Banach lattices in convex geometry

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Valuations

Given a class of sets $\mathcal{S}$, a valuation on $\mathcal{S}$ is a mapping $V : \mathcal{S} \longrightarrow \mathbb{R}$ such that for $A, B \in \mathcal{S}$,

$$V(A) + V(B) = V(A \cup B) + V(A \cap B),$$

whenever $A \cup B$, $A \cap B \in \mathcal{S}$.

Valuations are an important tool in convex geometry.

Hadwiger's theorem

$\mathcal{K}^n$: convex bodies in $\mathbb{R}^n$, with Hausdorff metric.
Quermassintegrals: $W_j : \mathcal{K}^n \to \mathbb{R}$ given by Steiner's formula, for $K \in \mathcal{K}^n$ and $t \geq 0$

$$Vol_n(K + tB) = \sum_{j=0}^{n} \binom{n}{j} W_j(K) t^j.$$

W_j is an $(n-j)$-homogenous valuation invariant under rigid motions.

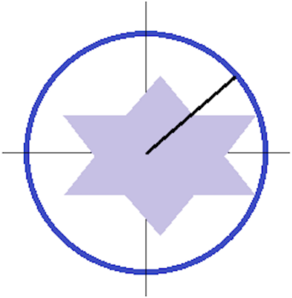
Theorem

If $V : \mathcal{K}^n \longrightarrow \mathbb{R}$ is a continuous valuation invariant under rigid motions, then there exist $(\lambda_j)_{j=0}^n$ such that for every $K \in \mathcal{K}^n$

$$V(K) = \sum_{j=0}^{n} \lambda_j W_j(K).$$

Star sets

$L \subset \mathbb{R}^n$ is a star set if $x \in L \Rightarrow tx \in L$ for every $t \in [0, 1]$

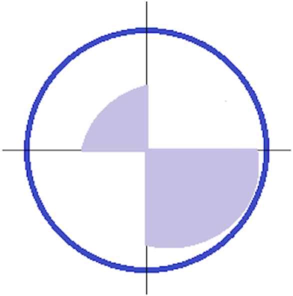


Given a star set $L \subset \mathbb{R}^n$, we define its radial function $\rho_L : S^{n-1} \to \mathbb{R}_+$ by

$$\rho_L(t) = \sup\{c \geq 0 \,:\, ct \in L\},$$

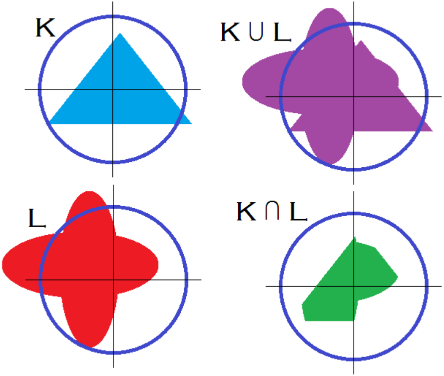
for $t \in S^{n-1}$.

Star sets



A star set L is called a star body if ρ_L is continuous.
Note, for every positive continuous function $f : S^{n-1} \longrightarrow \mathbb{R}^+ = [0, \infty)$ there exists a star body L_f such that $f = \rho_{L_f}$.
A star set L is a bounded Borel star set if ρ_L is a bounded Borel function.
$\mathcal{S}_0^n :=$ the set of n-dimensional star bodies.
$\mathcal{S}_b^n :=$ the set of n-dimensional bounded Borel star sets.

Star sets



Given $K, L \in \mathcal{S}_0^n$ (respectively $\mathcal{S}_b^n$), both $K \cup L$ and $K \cap L$ are in $\mathcal{S}_0^n$ (respectively $\mathcal{S}_b^n$), and it is easy to see that

$$\rho_{K\cup L} = \rho_K \vee \rho_L, \qquad \rho_{K\cap L} = \rho_K \wedge \rho_L.$$

Rotationally invariant valuations

Let $\theta : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a continuous function. Then the application $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ given by

$$V(K) = \int_{S^{n-1}} \theta(\rho_K(t))dm(t)$$

is a radial continuous rotationally invariant valuation (where m is Lebesgue's measure on S^{n-1}).

Theorem (Villanueva 2016)

If $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}_+$ is a positive, rotationally invariant, radial continuous valuation on the n-dimensional star bodies $\mathcal{S}_0^n$, with $V(\{0\}) = 0$, then there exists a continuous function $\theta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\theta(0) = 0$, such that for $K \in \mathcal{S}_0^n$,

$$V(K) = \int_{S^{n-1}} \theta(\rho_K(t))dm(t).$$

Valuations on Banach lattices

A valuation on a Banach lattice E is a mapping $V : E_+ \to \mathbb{R}$ such that for any $f, g \in E$

$$V(f) + V(g) = V(f \vee g) + V(f \wedge g).$$

A valuation $V : E_+ \to \mathbb{R}$ is continuous if $V(f_n) \to V(f)$ whenever $\|f_n - f\| \to 0$.
[Valuations on star bodies correspond to valuations on $C(S^{n-1})$.]
Properties that can be of interest:

1. boundedness of a valuation
2. decomposition of a valuation
3. integral representations
4. uniform continuity on certain sets

Boundedness

Theorem

Every continuous valuation $V : E \to \mathbb{R}$ is bounded on order bounded sets.

Theorem

Let E be a Banach lattice of measurable functions over a σ-finite measure space (Ω, Σ, μ) with finitely many atoms. If E satisfies a lower q-estimate for some $q < \infty$, then every valuation on E which is continuous at 0 is bounded on norm bounded sets.

Recall a Banach lattice E satisfies a lower q-estimate if $\exists M > 0$ such that for every choice of pairwise disjoint $(x_i)_{i=1}^n$

$$\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{\frac{1}{q}} \leq M \Big\| \sum_{i=1}^n x_i \Big\|.$$

Jordan-like decomposition

Let (K, d) be a compact metric space.

Definition

Given a set $A \subset K$, and $\omega > 0$, the ω-rim around A is the set

$$R(A, \omega) = \{t \in K : 0 < d(t, A) < \omega\}.$$

Lemma

Let $V : C(K)_+ \to \mathbb{R}$ be a continuous valuation. Let $A \subset K$ be any Borel set and $\lambda \in \mathbb{R}^+$.

$$\lim_{\omega \to 0} \sup\{|V(f)| : f \leq \lambda \chi_{R(A,\omega)}\} = 0.$$

Jordan-like decomposition

Theorem

Let $V : E_+ \longrightarrow \mathbb{R}$ be a radial continuous valuation with $V(0) = 0$. Then, there exist continuous valuations $V^+, V^- : E_+ \longrightarrow \mathbb{R}_+$ with $V^+(0) = V^-(0) = 0$ and

$$V = V^+ - V^-.$$

Sketch of proof

Define $\qquad V^+(f) = \sup\{V(g) : 0 \leq g \leq f\}.$

Enough to show:

- $V^+(f_1 \vee f_2) + V^+(f_1 \wedge f_2) \overset{(\leq)}{=} V^+(f_1) + V^+(f_2)$,
- V^+ continuous.

Then take $V^- := V^+ - V$.

Note: if V is rotationally invariant, so are V^+ and V^-.

[From now on, we can assume V positive.]

Control measure and extension to Borel sets

For each $\lambda \geq 0$, define the *outer* measure μ^*_λ: For $G \subset S^{n-1}$ open, set

$$\mu^*_\lambda(G) = \sup\{V(f) : f \leq \lambda \chi_G\}.$$

The following defines a finite Borel measure on S^{n-1}:

$$\mu_\lambda(A) = \inf\{\mu^*_\lambda(G) : A \subset G,\ G \text{ an open set }\}.$$

Theorem

Let $V : C(S^{n-1})_+ \to \mathbb{R}$ be a continuous valuation. There is a continuous valuation $\overline{V} : B(S^{n-1})_+ \to \mathbb{R}$ extending V.

Theorem

If $V : C(S^{n-1})_+ \to \mathbb{R}$ is a continuous valuation, then it is uniformly continuous on bounded sets.

Integral representation

Theorem

$V : C(S^{n-1})_+ \longrightarrow \mathbb{R}$ is a continuous valuation if and only if there exist a finite Borel measure μ on S^{n-1} and a function $K : \mathbb{R}_+ \times S^{n-1} \to \mathbb{R}$ such that

(a) *K satisfies the strong Carathéodory condition:*
  - *$K(s, \cdot)$ is Borel measurable for every $s \in \mathbb{R}_+$,*
  - *$K(\cdot, t)$ is continuous for μ-almost every $t \in S^{n-1}$;*

(b) *for every $\lambda > 0$ there is $G_\lambda \in L^1(\mu)$ such that $K(s,t) \leq G_\lambda(t)$ for $s < \lambda$ and μ-almost every $t \in S^{n-1}$,*

and

$$V(f) = \int_{S^{n-1}} K(f(t), t) d\mu(t).$$

Integral representation

Theorem

Let X be an order continuous Banach lattice represented as a function space on a σ-finite measure space (Ω, Σ, μ), and let $V : X_+ \longrightarrow \mathbb{R}$ be a continuous valuation. Then, there exists a strong Carathéodory function $K(\lambda, t) : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ such that, for every $f \in X_+$,

$$V(f) = \int_\Omega K(f(t), t) d\mu(t).$$

Global measures and variation of a valuation

A valuation $V : C(S^{n-1})_+ \longrightarrow \mathbb{R}$ has bounded variation if, for every $f, g \in C(S^{n-1})_+$ with $f \leq g$, it holds that

$$|V|([f,g]) = \sup \left\{ \sum_{k=1}^{m} |V(f_k) - V(f_{k-1})| \right\} < \infty$$

where the supremum is taken over all finite sequences $(f_k)_{k=0}^m$ contained in $C(S^{n-1})_+$ such that $f = f_0 \leq f_1 \leq \cdots \leq f_m = g$.
Given a valuation $V : C(S^{n-1})_+ \longrightarrow \mathbb{R}$ with bounded variation, we can associate the variation function $|V| : C(S^{n-1})_+ \longrightarrow \mathbb{R}_+$ given by

$$|V|(f) = |V|([0,f]).$$

It is clear that $|V|$ is increasing, in the sense that $|V|(f) \leq |V|(g)$ whenever $f \leq g$.

Lemma

$|V| : C(S^{n-1})_+ \to \mathbb{R}$ is a valuation, and if V is continuous so is $|V|$.

Global measures and variation of a valuation

Theorem

Let $V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ be a radial continuous valuation. Then the following are equivalent:

1. *There exists a (signed) countably additive Borel measure ν on $\mathbb{R}^n$ such that, for every $L \in \mathcal{S}_0^n$,*

$$V(L) = \nu(L).$$

2. *V has bounded variation.*
3. *V is the difference of two increasing radial continuous valuations.*

Thank you for your attention.