The Hardy-Littlewood maximal operator on graphs

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Notation

Let $G = (V, E)$ simple, connected, and finite graph.
Shortest path distance

$$d_G(v, v') = \min\{k : \exists (v_j)_{j=0}^k,\ v_0 = v,\ v_k = v',\ (v_{j-1}, v_j) \in E\ \forall j \leq k\}.$$

$B(v, r)$ ball of center $v \in V$ and radius $r \geq 0$.
$(V, d_G, |\cdot|)$ metric measure space.
Given $f : V \to \mathbb{R}$ let

$$M_G f(v) = \sup_{r \geq 0} \frac{1}{|B(v,r)|} \sum_{w \in B(v,r)} |f(w)|.$$

(Centered) Hardy-Littlewood maximal function

Theorem

Let G_1 and G_2 be two graphs with $V(G_1) = V(G_2) = \{1, \ldots, n\}$. The following are equivalent:

(i) $G_1 = G_2$.

(ii) *For every* $f : \{1, \ldots, n\} \to \mathbb{R}$, $M_{G_1} f = M_{G_2} f$.

(iii) *For every* $k \in V$, $M_{G_1} \delta_k = M_{G_2} \delta_k$.

In general, it is not true that if $G_1 \subset G_2$ (i.e., $V(G_1) = V(G_2)$ and $E(G_1) \subset E(G_2)$), then $M_{G_2} f \leq M_{G_1} f$. For example, if $V = \{1, 2, 3, 4\}$, G_1 is a linear tree with leafs 1 and 4, G_2 is the 4-cycle C_4 (with a clockwise orientation of V), then $G_1 \subset G_2$, but $M_{G_2} \delta_4(1) = 1/3 > 1/4 = M_{G_1} \delta_4(1)$.

Lemma

Let G be graph with n vertices, and $T : \ell^p(G) \to \ell^p(G)$ *be a sublinear operator, with* $0 < p \leq 1$. *Then,*

$$\|T\|_p = \max_{k \in V} \|T\delta_k\|_p.$$

$$\begin{array}{ccccc} & & K_4 & & \\ & & \cup & & \\ P_4 & \subset & D_4 & \supset & C_4 \\ \cup & & & & \cup \\ S_4 & & & & L_4 \end{array} \qquad \begin{array}{ccccc} & & \|M_{K_4}\|_1 & & \\ & & \wedge & & \\ \|M_{P_4}\|_1 & > & \|M_{D_4}\|_1 & = & \|M_{C_4}\|_1 \\ \wedge & & & & \wedge \\ \|M_{S_4}\|_1 & & & & \|M_{L_4}\|_1 \end{array}$$

Proposition

(i) *If* $0 < p \leq 1$, *then* $\|M_{K_n}\|_p = \left(1 + \frac{n-1}{n^p}\right)^{1/p}$.

(ii) *If* $1 < p < \infty$, *then*

$$\left(1 + \frac{n-1}{n^p}\right)^{1/p} \leq \|M_{K_n}\|_p \leq \left(1 + \frac{n-1}{n}\right)^{1/p} \qquad i.e.\ \|M_{K_n}\|_p \approx 1.$$

(iii) *For* $n \geq 3$, *we have* $\|M_{S_n}\|_1 = \frac{n+1}{2}$.

(iv) *For* $1 < p < \infty$, *then*

$$\left(1 + \frac{n-1}{2^p}\right)^{1/p} \leq \|M_{S_n}\|_p \leq \left(\frac{n+5}{2}\right)^{1/p}, \qquad i.e.\ \|M_{S_n}\|_p \approx n^{1/p}.$$

(v) *For* $n \geq 2$ *we have*

$$\|M_{L_n}\|_p \approx \begin{cases} \left(\frac{n^{1-p}-1}{1-p}\right)^{1/p}, & 0 < p < 1, \\ \log n, & p = 1. \end{cases}$$

Theorem

Let G be a graph with n vertices and $0 < p \leq 1$. Then, the following optimal estimates hold:

$$\left(1 + \frac{n-1}{n^p}\right)^{1/p} \leq \|M_G\|_p \leq \left(1 + \frac{n-1}{2^p}\right)^{1/p}.$$

Moreover,

(i) $\|M_G\|_p = \left(1 + \frac{n-1}{n^p}\right)^{1/p}$ *if and only if* $G = K_n$;

(ii) $\|M_G\|_p = \left(1 + \frac{n-1}{2^p}\right)^{1/p}$ *if and only if* $G \sim S_n$.

Weak p-estimates

$$\|f\|_{p,\infty} := \sup_{t>0} t\left|\{j \in V : f_j > t\}\right|^{1/p} = \max_{j\in V} j^{1/p} f_j^*.$$

$$\|M_G\|_{p,\infty} = \sup_f \frac{\|M_G f\|_{p,\infty}}{\|f\|_p}.$$

Theorem

For $0 < p < \infty$, we have

$$\|M_{K_n}\|_{p,\infty} = \begin{cases} n^{1/p-1}, & \text{if } 0 < p \leq 1, \\ 1, & \text{if } p \geq 1. \end{cases}$$

$$\max\{n^{1/p}/2, 1\} \leq \|M_{S_n}\|_{p,\infty} \leq n^{1/p}.$$

(In particular, $\|M_{S_n}\|_{p,\infty} \approx n^{1/p}$, for every $n \geq 1$ and $0 < p < \infty$)

Dilation index

Definition

Given a graph G we define its dilation index as

$$\mathcal{D}(G) = \max\left\{\frac{|B(x,3r)|}{|B(x,r)|} : x \in V,\ r \in \mathbb{N},\ r \leq \mathrm{diam}(G)\right\}.$$

Example

- Complete graph: $\mathcal{D}(K_n) = 1$.
- Star graph: $\mathcal{D}(S_n) = \frac{n}{2}$.
- Linear tree: easy to check that $\mathcal{D}(L_n) < 3$ for all $n \in \mathbb{N}$, and $\lim_{n\to\infty} \mathcal{D}(L_n) = 3$. For small n: $\mathcal{D}(L_3) = 3/2$, $\mathcal{D}(L_4) = 2$, $\mathcal{D}(L_5) = 2$, $\mathcal{D}(L_6) = 2$, $\mathcal{D}(L_7) = 7/3\ldots$

Overlapping index

Definition

Given a graph G we define its overlapping index as

$$\mathcal{O}(G) = \min \Big\{ r \in \mathbb{N} : \forall \{B_j\}_{j\in J},\ B_j \text{ a ball in } G,\ \exists I \subset J, \bigcup_{j\in J} B_j = \bigcup_{i\in I} B_i \text{ and } \sum_{i\in I} \chi_{B_i} \leq r \Big\}.$$

Example

$$\mathcal{O}(K_n) = 1,\ \ \forall n \in \mathbb{N}; \qquad \mathcal{O}(S_n) = n-1,\ \forall n \geq 2;$$

$$\mathcal{O}(L_n) = \begin{cases} 1 & n \leq 2, \\ 2 & n \geq 3; \end{cases} \qquad \mathcal{O}(C_n) = \begin{cases} 1 & n \leq 3, \\ 2 & n \geq 4. \end{cases}$$

Theorem

Given a graph G, we have

$$\|M_G\|_{1,\infty} \leq \min\left\{\mathcal{D}(G), \mathcal{O}(G)\right\}.$$

Proposition

For the linear graph L_n, we have that $\lim_{n\to\infty} \|M_{L_n}\|_{1,\infty} = 2$.

Note $\lim_{n\to\infty} \|M_{L_n}\|_{1,\infty} = 2 = \|\mathcal{M}\|_{1,\infty}$, where $\mathcal{M}$ is the uncentered maximal function in $\mathbb{R}$. Compare to the fact that for the centered Hardy-Littlewood maximal operator M in $\mathbb{R}$ and the discrete measures

$$\mathcal{D} = \left\{\mu = \sum_{k=1}^{N} \delta_{a_k} : a_k \in \mathbb{R},\ a_{k+1} = a_k + H,\ H \text{ fixed}, N \in \mathbb{N}\right\},$$

$$\sup_{\mu\in\mathcal{D}} \frac{\|M\mu\|_{1,\infty}}{\|\mu\|} = \frac{3}{2}.$$

[M. T. Menárguez and F. Soria, 1992]

Thank you for your attention.