The convex hull of a Banach-Saks set

Pedro Tradacete

Universidad Carlos III de Madrid

Joint work with J. Lopez-Abad (CSIC) and C. Ruiz-Bermejo (UCM)

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Convergent sequences

Let $(X, \|\cdot\|)$ be a Banach space. $(x_n)_n \subset X$

$$x_n \longrightarrow x \qquad \Leftrightarrow \qquad \|x_n - x\| \to 0$$

$$x_n \stackrel{Cesaro}{\longrightarrow} x \qquad \Leftrightarrow \qquad \Big\| \frac{1}{n} \sum_{j=1}^{n} x_j - x \Big\| \to 0$$

$$x_n \stackrel{w}{\longrightarrow} x \qquad \Leftrightarrow \qquad x^*(x_n - x) \to 0,\ \forall x^* \in X^*$$

$$x_n \longrightarrow x \quad \Rightarrow \quad x_n \stackrel{Cesaro}{\longrightarrow} x \quad \Rightarrow \quad x_n \stackrel{w}{\longrightarrow} x$$

Definition

A subset $A \subseteq X$ of a Banach space is called Banach-Saks if every sequence in A has a Cesàro convergent subsequence.

Examples:

1. The unit basis of c_0, ℓ_p, $p > 1$ are Banach-Saks (and weakly-null)
2. The unit basis of ℓ_1 is not (and it is not weakly-null)
3. The unit basis of the Schreier space $X_{\mathcal{S}}$ is not, but it is weakly-null.

Recall $X_{\mathcal{S}}$ is the completion of c_{00} under the norm given by

$$\|(a_n)\|_{X_{\mathcal{S}}} = \sup_{E \in \mathcal{S}} \sum_{n \in E} |a_n|,$$

where $\mathcal{S}$ is the class of finite sets of the form $\{n_1 < n_2 < \cdots < n_k\}$ with $k \leq n_1$.

Convex hulls

A compact $\Rightarrow$ A Banach-Saks $\Rightarrow$ A weakly-compact.

Given $A \subset X$,

$$\overline{co}(A) := closure\Big\{ \sum_i \lambda_i x_i : \lambda_i \geq 0, \sum_i \lambda_i \leq 1, x_i \in A \Big\}$$

- A compact $\Rightarrow$ $\overline{co}(A)$ compact (Mazur).
- A weakly-compact $\Rightarrow$ $\overline{co}(A)$ weakly-compact (Krein-Smulian)
- Question: A Banach-Saks $\Rightarrow$ $\overline{co}(A)$ Banach-Saks?

Positive results

A Banach space has the weak Banach-Saks property if every weakly convergent sequence has a Cesàro convergent subsequence.

Examples: L_p $(1 \leq p < \infty)$, $c_0, \ldots$

Proposition

Let X have the weak Banach-Saks property. $A \subset X$ is Banach-Saks if and only if $\overline{co}(A)$ is Banach-Saks.

Positive results

A sequence $(x_n)_n$ in a Banach space X is weakly uniformly convergent to $x \in X$ if for every $\varepsilon > 0$, there is $n(\varepsilon) \in \mathbb{N}$ such that for every $x^* \in X^*$

$$\#\{n \in \mathbb{N} : |x^*(x_n - x)| \geq \varepsilon\} \leq n(\varepsilon).$$

Theorem (Mercourakis)

$(x_n)_n$ converges uniformly weakly to $x \Leftrightarrow \forall (x_{n_k})_k,\ x_{n_k} \overset{Cesaro}{\longrightarrow} x$.

Proposition

If $(x_n)_n$ is uniformly weakly convergent $\Rightarrow$ $\overline{co}(\{x_n\})$ is Banach-Saks.

Schreier spaces

Theorem (González-Rodríguez)

$A \subseteq X_{\mathcal{S}}$ is Banach-Saks if and only if $\overline{co}(A)$ is Banach-Saks.

The Schreier family $\mathcal{S}$ can be extended by induction

$$\mathcal{S}_2 = \mathcal{S} \otimes \mathcal{S} = \big\{ s_1 \cup \cdots \cup s_n : s_i \in \mathcal{S},\ s_1 < \cdots < s_n,\ \{\min(s_1), \ldots, \min(s_n)\} \in \mathcal{S} \big\}$$

$$\mathcal{S}_3 = \mathcal{S}_2 \otimes \mathcal{S}$$

$$\ldots$$

$\mathcal{S}_\alpha$ can be defined for any countable ordinal α.

Theorem

$A \subseteq X_{\mathcal{S}_\alpha}$ is Banach-Saks if and only if $\overline{co}(A)$ is Banach-Saks.

Towards a counterexample

Let $\mathcal{F} \subseteq [\mathbb{N}]^{<\omega}$ be a compact family.

$$\|(a_n)\|_{X_{\mathcal{F}}} = \sup_{E \in \mathcal{F}} \sum_{n \in E} |a_n|$$

Definition

A family $\mathcal{F}$ is large in M when for every $n \in \mathbb{N}$ and $N \subseteq M$ there is $s \in \mathcal{F}$ such that $\#(s \cap N) \geq n$.

Definition

A T-family is a compact and hereditary family $\mathcal{F}$ on $\mathbb{N}$ such that:

(1) $\mathcal{F}$ is never large in any $M \subseteq \mathbb{N}$.

(2) There is a partition $\bigcup_n I_n = \mathbb{N}$ in finite sets I_n and $\delta > 0$ such that

$$\mathcal{G}_\delta(\mathcal{F}) := \{t \subseteq \mathbb{N} : \text{there is } s \in \mathcal{F} \text{ with } \#(s \cap I_n) \geq \delta \# I_n \text{ for all } n \in t\}$$

is large.

Theorem

There is a T-family. In fact, for every $\varepsilon > 0$ there is a family $\mathcal{F}$ such that

1. *$\mathcal{F}$ is not 4-large in any M.*
2. $\mathcal{G}_{1-\varepsilon}(\mathcal{F}) = \mathcal{S}$.

Therefore, in $X_{\mathcal{F}}$ every subsequence of the unit basis $(u_n)_n$ has a subsequence 4-equivalent to the unit basis of c_0. In particular,

$(u_n)_n$ is Banach-Saks.

While, the sequence $x_n = \frac{1}{\#I_n}\sum_{j\in I_n} u_j \in \overline{co}(\{u_n\})$ is equivalent to the unit basis of Schreier space $X_{\mathcal{S}}$. Thus,

$(x_n)_n$ is not Banach-Saks.

Proof of the Theorem:

Please, go to http://arxiv.org/abs/1209.4851

Proposition (Gillis)

For every $\varepsilon > 0$, $\delta > 0$ *and* $m \in \mathbb{N}$, *there is* $n := n(\varepsilon, \delta, m)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mu)$ *and every sequence* $(A_i)_{i<n}$ *with* $\mu(A_i) \geq \varepsilon$ *for all* $i < n$, *there is* $s \subset \{1, \ldots, n\}$ *with* $\sharp s = m$ *such that*

$$\mu(\bigcap_{i \in s} A_i) \geq (1 - \delta)\varepsilon^m.$$

A key idea in the proof of our theorem is the following construction by Erdős and Hajnal:
Let $r, n \in \mathbb{N}$. Given $i < j < n$, let

$$A_{i,j} := \{(a_k)_{k<n} \in r^n \ : \ a_i \neq a_j\}.$$

Clearly $\#A_{i,j} = r^{n-1}(r-1)$. Now if $s \subseteq n$ has cardinality $\geq r+1$, then

$$\bigcap_{\{i,j\} \in [s]^2} A_{i,j} = \emptyset.$$

This provides a counterexample for double-indexed sequences of the expected generalization of Gillis' result.

Thank you very much for your attention.