Disjointly homogenous Banach lattices: duality and complementation

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Disjointly homogeneous Banach lattices:

Definition

E is disjointly homogeneous (DH) $\Leftrightarrow$ $\forall\,(x_n)$, (y_n) disjoint in E, $\exists\,(n_k)$ such that

$$\Big\|\sum_{k=1}^{\infty} a_k x_{n_k}\Big\| \sim \Big\|\sum_{k=1}^{\infty} a_k y_{n_k}\Big\|$$

Examples: L_p, $L_{p,q}$, $\Lambda(W,p)$, $\ell_p(X_n)\dots$

Definition

E is p-disjointly homogeneous (p-DH) if every disjoint sequence (x_n) in E has a subsequence such that

$$\Big\|\sum_{k=1}^{\infty} a_k x_{n_k}\Big\| \sim \Big(\sum_{k=1}^{\infty} |a_k|^p\Big)^{1/p} \big(\sup_k |a_k| \text{ in case } p=\infty\big)$$

Remark: Not every DH Banach lattice is p-DH (Ex. Tsirelson)

Applications of DH Banach lattices

Theorem

E DH with finite cotype and unconditional basis.

$$T \in \mathcal{SS}(E) \Rightarrow T^2 \in \mathcal{K}(E)$$

Theorem

E 1-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{DP}(E)$

Theorem

E 2-DH with finite cotype. $T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{K}(E)$

Theorem

E discrete with a disjoint basis and DH.

$$T \in \mathcal{SS}(E) \Rightarrow T \in \mathcal{K}(E)$$

Duality

Question: Is the property DH stable by duality?

Known-facts:

- E ∞-DH $\Rightarrow$ E^* 1-DH.
- $L_{p,1}$ is 1-DH but $L^*_{p,1} = L_{p',\infty}$ is not DH.
- Maybe for E reflexive?

We will show that in general the answer is negative (even in the reflexive case), but will also provide positive results.

Positive results

Definition

A Banach lattice E has property $\mathfrak{P}$ if for every disjoint positive normalized sequence $(f_n) \subset E$ there exists a positive operator $T : E \to [f_n]$, such that $\|T^* f_n^*\| \nrightarrow 0$.

Theorem

Let E be a reflexive Banach lattice with property $\mathfrak{P}$. If E^* is DH, then E is DH. Moreover, in the particular case when E^* is p-DH, for some $1 < p < \infty$, then E is q-DH with $\frac{1}{p} + \frac{1}{q} = 1$.

Corollary

Let E be a reflexive Banach lattice satisfying an upper p-estimate. If E^* is q-DH (with $\frac{1}{p} + \frac{1}{q} = 1$), then E is p-DH.

Orlicz spaces

Theorem

An Orlicz space $L_\varphi(0,1)$ *is* p*-DH* $\Leftrightarrow$ $E_\varphi^\infty \cong \{t^p\}$. *Here,* $E_\varphi^\infty = \bigcap_{s>0} \overline{\left\{ \frac{\varphi(rt)}{\varphi(r)} : r \geq s \right\}}$. *In particular, this holds if* $\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p$.

Remark: $L_\varphi(0,1)$ is p-DH $\Leftrightarrow$ $L_\varphi^*(0,1)$ is q-DH ($\frac{1}{p} + \frac{1}{q} = 1$).

Theorem

A separable Orlicz space $L_\varphi(0,\infty)$ *is* p*-DH (for* $1 \leq p < \infty$*) if and only if* $C_\varphi(0,\infty) \cong \{t^p\}$. *Where*

$$C_\varphi(0,\infty) = \overline{\mathrm{conv}}\left\{F \in C(0,1) \mid F(\cdot) = \frac{\varphi(s\cdot)}{\varphi(s)},\ \textit{for some}\ \ s \in (0,\infty)\right\}.$$

Example

Let $1 < p < \infty$ and an Orlicz function $\varphi(t)$ agrees with t^p on $[0,1]$ and $\varphi(t) \simeq t^p \log(1+t)$ on $[1,\infty)$. Then the Orlicz space $L^\varphi(0,\infty)$ is a reflexive p-DH Banach lattice whose dual is not DH.

Projections onto disjoint sequences

Question: Does every reflexive Banach lattice contain a complemented positive disjoint sequence?

Theorem

Let E be a DH Banach lattice. E has property $\mathfrak{P}$ if and only if E contains a complemented positive disjoint sequence.

Theorem

If E is a separable non-reflexive DH Banach lattice, then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

Theorem

Let E be a p-DH Banach lattice which is p-convex with $1 < p < \infty$. Then every disjoint sequence in E has a subsequence spanning a complemented subspace in E.

References

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Thank you all for your attention...

and Happy Birthday Prof. Defant