat output

To integrate the control system we do not have to integrate, we differentiate only

Flat systems of minimal differential weight

# Summary

## 1) Introduction

- 2) State-space equivalence and linearization
- 3 Feedback equivalence and linearization
- 4) Orbital feedback equivalence and linearization
- 5 Linearization via dynamic feedback and flatness
- 6 4 Definitions of flatness

Flat systems of minimal differential weight

### Conclusion

Flat systems of minimal differential weight

# Linear systems

### Theorem

- (i) A linear control system  $\Lambda$ :  $\dot{x} = Ax + Bu$  is flat if and only if it is controllable.
- (ii) Flat outputs are  $\varphi = z_{11}, \ldots, \varphi_m = z_{m1}$ , the top variables of the Brunovsky canonical form.

For m = 1, define  $c \neq 0$  by

$$cb = cAb = \cdots = cA^{n-2}b = 0.$$

Then h = cx is a flat output

Flat systems of minimal differential weight

# Nonlinear single-input systems, m = 1

### Theorem

The following conditions are equivalent for a single-input system  $\Sigma$ 

- (i)  $\Sigma$  is flat;
- (ii)  $\Sigma$  is *F*-linearizable
- (iii)  $\Sigma$  satisfies

(FL1) dim  $\mathcal{D}^{j}(x)$ =const.

(FL2) dim  $\mathcal{D}^n(x) = n$ 

(FL3)  $\mathcal{D}^{j}$  are involutive, for  $0 \leq j \leq n$  ( $\mathcal{D}^{n-1}$  is involutive) Moreover, a flat output is any function  $\varphi$  satisfying

(S) 
$$\frac{\partial \varphi}{\partial x} A(x) = 0$$
, where  $L(x) = (g(x), \dots, ad_f^{n-2}g(x))$ 

Flat systems of minimal differential weight

# F-linearizable systems

- For multi-input systems m ≥ 2, F-linearizability is sufficient for flatness but not necessary.
- Moreover, flat ouputs are the top variables of the Brunovsky canonical form.

### Minimal flat outputs and differential weight

For any flat output  $\varphi$  of  $\Xi$  there exist integers  $s_1, \ldots, s_m$  such that

$$\begin{aligned} \mathbf{x} &= \gamma(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\mathbf{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\mathbf{s}_m)}) \\ \mathbf{u} &= \delta(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\mathbf{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\mathbf{s}_m)}). \end{aligned}$$

We can choose  $(s_1, \ldots, s_m)$  such that if for any other *m*-tuple  $(\tilde{s}_1, \ldots, \tilde{s}_m)$ 

$$\begin{aligned} \mathbf{x} &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{\mathbf{s}}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{\mathbf{s}}_m)}) \\ \mathbf{u} &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{\mathbf{s}}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{\mathbf{s}}_m)}), \end{aligned}$$

then  $s_i \leq \tilde{s}_i$ , for  $1 \leq i \leq m$ .

• Differential weight of  $\varphi = \sum_{i=1}^{m} (s_i + 1) = \sum_{i=1}^{m} s_i + m$ ,

i.e., minimal number of derivatives of  $\varphi_i$  needed to express *x* and *u*.

- $\varphi$ : minimal flat output if its differential weight is the lowest among all flat outputs of  $\Xi$ .
- Differential weight of  $\Xi$ : the differential weight of a minimal flat output.

### Static feedback linearizable (F-linearizable) systems

F-linearizable systems are the only flat systems of differential weight n + m.

- The representation of x and u uses the minimal possible, which is n + m, number of time-derivatives of  $\varphi_i$ .
- · For any flat system, that is not F-linearizable, the differential weight

 $\hookrightarrow$  is bigger than n + m.

 $\leftrightarrow$  measures the smallest possible dimension of a precompensator linearizing dynamically the system.

- In general, a flat system is not F-linearizable, except the single-input case where flatness reduces to F-linearization.
- The simplest flat systems that are not F-linearizable are systems that become F-linearizable via one-dimensional precompensator  $\Rightarrow$  differential weight n + m + 1.

### Our goal

· To give a geometric characterization of control-affine systems

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x),$$

that become F-linearizable after a one-fold prolongation of a suitably chosen control (the simplest dynamic feedback).

 $\hookrightarrow$  verifiable conditions (like involutivity conditions).

 $\hookrightarrow$  describe and understand the geometry of this class of systems.

Flat systems of minimal differential weight

### Main theorem (Nicolau - Respondek) m=2

Assume  $k \ge 1$  and  $\overline{\mathcal{D}}^k \neq TX$ .  $\Sigma$  is x-flat at  $x_0 \in X$ , of differential weight n + 3, if and only if

(A1)  $rk \, \bar{\mathcal{D}}^k = 2k + 3;$ 

(A2) 
$$rk(\bar{\mathcal{D}}^k + [f, \mathcal{D}^k]) = 2k + 4 \Rightarrow \exists g_c \in \mathcal{D}^0 \text{ such that } ad_f^{k+1}g_c \in \bar{\mathcal{D}}^k;$$

(A3)  $\mathcal{B}^i$ , for  $i \ge k$ , is involutive, where  $\mathcal{B}^k = \mathcal{D}^{k-1} + \operatorname{span} \{ ad_f^k g_c \}$  and  $\mathcal{B}^{i+1} = \mathcal{B}^i + [f, \mathcal{B}^i];$ 

(A4) There exists  $\rho$  such that  $\mathcal{B}^{\rho} = TX$ .

#### Comparison with the F-linearizable case

• Geometry of flat systems of differential weight *n*+3

Geometry of F-linearizable systems

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \cdots \subset \mathcal{D}^{n-1} = TX$$
 63/75

### Remarks

- General result (the particular cases k = 0 and  $\overline{D}^k = TX$  have slightly different geometry).
- Enables us to define, up to a multiplicative function, the control to be prolonged:

$$g_c = \beta_1 g_1 + \beta_2 g_2 \in \mathcal{D}^0 \Rightarrow v_1 = \frac{d}{dt} (\beta_2 u_1 - \beta_1 u_2) = \frac{d}{dt} \tilde{u}_1$$

• All conditions are verifiable, verification involves derivations and algebraic operations only (without solving PDE'S).

### Calculating flat outputs

- $\mu$ : the largest integer such that corank  $(\mathcal{B}^{\mu-1} \subset \mathcal{B}^{\mu}) = 2$ .
- $\rho$ : the smallest integer such that  $\mathcal{B}^{\rho} = TX$ .

Proposition (Nicolau - Respondek)

(i) Assume  $\overline{\mathcal{D}}^k \neq TX$  or  $\overline{\mathcal{D}}^k = TX$  and  $[\mathcal{D}^{k-1}, \mathcal{D}^k] \notin \mathcal{D}^k$ .  $(\varphi_1, \varphi_2)$  is a minimal *x*-flat output at  $x_0$  if and only if

$$darphi_1\perp \mathcal{B}^{
ho-1} \qquad darphi_2\perp \mathcal{B}^{\mu-1}, \ darphi_2\wedge darphi_1\wedge dL_f arphi_1\wedge \cdots \wedge dL_f^{
ho-\mu}arphi_1(x_0)
eq 0.$$

The pair  $(\varphi_1, \varphi_2)$  is unique, up to a diffeomorphism.

(ii) Assume  $\overline{\mathcal{D}}^k = TX$  and  $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ .

 $(\varphi_1, \varphi_2)$  is a minimal x-flat output at  $x_0$  if and only if  $(d\varphi_1 \wedge d\varphi_2)(x_0) \neq 0$ and the involutive distribution  $\mathcal{L} = (\text{span } \{d\varphi_1, d\varphi_2\})^{\perp}$  satisfies

 $\mathcal{D}^{k-1} \subset \mathcal{L} \subset \mathcal{D}^k.$ 

## Remarks for the case $(ii)\overline{\mathcal{D}}^k = TM$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$

• For any function  $\varphi_1$ , satisfying

$$darphi_1\perp \mathcal{D}^{k-1}$$
 ,

there exists  $\varphi_2$  such that the pair  $(\varphi_1, \varphi_2)$  is a minimal *x*-flat output and the choice of  $\varphi_2$  is unique, up to a diffeomorphism.

• There is as many flat outputs as functions of three variables (since  $\mathcal{D}^{k-1}$  is involutive and of corank 3).

### Induction motor - first model with $\theta$ , the mechanical position

$$\begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = \mu \psi_d i_q - \frac{\tau_L}{J} \\ \dot{\psi}_d = -\eta \psi_d + \eta M i_d \\ \dot{\rho} = n_p \omega + \frac{\eta M i_q}{\psi_d} \\ \dot{i}_d = -\gamma i_d + \frac{\eta M \psi_d}{\sigma L_R L_S} + n_p \omega i_q + \frac{\eta M i_q^2}{\psi_d} + \frac{u_d}{\sigma L_S} \\ \dot{i}_q = -\gamma i_q - \frac{M n_p \omega \psi_d}{\sigma L_R L_S} - n_p \omega i_d - \frac{\eta M i_d q}{\psi_d} + \frac{u_d}{\sigma L_S} \end{cases}$$

•  $u_d$ ,  $u_q$  are the inputs (the stator voltages);

- *i<sub>d</sub>* and *i<sub>q</sub>* are the stator currents;
- $\psi_d$  and  $\rho$  are two well-chosen functions of the rotor fluxes;
- $\omega$  is the rotor speed;
- $\theta$  is the mechanical position.

The system is flat of differential weight 9 = 6 + 2 + 1 = n + m + 1. k = 1 and  $\bar{D}^1 \neq TX \xrightarrow{Prop.2(i)} (\varphi_1, \varphi_2) = (\omega, \rho)$  and the pair  $(\varphi_1, \varphi_2)$  is unique.

### Induction motor - second model without $\theta$ , the mechanical position

$$\begin{cases} \dot{\omega} = \mu \psi_d i_q - \frac{\tau_L}{J} \\ \dot{\psi}_d = -\eta \psi_d + \eta M i_d \\ \dot{\rho} = n_p \omega + \frac{\eta M i_q}{\psi_d} \\ \dot{i}_d = -\gamma i_d + \frac{\eta M \psi_d}{\sigma L_R L_S} + n_p \omega i_q + \frac{\eta M i_q^2}{\psi_d} + \frac{u_d}{\sigma L_S} \\ \dot{i}_q = -\gamma i_q - \frac{M n_p \omega \psi_d}{\sigma L_R L_S} - n_p \omega i_d - \frac{\eta M i_d i_q}{\psi_d} + \frac{u_q}{\sigma L_S} \end{cases}$$

The system is flat of differential weight 8 = 5 + 2 + 1 = n + m + 1.  $k = 1, \overline{D}^1 = TX$  and  $[D^0, D^1] \subset D^1 \xrightarrow{Prop.2(ii)}$  many flat outputs (the choice being parameterized by a function of three well defined variables)

• if 
$$\varphi_1 = \omega$$
, then  $\varphi_2 = \rho$ ;

• if 
$$\varphi_1 = \psi_d$$
, then  $\varphi_2 = \frac{\eta M}{\mu \psi_d^2} \omega - \rho$ ;

• if 
$$\varphi_1 = \rho + \frac{\eta M}{\mu \psi_d}$$
, then  $\varphi_2 = \psi_d - \omega$ .
Flat systems of minimal differential weight

### Main theorem (Nicolau - Respondek) $m \ge 3$

Assume  $k \ge 1$  and cork  $(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \ge 2$ . A control system  $\Sigma$  is x-flat, with the differential weight n + m + 1, if and only if it satisfies around:

(A1) There exists an involutive subdistribution  $\mathcal{H}^k \subset \mathcal{D}^k$ , of corank one;

(A2)  $\mathcal{H}^i$ , for  $i \ge k + 1$ , is involutive, where  $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}];$ 

(A3) There exists  $\rho$  such that  $\mathcal{H}^{\rho} = TX$ .

#### Comparison with the F-linearizable case

• Geometry of flat systems of differential weight n + m + 1

$$\mathcal{D}^{0} \subset \cdots \subset \mathcal{D}^{k-1} \quad \subset \quad \mathcal{D}^{k} \quad \subset \quad \bar{\mathcal{D}}^{k} \\ 1 \cup \qquad \cap \\ \mathcal{H}^{k} \quad \subset \quad \mathcal{H}^{k+1} \subset \cdots \subset \mathcal{H}^{\rho} = TX$$

Geometry of F-linearizable systems

$$\mathcal{D}^{0} \subset \mathcal{D}^{1} \subset \cdots \subset \mathcal{D}^{k-1} \subset \mathcal{D}^{k} \subset \mathcal{D}^{k+1} \subset \cdots \subset \mathcal{D}^{n-1} = TX$$

Flat systems of minimal differential weight

### Remarks

General result (the particular cases k = 0 and cork  $(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$  have slightly different geometry).

 $\hookrightarrow$  If k = 0: similar result, but in the chain of subdistributions

$$\mathcal{H}^0 \subset \mathcal{D}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2 \subset \cdots$$

the distribution  $\mathcal{H}^1$  is not defined as  $\mathcal{H}^{k+1} = \mathcal{H}^k + [f, \mathcal{H}^k]$ , but as

$$\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$$

and satisfies an additional nonsingularity condition  $\Rightarrow$  singularity in the control space.

Linearization of nonlinear control systems: state-space, feedback, orbital, and dynamic

Flat systems of minimal differential weight

#### Remarks

2 In order to verify conditions (A1)-(A3): check the existence of the involutive subdistribution  $\mathcal{H}^k$  of corank one in  $\mathcal{D}^k$ .

↔ Pasillas-Lépine and Respondek (2001): checkable conditions, based on Bryant (Ph.D. thesis, 1979), to verify the existence of an involutive subdistribution of corank one and an explicit way to construct it.

 $\hookrightarrow$  If cork  $(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) \ge 2$ : the subdistribution  $\mathcal{H}^k$  is unique.

 $\hookrightarrow$  If cork  $(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ : the subdistribution  $\mathcal{H}^k$  is no longer unique, but we can uniquely identify it by another argument.

- All conditions are verifiable, verification involves derivations and algebraic operations only (without solving PDE'S).
- Explicit construction of H<sup>k</sup> enables us to define the control to be prolonged (given up to a multiplicative function).

Linearization of nonlinear control systems: state-space, feedback, orbital, and dynamic

Conclusions

## Summary

### 1) Introduction

- 2) State-space equivalence and linearization
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### Conclusions

Conclusions

# What do we know about flatness?

- Via flatness we can solve the constructive controllability problem
- Although very useful, flatness is a highly non generic property: a slight perturbation of a flat system yields a non flat one (Tchoń)
- We know that a few classes of control systems are flat: accessible systems with *n* − 1 controls, accessible control-linear systems with *n* − 1 and *n* − 2 controls
- We know to characterize flat control systems of special forms: feedback linearizable systems, control-linear systems with 2 controls (chained form), *m*-chained form
- or of very special dimensions: 3 states and 2 controls (nonlinear) and 4 states and 2 controls (affine)
- We know to characterize flat systems of differential weight n + m + 1

Conclusions

# What don't we know about flatness?

- We do not know to characterize flatness in general.
- We do not know whether the problem is finite or infinite dimensional, that is, we do not know if there is a bound on the number of derivatives of controls
- We do not even know how to check flatness for control-affine systems with 2 controls nor for control-linear systems with 3 controls
- We know that the problem is difficult: Ellie Cartan (1914) has introduced the notion of absolute equivalence of underdetermined differential equations. His absolutely trivial equations are just flat systems. He proved that systems with 2 controls are flat (absolutely trivial) if and only if they are equivalent to the chained form (Goursat normal form). Cartan claimed that the general problem is difficult.
- Non flat systems exist! The first example is due David Hilbert (1912) who had also been working on absolute equivalence (integrating differential equations without integration). His example is, geometrically, the same as the unicycle towing a trailer but with a hook that is not at the mid-point.

Conclusions

# Conclusions

- We presented various definitions of the notion of flatness
- We provided geometric tools convenient (needed) to study flatness
- We presented geometric conditions for characterizing flatness (verifiable via differentiation and algebraic operations only) for a few classes of systems
- On not confuse linearization (static, dynamic) with linear approximation
- Whenever we can linearize the system (statically, dynamically), the control problems, we are dealing with, get substantially simplified
- Even if we do not apply linearizing transformations or the system is not linearizable (flat), our knowledge about the system is deeper: we identify intrinsic nonlinearities that cannot be removed via feedback (static, dynamic)