Optimal range and domain for Hardy type operators on rearrangement invariant spaces.

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Rearrangement invariant spaces

Let (Ω, Σ, μ) be a measure space
Given a meaurable function f we consider its distribution function

$$\mu_f(s) = \mu\{\omega \in \Omega : |f(\omega)| > s\} \quad (s \geq 0)$$

and its decreasing rearrangement

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}.$$

Definition

An r.i. Banach function space X over (Ω, Σ, μ) is the set of measurable functions $f : \Omega \to \mathbb{R}$ with $\|f\|_X < \infty$ satisfying:

(P1) $\|\cdot\|_X$ is a norm;

(P2) $0 \leq f^* \leq g^* \Rightarrow \|f\|_X \leq \|g\|_X$;

(P3) $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow \|f_n\|_X \uparrow \|f\|_X$ (Fatou property);

(P4) $E \in \Sigma$, $\mu(E) < \infty \Rightarrow \|\chi_E\|_X < \infty$;

(P5) $\mu(E) < \infty \Rightarrow \int_E |f| d\mu \leq C_E \|f\|_X$, for some $C_E < \infty$ independent of f.

Examples

- L_p **spaces:**

$$\|f\|_p = \Big(\int_\Omega |f(\omega)|^p \, d\mu\Big)^{\frac{1}{p}} = \Big(p\int_0^\infty s^{p-1}\mu_f(s)\, ds\Big)^{\frac{1}{p}} = \Big(\int_0^\infty f^*(t)^p \, dt\Big)^{\frac{1}{p}}.$$

- **Lorentz spaces:** Given a concave function φ on $\mathbb{R}_+$, we consider the space Λ_φ normed by

$$\|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(s)\, d\varphi(s).$$

- **Marcinkiewicz spaces:** Given a concave function φ on $\mathbb{R}_+$, we consider the space M_φ normed by

$$\|f\|_{M_\varphi} = \sup_{t>0} \frac{\varphi(t)}{t}\int_0^t f^*(s) ds.$$

- **Orlicz spaces**...

The fundamental function

$L^1 \cap L^\infty$ is an r.i. space with the norm

$$\|f\|_{L^1 \cap L^\infty} = \max\{\|f\|_{L^1}, \|f\|_{L^\infty}\}$$

$L^1 + L^\infty$ is an r.i. space with the norm

$$\|f\|_{L^1 + L^\infty} = \inf\{\|g\|_{L^1} + \|h\|_{L^\infty} : f = g + h\}$$

For every r.i. space X: $\qquad L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty$

The fundamental function of an r.i. space X is given by

$$\varphi_X(t) = \|\chi_E\|_X,$$

where $\mu(E) = t$.
φ_X is a concave function on $\mathbb{R}_+$.

$$\Lambda_{\varphi_X} \hookrightarrow X \hookrightarrow M_{\varphi_X}$$

Hardy operator

$$Sf(t) = \frac{1}{t} \int_0^t f(r)dr$$

Theorem (Hardy – Littlewood)

Let $1 < p \leq \infty$. $S : L_p(\mathbb{R}_+) \to L_p(\mathbb{R}_+)$ is bounded.

Given an r.i. space X, consider the dilation $E_t f(s) = f(st)$ and $h_X(t) = \|E_{1/t}\|_X$. The **upper Boyd index** of X is

$$\overline{\alpha_X} = \lim_{t\to\infty} \frac{\log h_X(t)}{\log t}$$

Theorem (Lorentz – Shimogaki)

Let X be an r.i. space on $\mathbb{R}_+$. $S : X \to X$ is bounded iff $\overline{\alpha_X} < 1$.

Optimal domain / range

For r.i. X with $\overline{\alpha_X} = 1$, it is not true that $S : X \to X$. However, it may happen that for some r.i. Y

$$S : X \to Y$$

If we fix X, it is natural to look for the minimal r.i. space Y such that $S : X \to Y$ (**r.i. optimal range**)

Analogously, if we fix Y, we look for the maximal r.i. space Z such that $S : Z \to Y$ (**r.i. optimal domain**)

R.i. optimal range

Theorem

Let X be an r.i. space. TFAE:

(i) There exists the r.i. optimal range for the Hardy operator S on X.

(ii) $S : X \to L^1 + L^\infty$ is bounded.

(iii) $\chi_{(0,1)}(t) \log\left(1/t\right) = \log^+(1/t) \in X'$.

(iv) $S'\chi_{(0,t)} \in X'$, for every $t > 0$.

Moreover, if any of these conditions holds, then the r.i. optimal range for Hardy operator on X is given by

$$\mathfrak{R}[S,X] = \left\{ f \in L^1 + L^\infty : \|f\|_{\mathfrak{R}[S,X]} = \sup_{g\downarrow} \frac{\int_0^\infty f(r)g(r)\,dr}{\|S'g\|_{X'}} < \infty \right\}.$$

$$S'f(t) = \int_t^\infty \frac{f(s)}{s} ds.$$

R.i. optimal range for Lorentz spaces

Proposition

Let φ be a concave function on $\mathbb{R}_+$. TFAE:

(i) There exists the r.i. optimal range for the Hardy operator S on Λ_φ (namely, $\mathfrak{R}[S, \Lambda_\varphi]$).

(ii) $\varphi(t) \gtrsim t \log(1 + 1/t)$

Although we don't know the explicit form of $\mathfrak{R}[S, \Lambda_\varphi]$ in general, its characteristic function satisfies:

$$\varphi_{\mathfrak{R}[S,\Lambda_\varphi]}(t) = \|\chi_{(0,t)}\|_{\mathfrak{R}[S,\Lambda_\varphi]} \approx \inf_{r>0} \frac{t\varphi(r)}{r \log(1 + \frac{t}{r})} =: \widetilde{\varphi}(t).$$

Proposition

$$\mathfrak{R}[S, \Lambda_\varphi] = \Lambda_{\widetilde{\varphi}} \Leftrightarrow \int_t^\infty \frac{\widetilde{\varphi}(s)}{s^2}\, ds \lesssim \frac{\varphi(t)}{t}.$$

R.i. optimal range for Marcinkiewicz spaces

Proposition

Let φ be a concave function on $\mathbb{R}_+$. TFAE:

(i) There exists the r.i. optimal range for the Hardy operator S on M_φ (namely, $\mathfrak{R}[S, M_\varphi]$).

(ii) $\frac{1}{\varphi}$ is locally integrable at zero.

The fundamental function:

$$\varphi_{\mathfrak{R}[S,M_\varphi]}(t) = \frac{t}{\int_0^t \frac{ds}{\varphi(s)}}$$

Theorem

$$\mathfrak{R}[S, M_\varphi] = M_{\frac{t}{\int_0^t \frac{ds}{\varphi(s)}}}.$$

R.i. optimal domain

Theorem

Given an r.i. space X, the following are equivalent:

(i) There exists the r.i. optimal domain for the Hardy operator into the space X.

(ii) $S : L^1 \cap L^\infty \to X$ is bounded.

(iii) $\frac{1}{1+s} \in X$.

Moreover, under any of the above assumptions, the r.i. optimal domain for the Hardy operator S into X is given by

$$\mathfrak{D}[S, X] = \left\{ f \in L^1 + L^\infty : Sf^* \in X \right\}$$

endowed with the norm $\|f\|_{\mathfrak{D}[S,X]} = \|Sf^*\|_X$.

Two new functors

$$\mathcal{D}_X = \mathfrak{D}[S, \mathfrak{R}[S, X]].$$

$$\mathcal{R}_X = \mathfrak{R}[S, \mathfrak{D}[S, X]].$$

Question:

Is it true that $\mathcal{D}_X = X = \mathcal{R}_X$?

(i) $\mathcal{R}_X \subset X \subset \mathcal{D}_X$.
(ii) If $\overline{\alpha}_X < 1$, then $\mathcal{R}_X = X = \mathcal{D}_X$.
(iii) If $X \subset Y$, then $\mathcal{D}_X \subset \mathcal{D}_Y$ and $\mathcal{R}_X \subset \mathcal{R}_Y$.
(iv) $\mathfrak{R}[S, \mathcal{D}_X] = \mathfrak{R}[S, X]$ and $\mathfrak{D}[S, \mathcal{R}_X] = \mathfrak{D}[S, X]$.
(v) $\mathcal{R}_{\mathcal{R}_X} = \mathcal{R}_X$ and $\mathcal{D}_{\mathcal{D}_X} = \mathcal{D}_X$.

Theorem

If $\exists g \downarrow$ s.t. $\varphi(t)/t \approx SS'g(t)$, then $\mathcal{D}_{\Lambda_\varphi} = \Lambda_\varphi$

Thank you for your attention!

http://arxiv.org/abs/1311.3582