Invariant subspaces for invertible operators on Banach spaces

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Contents

Chapter 1

Introduction

In functional analysis, the Invariant Subspace Problem (ISP) is the question whether any bounded linear operator $T : E \to E$, with E a Banach space, has a non-trivial closed invariant subspace, that is $F \subset E$, closed and $F \neq \{0\}, E$ such that $TF \subset F$.

In other words, to answer the question means either to find an operator T as before, which has no invariant subspaces or to prove that every bounded linear operator has at least one invariant subspace. In the case of general Banach spaces the answer is already known and it is negative, not every linear and bounded operator defined on Banach spaces has an invariant subspace and the first counterexample was found by the swedish mathematician Per Enflo in 1976, when he announced the existence of such operator in the Seminaire Maurey-Schwarz (1975-1976) [6] but it was not until 1981 when he submitted a paper for publication in Acta Mathematica which remained unrefereed for more than five years because apparently they say that the paper was quite difficult and not well writen. The paper was finally accepted with small changes in 1985 and it actually appeared in 1987 [7]. But there are several other examples of operators without invariant subspaces constructed by C. Read.

It is clear that if T has an eigenvalue then T has a non-trivial invariant subspace, namely the closure of $\mathrm{Ker}(\lambda - T)$ where λ is such eigenvalue and of course, T is not a λ-multiple of the identity. So, if the operator is defined on a finite-dimensional Banach space, T has indeed an eigenvalue. On the other hand, if the space is not separable, that means that fixed a non-zero element $x \in E$, the closure of the linear span of $\{T^n x; n \in \mathbb{N}\}$ can not be the whole space since it would contradict the fact that E is not separable, and hence T would have non-trivial invariant subspace. So, solving the ISP would be equivalent to find a non-zero vector $x \in E$ such that the closure of the span of the set $\{T^n x\}_{n\in\mathbb{N}}$ is not the whole space. So at this point the answer is:

Question: *Does every bounded linear operator defined on a complex, infinite-dimensional separable Banach space have a non-trivial invariant subspace?.*

The problem remains still open for the case of Hilbert spaces. As we mentioned, for the case of Banach spaces the answer is no. In this work our purpose will be to go over the main results obtained by several mathematicians in the case of $T : E \to E$ bounded linear, defined on E a complex Banach space, and T will have a bounded inverse.

One of the first, and most important results that asserts that T has invariant subspace is due to V. I. Lomonosov (1973) [13] who gave a proof of the following assertion:

Theorem: *If a non-scalar bounded and linear operator T on a Banach space commutes with a compact operator, then T has a non-trivial hyperinvariant subspace.*

Here, Lomonosov gave a stronger result than the ISP. An hyperinvariant subspace for T is an invariant subspace for every operator that commutes with T. Therefore, finding hyperinvariant subspaces implies finding invariant subspaces.

At this point, it seemed reasonable that Lomonosov's Theorem could lead us to the affirmative answer for the ISP. Imagine every linear bounded operator T commutes with a non-zero compact operator, then the problem is solved. Nevertheless, seven years later, in 1980, D. W. Hadvin, E. A. Nordgren, H. Radjavi and P. Rosenthal gave an exampe of an operator which does not commute with any non-zero compact operator and has invariant subspaces.

In the first chapter we will present several results from analysis, more precisely, we will use tools from functional and harmonic analysis, Banach algebras, operator theory and complex analysis. Then, in the next chapters, we will study more carefully the conditions and properties of an operator to ensure the existence of non-trivial invariant, or more generally, hyperinvariant subspaces.

The first main work about this issue is due to John Wermer. In 1952, [18] he proved, for an *invertible* operator, that under some conditions on the behavior of the norm of the iterates of an operator and its spectrum, we can assure the existence of invariant subspaces. We will give a detailed proof of this fact as it is the main part of the work and it is the origin of later works regarding the improvement of the result.

Later on, the mathematician Aharon Atzmon presented a better result for the existence of hyperinvariant subspaces which included as a particular case Wermer's Theorem which actually also gives the existence of hyperinvariant subspaces [3]. By better, we mean that the hypothesis needed to assure the existence of hyperinvariant subspaces are weaker than the ones given by Wermer. We will take a look at the main results of Atzmon's work in Chapter 3.

Then, a work by the mathematician K. Kellay [11] is discussed in Chapter 3 which establishes a slightly weaker condition than the one given by A. Atzmon and hence again, an improvement on the hypothesis of Wermer's Theorem.

The final Chapter of this work is devoted to give C. J. Read's counterexample [15], that is, we will present a definition of an invertible operator without invariant subspaces.

Chapter 2

Preliminaries

2.1 The spectrum of an operator

Let, in general, T be an operator between locally convex spaces. We define the resolvent set of T as the set $\rho(T)$ of scalars $\lambda \in \mathbb{K}$ such that $\lambda - T$ is invertible, this is, the operator $R(\lambda, T) = (\lambda - T)^{-1}$ exists and it is continuous. We call $R(\lambda, T)$ the resolvent of T. In our case, we are interested in studying operators between Banach spaces and the set of scalars will be the complex numbers. As long as no confusion arises, we will just write $R(\lambda)$ to refer to the resolvent of an operator T when the operator is fixed.

Definition 2.1.1. *We define the spectrum of an operator $\sigma(T)$ as the complementary set of $\rho(T)$, this means, the set of scalars λ such that $\lambda - T$ is not invertible.*

Next, we will state some topological properties of the spectrum.

Proposition 2.1.1. *The spectrum of an operator has the following properties:*

(a) The resolvent set is an open subset of the complex plane.

(b) The function $R : \rho(T) \longrightarrow \mathcal{L}(E)$ is analytic on $\rho(T)$, and $\|R(\lambda)\| \geqslant 1/d(\lambda)$ where $d(\lambda) = dist(\lambda, \sigma(T))$.

(c) The spectrum $\sigma(T)$ is contained in the disk $\overline{D}(0, \|T\|)$, has at least one point, and is compact.

Proof. Fix $\lambda \in \rho(T)$, then there exists a radius $r > 0$ such that the ball $B(\lambda, r) = \lambda + B(0, r) \subset \rho(T)$. We take an element $\mu \in B(0, r)$ and see that $\lambda + \mu \in \rho(T)$ for a convenient r.

Define,

$$S(\mu) = \sum_{k \geqslant 0} \mu^k (\lambda - T)^{-(k+1)} = \sum_{k \geqslant 0} \mu^k R(\lambda)^{k+1}.$$

Now we may require that $\|\mu R(\lambda)\| < 1$ then $|\mu| < 1/\|R(\mu)\|$. In this case, the series converges in operator norm and,

$$\begin{aligned} (\mu + \lambda - T) S(\mu) &= (\lambda - T) S(\mu) - \mu S(\mu) \\ &= \sum_{k \geqslant 0} \mu^k (\lambda - T)^{-k} - \sum_{k \geqslant 0} \mu^{k+1} (\lambda - T)^{-(k+1)} \\ &= I, \end{aligned}$$

since S commutes with $\lambda + \mu - T$, then it is its inverse. Hence, $B(\lambda, 1/\|R(\lambda)\|) = \lambda + B(\mu, 1/\|R(\lambda)\|) \subset \rho(T)$ which proves $\rho(T)$ is open.

Analycity comes for free, since we found a power series $S(\mu)$ of the function $R(\lambda+\mu)$ on a neigbourhood of $\mu = 0$.

For (c), similarly we set $p(\lambda) = \sum_{k\geqslant 0} \frac{T^k}{\lambda^{k+1}}$. This series converges as long as $\|T\| < |\lambda|$, and $(\lambda - T)p(\lambda) = I$, so $p(\lambda) = R(\lambda)$ for $\|T\| < |\lambda|$, and hence the λ such that $\lambda - T$ is not invertible lie inside the disk $\overline{D}(0, \|T\|)$ and since $\sigma(T)$ is closed, it is then compact.

Now since the definition of p makes sense and $\|p(\lambda)\| \to 0$ as $|\lambda| \to \infty$, by the maximum modulus principle the function must be constantly 0 which is contradictive, therefore $\sigma(T)$ has at least one element. □

The element in $\sigma(T)$ with maximum modulus is called the spectral radius of T, which always exists since $\sigma(T)$ is compact, we denote it by $r(T)$ and we see, in the next proposition, a way of computing it.

Proposition 2.1.2. *For a bounded operator we have,*

(a) (Gelfand's formula) $r(T) = \lim_{k\to\infty} \|T^k\|^{1/k} \leqslant \|T\|$

(b) The series $R(\lambda) = \sum_{k\geqslant 0} \frac{T^k}{\lambda^{k+1}}$ *converges in operator norm if* $|\lambda| > r(T)$

Proof. Part (b) is clear.

Part (a) comes from the radius of convergence of the Laurent series $R(\lambda)$, the unique point is to ensure the existence of the limit.

We have $r(T) = \limsup \|T^k\|^{1/k}$ which is the radius of convergence. Now, if $\lambda \in \sigma(T)$ then $\lambda - T = (\lambda^k - T^k)p_k(\lambda, T)$ for a polynomial p_k is not invertible and then $\lambda^k \in \sigma(T^k)$, so $|\lambda|^k \leqslant \|T^k\|$ and hence $|\lambda| \leqslant \liminf \|T^k\|^{1/k}$ which proves the existence of the limit and therefore gives the formula.

The fact that $r(T) \leqslant \|T\|$ is obvious but can also be seen by the Stolz criterion applied to the logarithm of the limit. □

Now, we will see how the spectrum of an operator can be decomposed in smaller disjoint parts according to some diferent properties of its elements.

Definition 2.1.2. *We can decompose the spectrum of an operator as follows,*

- $\sigma_p(T)$ *pointwise spectrum of* T*, the set of eigenvalues of* T*.*
- $\sigma_c(T)$ *continuous spectrum of* T*,*

$$\sigma_c(T) = \{\lambda \in \sigma(T), \lambda - T \textit{ is injective, non surjective but has dense range}\}$$

- $\sigma_r(T)$ *residual spectrum of* T*,*

$$\sigma_r(T) = \{\lambda \in \sigma(T), \lambda - T \textit{ is injective, non surjective and does not have dense range}\}$$

- $\sigma_a(T)$ *approximate point spectrum of* T*, is the set of* $\lambda \in \sigma(T)$ *such that there exists a sequence* $(x_n)_{n\geqslant 0}$ *of points in* E *with* $\|x_n\| = 1$ *for all* n*, and* $Tx_n - \lambda x_n \to 0$ *when* $n \to \infty$*. The sequence* $(x_n)_{n\geqslant 0}$ *will be called a sequence of almost eigenvectors for* λ*.*

An immediate observation is,

$$\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)$$

and obviusly,

$$\sigma_p(T) \subset \sigma_a(T)$$

The next proposition gives us some basic properties of the subsets of the spectrum defined above.

Proposition 2.1.3.

(a) $\sigma_r(T) = \sigma_p(T^t)$.

(b) $\sigma_p(T) \subset \sigma_r(T^t)$, *equality holds when* E *is reflexive.*

(c) $\sigma_a(T)$ *is a closed subset of* $\sigma(T)$ *which contains* $\sigma_p(T)$, $\sigma_c(T)$ *and the boundary of* $\sigma(T)$.

Our concern will be to find, given an operator T from a Banach space E into itself, a non-trivial closed subspace Y such that $T(Y) \subseteq Y$. So, we observe that if $\lambda \in \sigma_p(T)$, $\lambda \neq 0$ then there exists an element $x \in E$ different from 0 and indeed $Y = [x]$ will be invariant.

Also, if $\lambda \in \sigma_r(T)$ then taking $Y = \overline{(\lambda - T)(E)}$ will be a closed invariant subspace. Indeed, take $x \in E$ and set $y = (\lambda - T)x \in Y$ then $Ty = T(\lambda - T)x = (\lambda - T)(Tx)$. Therefore, $T(\overline{(\lambda - T)(E)}) \subseteq \overline{(\lambda - T)(E)}$ since T is continuous and this is not the whole space because the range is not dense by definition of $\sigma_r(T)$.

The following proposition gives us another possibility to find eigenvalues in $\sigma_a(T)$.

Proposition 2.1.4. *Let* T *be an operator defined on a reflexive Banach space* E. *Fix* $\lambda \in \sigma_a(T)$ *different from 0, and* $(x_n)_{n\geqslant 0}$ *a sequence of almost eigenvectors for* λ. *Then, either* λ *is not an eigenvalue, and* $(x_n)_{n\geqslant 0}$ *tend to 0 weakly, or* λ *is an eigenvalue, and every non-zero weak accumulation point of this sequence is an eigenvector for* λ.

Proof. Since E is reflexive, this is, we can identify E with its bidual, then there exists a subsequence x_{n_k} weakly convergent to some element $y \in E$. Thus,

$$Tx_{n_k} - \lambda x_{n_k} \to 0 \text{ in norm}$$

$$Tx_{n_k} \to Ty \text{ weakly}$$

and then,

$$\lambda x_{n_k} \to Ty \text{ weakly}$$

but,

$$\lambda x_{n_k} \to \lambda y \text{ weakly}$$

So, since λ is non zero, $Ty = \lambda y$. If we find a subsequence we find $y = 0$, then $(x_n)_{n\geqslant 0}$ converges to 0 weakly, or for some subsequence the point $y \neq 0$ then λ is an eigenvalue and y is its correspondent eigenvector. □

2.2 The analytic functional calculus

In this section we will introduce some notions on analytic functional calculus. We will introduce the Cauchy formula for operators and see what are the properties of the sprectrum inherited by this transformation. The main result of this section will be the spectral mapping theorem which gives us the answer of how spectrum varies when we apply an analytic function to our operator T.

Let $\mathcal{F}(T)$ denote the set of functions taking values on $\mathbb{C}$ analytic on some neighbourhood of $\sigma(T)$. If $f \in \mathcal{F}(T)$ and Ω an open set containing $\sigma(T)$ whose boundary $\Gamma = \partial\Omega$ consists of a finite number of rectificable Jordan curves oriented in the positive sense and we assume that f is analytic on $\overline{\Omega}$ then, we define:

$$f(T) = \frac{1}{2\pi i}\oint_\Gamma f(\lambda)R(\lambda)d\lambda$$

This definition is independent of the domain Ω, if we take another one, $R(\lambda)$ will also be analytic between boths boundaries and then by the Cauchy formula we will have $f(T) = 0$ on between. Observe also, that if Γ consists of a circle centered at the origin with a radius strictly larger than the spectral radius, then,

$$f(T) = \frac{1}{2\pi i}\sum_{n\geqslant 0} T^n \oint_\Gamma \frac{f(\lambda)}{\lambda^{n+1}}d\lambda$$

since the series converges in operator norm.

Let us now turn to some basic properties of these functions,

Proposition 2.2.1. *Let $f, g \in \mathcal{F}(T)$ and $\alpha, \beta \in \mathbb{C}$, then:*

(a) $\alpha f + \beta g \in \mathcal{F}(T)$ and $(\alpha f + \beta g)(T) = \alpha f(T) + \beta g(T)$

(b) $f \cdot g \in \mathcal{T}$ and $(f \cdot g)(T) = f(T)g(T)$

(c) if $f(\lambda) = \sum_{k=0}^\infty a_k\lambda^k$ converges in a neighbourhood of $\sigma(T)$ then $f(T) = \sum_{k=0}^\infty a_k T^k$

(d) also $f \in \mathcal{F}(T^t)$ and $f(T^t) = f(T)^t$

(e) if S commutes with T then S also commutes with $f(T)$.

(f) For T an operator on a Hilbert space, and $f \in \mathcal{F}(T)$, denote $f^\#(\lambda) = \overline{f(\overline{\lambda})}$ then $f^\# \in \mathcal{F}(T)$ and $f(T)^* = f^\#(T^*)$

We will now introduce the spectral mapping theorem which gives a satisfactory description of how spectrum is transformed by an analytic function of the operator. We will also give a short and easy proof of the theorem.

Theorem 2.2.2. *[Spectral Mapping Theorem] If $f \in \mathcal{F}(T)$,*

$$f(\sigma(T)) = \sigma(f(T)),$$

where $f(\sigma(T)) = \{f(\lambda); \lambda \in \sigma(T)\}$.

Proof. Take $\lambda \in \sigma(T)$ and define a function g by,

$$g(\zeta) := \frac{f(\lambda) - f(\zeta)}{\lambda - \zeta}$$

Then $g \in \mathcal{F}(T)$ and satisfies,

$$(\lambda - \zeta)g(\zeta) = f(\lambda) - f(\zeta)$$

applying now T,

$$(\lambda - T)g(\zeta) = f(\lambda) - f(T)$$

If $\lambda \notin \sigma(f(T))$, this is, $f(\lambda) - f(T)$ were invertible, we call Λ its inverse. Then $\Lambda g(T)$ would be the inverse of $\lambda - T$ and $\lambda \notin \sigma(T)$ which contradicts our initial assumption, so $f(\lambda) \in \sigma(f(T))$ and hence $f(\sigma(T)) \subset \sigma(f(T))$.

Conversely, assume that $\mu \notin f(\sigma(T))$. Then the function,

$$h(\zeta) := \frac{1}{f(\zeta) - \mu}$$

is in $\mathcal{F}(T)$ and satisfies $h(\zeta)(f(\zeta) - \mu) = I$. So, $h(T)(f(T) - \mu) = I$ and $\mu \notin \sigma(f(T))$. This shows that $f(\sigma(T)) \supset \sigma(f(T))$. □

2.3 Analytic Continuation

The target of this section will be to give the reader a quick overview on analytic continuation and some useful definitions and notions about this issue which will be essencial for the proof of J. Wermer's result later in the first chapter. We will define the spectrum of an element in a concrete Banach algebra in terms of analytic continuation properties. First, let us give a slight idea of what analytic continuation means.

Suppose we have an open and connected subset $\Omega \subset \mathbb{C}$, and a holomorphic function $f : \Omega \to \mathbb{C}$. It is well-known that if we have $U \subset \Omega$ open and another function $g \in \mathcal{H}(U)$ and $f = g$ on U then $f = g$ on the whole Ω and this is called the uniqueness property [1, pp, 122-123]. In other words, if f vanishes on a set $U \subset \Omega$ of positive measure then $f \equiv 0$ everywhere on Ω. So, given a function f in $U \subset \Omega$, there is a unique way to extend it analitically over the whole Ω if it were possible. For instance, define,

$$f(z) = \sum_{n=0}^{+\infty} z^n.$$

We know that this power series converges on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. So we could think that $\mathbb{D}$ is the natural domain to define this function, but, can we extend it to a larger set? The answer is yes. This is a geometric power series. Let us define the function,

$$g(z) = \frac{1}{1 - z}.$$

Observe now that this function is holomorphic on the whole plane except for the point $z = 1$ and this function agrees with f, that is, $f(z) = g(z)$ for all $z \in \mathbb{D}$. So, we can extend f to the whole plane except for the value $z = 1$. Thus, we see that the natural domain to define f is the larger set $\mathbb{C} \setminus \{1\}$.

We will consider now functions defined on disks and we will define the notion of analytic continuation on disks. Let us consider functions $f_i : D_i \to \mathbb{C}$, with $D_i \subset \mathbb{C}$ two disks for $i = 1, 2$. Then, we say that f_1 and f_2 are direct analytic continuation one of each other if $D_1 \cap D_2 \neq \emptyset$ and $f_1 = f_2$ on $D_1 \cap D_2$.

Iterating this reasoning, given the couples $\{(f_i, D_i)\}_{i=1}^{k}$ holomorphic functions each one defined on D_i and $D_i \cap D_{i+1} \neq \emptyset$ for all $i = 1, ..., k-1$. Then we say that (f_1, D_1) is a direct analytic continuation of (f_k, D_k) if, and only if, (f_i, D_i) is a direct analytic continuation of (f_{i+1}, D_{i+1}) for all $i = 1, ..., k-1$. Thus, the analytic continuation property is a reflexive, symmetric and transitive property.

2.3.1 Analytic Continuation along a Curve

Following the previous method we can as well continue a function analytically over a given curve or arc. Let $\gamma : [0, 1] \to \mathbb{C}$ be a curve piecewise smooth, and (f, D) an analytic function defined on a disk D centered at $\gamma(0)$. An analytic continuation of f along the curve γ is a family of couples (f_t, D_t) where each f_t is defined on D_t for all $t \in [0, 1]$ such that,

(i) $(f_0, D_0) = (f, D)$.

(ii) D_t a disk is centered at $\gamma(t)$ for each $t \in [0, 1]$.

(iii) For each $t \in [0, 1]$, there exists $\delta > 0$ such that for all $s \in (t - \delta, t + \delta)$, $\gamma(s) \in D_t$ and therefore $D_t \cap D_s \neq \emptyset$, and $f_t \equiv f_s$ on $D_t \cap D_s$.

We will denote

$$f_1 \#_\gamma f_2$$

meaning that f_1 and f_2 are analytic continuation one of each other along the curve γ or the other way around.

2.4 The Invariant Subspace Problem

As we mentioned in the Introduction of this work, although the Invariant Subspace Problem is somehow easy to state the answer is still open in the case of Hilbert spaces. The problem is the simple question: "Does every bounded operator T on a separable Hilbert space H over $\mathbb{C}$ have a non-trivial invariant subspace?" The answer is no, in general, for separable complex Banach spaces. The solution for Banach spaces was shown by the mathematician Per Enflo in 1976, who announced the existence of a Banach space and a bounded linear operator on it with no trivial invariant subspaces. Further manipulations were done by other mathematicians in order to sharpen his proof or give other counterexamples following Enflo's ideas. C. J. Read provided a shorter proof of the problem and also the french mathematician Bernard Beauzamy refined the techniques of Enflo and produced a counterexample. In this work, we will be more concerned on studying the concrete case of isomorphic operators, this is, continuous linear operators such that the inverse exists and it is continuous.

As we mentioned earlier, if $\lambda \in \sigma_p(T)$ different from zero, and x is the eigenvector associated to λ, the subspace $F = \overline{\{\lambda x; \lambda \in \mathbb{C}\}}$ gives us a solution. Yet another, if we find $\lambda \in \sigma_r(T)$ the set $F = \overline{Im(\lambda - T)}$ provides a non-trivial closed invariant subspace.

A more naiv way to search for invariant subspaces is to look at the behaviour of orbits of points under the operator T, if we have a point $x \in E$ such that the orbits $\mathcal{O}_T(x) = \{x, Tx, T^2x, \dots\}$ does not have dense span then $F_x = \overline{span}\{\mathcal{O}_T(x)\}$ is a closed and invariant subspace under T. Such a subspace will be called an elementary invariant subspace. If F_x is the whole space the point x is said to be cyclic, or non-cyclic otherwise.

Another concept, which is stronger, is the hyperinvariant subspace. A subspace is said to be hyperinvariant if it is invariant by all operators which commute with T. Following this definition we can consider the set,

$$G_x = \overline{span}\{\mathcal{O}_S(x)\} \text{ such that } S \text{ commutes with } T$$

Then G_x is an elementary hyperinvariant subspace. The aim, then, will be to look for non-trivial invariant or hyperinvariant subspaces. The term "non-trivial" will always be implicitly assumed.

The following result gives us an answer on whether an operator has hyperinvariant subspaces by means of a topological property of the spectrum.

Theorem 2.4.1 (F. Riesz). *If $\sigma(T) = \sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are disjoint closed subsets. Then T has hyperinvariant subspaces F_1 and F_2, and*

$$\sigma_1 = \sigma(T|_{F_1}), \quad \sigma_2 = \sigma(T|_{F_2})$$

where $T|_F$ is the restriction of T on the subspace F.

Proof. Let U_1 and U_2 be disjoint open sets containing σ_1 and σ_2 respectively. Let $f_1 = 1$ in U_1 and $f_1 = 0$ in U_2, and $f_2 = 1$ on U_2 and $f_2 = 0$ on U_1. Then $f_1, f_2 \in \mathcal{T}$ and we have,

$$f_1(T) + f_2(T) = I$$

$$f_1^2(T) = f_1(T) \text{ and } f_2^2(T) = f_2(T)$$

Therefore, f_1 and f_2 are projections, which commute with T. Their ranges F_1 and F_2 are the hyperinvariant subspaces by proposition () property (e), and indeed closed and non-trivial.

Now, on F_1, we have $T = Tf_1(T)$. So, $\sigma(Tf_1(T)) = \sigma(T)f_1(\sigma(T))$ by the spectral mapping theorem, Theorem 2.2.2, applied to the analytic function $f(\lambda) = \lambda f_1(\lambda)$. But, $\sigma(T)f_1(\sigma(T)) = \sigma_1$ and the same for F_2. □

2.5 Orbits of a Linear Operator

2.5.1 The image of a ball by a linear operator

The successive images of a ball by a linear operator will be very useful concerning the study of the iterates of a points by a linear operator. The shape and position of a ball $(T_nB)_{n\geqslant 0}$, for a fixed ball B will be of important consideration. Of course, our goal is to predict the behaviour of the itaterates of a given point x, this is $(T_n x)_{n\geqslant 0}$ and try to find a point such that the iterates have a controlled behaviour. For instance, if the closure of the span of the iterates $(T_n x)_{n\geqslant 0}$ is not the whole space, then we have found a non trivial closed invariant subspace.

The image of a ball determines uniquely the operator. Indeed, no matter how small the ball is, giving its image, by linearity and continuity this fact makes possible to recover the operator T for which the ball was applied to.

If B is any closed ball in a Banach space E, then TB is a convex set which is closed in the case of E being reflexive or if the operator is weakly compact. For example, in the case of Hilbert spaces.

The inverse image $T^{-1}B$ is a closed convex set, since T is continuous. It will be bounded if $T^{-}1$ exists as a continuous operator.

The image of the ball will also be balanced (if $|\lambda| \leqslant 1$ then $\lambda TB \subset TB$).

2.5.2 Baire property for operators

We will now give a more general version of Baire Property involving linear operators on Banach spaces.

Proposition 2.5.1. *Let T be an operator on a Banach space E, with danse range. Let $(G_n)_{n\geqslant 0}$ be a countable family of open sets. Then the intersection $\cap_{n\geqslant 0} T^n G_n$ is dense.*

Corollary 2.5.2. *If one takes $T = Id$, one gets the usual Baire Property.*

Corollary 2.5.3. *Let T be an operator with sense range. Then, there is a dense set of points x which have an infinite chain of backwards iterates, that is: for all $n > 0$, there is y_n such that $T^n y_n = x$.*

Proof. This follows from the proposition 2.5.1 taking $G_n = E$ for all $n \geqslant 0$. □

2.5.3 C. Rolewicz example of an operator with one hypercyclic point

We will now show an example of an operator with one hypercyclic point, this means, a point such that the iterates are dense in the whole space. C. Rolewicz example was provided in 1969 and the one we will show is C. Rolewicz version modified by B. Beauzamy in order to get a class of examples with several supplementary properties.

Let us consider the space of square summable sequences on $\mathbb{Z}$, that is $l^2(\mathbb{Z}) = \{(a_n)_{n\in\mathbb{Z}} \subset \mathbb{Z};\ \|(a_n)_n\|_2 := \left(\sum_n |a_n|^2\right)^{1/2} < +\infty\}$. Consider the canonical basis, e_n that consists of the sequence having all 0's and 1 on the n-th position. The support of $x \in l^2(\mathbb{Z})$ is the set $\{k; x_k \neq 0\}$.

We define weighted shift operator on $l_2(\mathbb{Z})$,

$$Te_k = w_k e_{k-1}, \text{ for } k \in \mathbb{Z},$$

the integers $w_k \in \mathbb{Z}$ are the weights. This operator consists of translating the terms of the sequence $(e_n)_n$ one place to the left and multiplying by an integer w_k. We now introduce a condition on the weights that ensures there exists an hypercyclic point.

Theorem 2.5.4. *If the weights w_k satisfy:*

$$\lim_{n\to\infty} \prod_{k=0}^{n} w_k = +\infty, \text{ and } w_k \geqslant 1 \text{ for } k \geqslant 0, \tag{2.1}$$

$$\lim_{n\to\infty} \prod_{k=0}^{-n} w_k = 0, \text{ and } 0 < w_k < 1 \text{ for } k < 0, \tag{2.2}$$

the operator T has an hypercyclic point.

Proof. We observe that the inverse of T is S:

$$Se_k = \frac{1}{w_{k+1}} e_{k+1}.$$

Computing the iterates,

$$\begin{aligned}
Te_k &= w_k e_{k-1} \\
T^2 e_k &= T(w_k e_{k-1}) = w_k T(e_{k-1}) = w_k w_{k-1} e_{k-2} \\
\cdots & \\
T^n e_k &= T^{n-1}(w_k e_{k-1}) = w_k T^{n-1}(e_{k-1}) = \cdots = w_k w_{k-1} \cdots w_{k-n} T e_{k-(n-1)} = \\
&= \prod_{j=0}^{n-1} w_{k-j} e_{k-n},
\end{aligned}$$

and similar for $S^n e_k$, from the hypothesis 2.2 and 2.1 we have that,

$$\lim_{n\to\infty} T^n e_k = 0 \text{ and } \lim_{n\to\infty} S^n e_k = 0 \text{ for all } k \in \mathbb{Z} \tag{2.3}$$

Let us now consider a dense sequence $(x^{(n)})_n \subset l^2(\mathbb{Z})$, each $x^{(n)}$ with finite support and let us call $k(n) = \max_k\{k : x_k^{(n)} \neq 0\}$ the last integer of its support.

Now, for $n \geqslant 0$, let $r(n)$ be an integer such that, if $r > r(n)$, we have for $i = 1, \cdots, n-1$,

$$\|T^r x^{(i)}\| < 1/2^n, \tag{2.4}$$

$$\|S^r x^{(n)}\| < 1/2^n. \tag{2.5}$$

These integers exist due to (2.3). Set $p(n) = \sum_{i=1}^{n} r(i)$, and consider $z = \sum_{k=1}^{\infty} S^{p(k)} x^{(k)}$. Then:

$$\begin{aligned}
T^{p(n)} z &= T^{p(n)} \left(S^{p(1)} x^{(1)} + S^{p(2)} x^{(2)} + \cdots + S^{p(k)} x^{(k)} + \cdots \right) = \\
&= T^{p(n)} S^{p(1)} x^{(1)} + T^{p(n)} S^{p(2)} x^{(2)} + \cdots + T^{p(n)} S^{p(k)} x^{(k)} + \\
&+ \cdots + T^{p(n)} S^{p(n-1)} x^{(n-1)} + x^{(n)} + \sum_{m=n+1}^{\infty} S^{p(m)-p(n)} x^{(m)} = \\
&= T^{p(n)-p(1)} x^{(1)} + \cdots + T^{p(n)-p(k)} x^{(k)} + \cdots + T^{p(n)-p(n-1)} x^{(n-1)} + \\
&+ x^{(n)} + \sum_{m=n+1}^{\infty} S^{p(m)-p(n)} x^{(m)}.
\end{aligned}$$

But for $k = 1, \cdots, n-1$ by (2.4) and (2.5) respectively we have,

$$\|T^{p(n)-p(k)}x^{(k)}\| = \|T^{r(k+1)+\cdots+r(n)}\| \leqslant 1/2^n$$

$$\|\sum_{m=n+1}^{\infty} S^{p(m)-p(n)}x^{(m)}\| \leqslant \sum_{m=n+1}^{\infty} \|S^{r(m)+\cdots+r(n+1)}x^{(m)}\| < \sum_{m=n+1}^{\infty} (1/2)^m = 1/2^n.$$

Finally,

$$\|T^{p(n)}z - x^{(n)}\| \leqslant \|\sum_{k=1}^{n-1} T^{p(n)-p(k)}x^{(k)}\| + \|\sum_{m=n+1}^{\infty} S^{p(m)-p(n)}x^{(m)}\| < (n-1)/2^n + 1/2^n = n/2^n,$$

since the sequence $x^{(n)}$ is dense, so is $T^{(n)}z$ for $n \geqslant 1$. Hence z is an hypercyclic point. □

2.6 Banach Algebras

In this section we will give some overview on banach algebras, in particular, operator algebras and some important facts that will be necessary later in the preliminaries of Wermer's result.

A complex Banach algebra A, is a complex Banach space, endowed with an operation

$$A \times A \to A$$

$$(x, y) \mapsto xy,$$

satisfying, for all $x, y, z \in A$ and $\lambda \in \mathbb{C}$,

(i) $x(y+z) = xy + xz$, and $(y+z)x = yx + zx$,

(ii) $(xy)z = x(yz)$,

(iii) $\lambda(xy) = (\lambda x)y = x(\lambda y)$,

(iv) $\|xy\| \leqslant \|x\|\|y\|$.

If also $xy = yx$ the algebra A is said to be commutitave, and if A has a unit, that is, an element $e \in A$ such that $ex = xe = x$ for all $x \in A$, the algebra A is said to be unitary. Usually the unit element is represented by 1.

A subalgebra $B \subset A$, is a vector subspace of A which is closed under the algebra operation, i,e: if $x, y \in B$ then $xy \in B$.

For instance, the space of all sequences $a = (a_n)_{n\in\mathbb{Z}} \subset \mathbb{C}$ satisfying that $\sum_{n\in\mathbb{Z}} |a_n| < +\infty$ is a Banach algebra with the following operation,

$$* : A \times A \to A$$

$$(a, b) \mapsto a * b,$$

where the element $a * b$ is another sequence, defined as,

$$(a * b)_n = \sum_{m\in\mathbb{Z}} a_{n-m}b_m.$$

This algebra is denoted by ℓ^1 and the norm associated to this space is $\|a\|_{\ell^1} = \sum_{n\in\mathbb{Z}} |a_n|$. This space, using weights, will be a useful tool for the proof of existence of invariant subspaces of isomorphisms between Banach spaces.

We say that an element $a \in A$ is invertible, if there exists $b \in A$ such that $ab = ba = 1$, usually we will denote it as a^{-1}.

We define the spectrum of an element $a \in A$ as the set of scalars such that $\lambda - a$ is not invertible, that is,

$$\sigma(a) = \{\lambda \in \mathbb{C}; \quad \lambda - a \text{ is not invertible}\}.$$

Finally, we give a very strong result on unitary Banach algebras whose elements, besides 0, are all invertible.

Theorem 2.6.1 (Gelfand-Mazur). *If A is a complex Banach algebra with unit and every non-zero element is invertible, then A is isometric to $\mathbb{C}$.*

Proof. We saw in (2.1.1) that the spectrum of an operator has at least one point, following the same argument one can show that $\sigma(a)$ has also at least one point. Now, since the spectrum of each element is non-empty, for each $a \in A$, there is a $\lambda \in \mathbb{C}$ such that $\lambda - a$ is not invertible. By hypothesis A is an algebra whose elements are all invertible, besides 0, then $\lambda - a = 0$ and hence $a = \lambda$, and it is of course an isometry since $\|a\| = |\lambda|$. □

2.6.1 Ideals and Homomorphisms of Banach algebras

A vector subspace I of a complex Banach algebra A is called an ideal, if it satisfies that for all $a \in A$ and $x \in I$, then $ax \in I$ and $xa \in I$. An ideal I is said to be a proper ideal, if $I \neq A$, and it is said to be maximal if there are no other proper ideals containing I, that is, if $I' \subset A$ is another proper ideal such that $I \subset I'$ then $I' = I$.

Remark 2.6.1. Observe that a proper ideal can not contain any invertible element, otherwise the unit element would belong to the ideal, so the ideal would coincide with the whole algebra.

Proposition 2.6.2. *Every maximal ideal is closed and every proper ideal is contained in a maximal ideal.*

Proof. Consider I a maximal ideal of A. Then, its closure $\overline{I}$ is also an ideal which contains I. Let us see that $\overline{I} \neq A$: We consider the unit ball centered at 1, that is,

$$B = \{a \in A; \quad \|a - 1\| < 1\}.$$

Then $I \cap B = \emptyset$, otherwise, if $b \in B$, then $\|1 - b\| < 1$ so b would be invertible, and $b \in I$ invertible would imply $I = A$ which is a contradiction. Now, since B is open, $\overline{I} \cap B = \emptyset$. This shows that $\overline{I}$ is proper, and since I is maximal, then $I = \overline{I}$, that is, closed.

Let now I be a proper ideal, and consider the family of all ideals containing I, ordered by inclusion, $I \subset I_1 \subset I_2 \subset \ldots$. We consider a totally ordered subfamily (I_i) of these ideals. Then the set $\cup_i I_i$ is a proper ideal since none of them intersects B as we have seem before, and $I \subset \cup_i I_i$ so, it is a majorant for I. Then, by Zorn's lemma there are maximal elements in $\cup_i I_i$. □

An homomorphism φ between two Banach algebras A and B is a mapping,

$$A \xrightarrow{\varphi} B,$$

satisfying,

(i) $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in A$,

(ii) $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x \in A$ and $\lambda \in \mathbb{C}$,

(iii) $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in A$.

Remark 2.6.2. The multiplicativity property provides directly the continuity as we shall see in the coming proposition.

Proposition 2.6.3. *Every homomorphism φ from A into $\mathbb{C}$ is continuous, and if $\varphi \neq 0$ then $\varphi(1) = 1$ and $\|\varphi\| = 1$. Moreover $\varphi(a) \in \sigma(a)$ for all $a \in A$.*

Proof. Since φ is a linear mapping, it is enough to check continuity at zero. We have that the kernel of φ is an ideal. Indeed, given $x, y \in \mathrm{Ker}\varphi$, then $\varphi(xy) = \varphi(x)\varphi(y) = 0$, so $xy \in \mathrm{Ker}\varphi$.

Moreover, it is a hyperplane by the Hahn Banach theorem, so $\mathrm{Ker}\varphi$ is a maximal ideal and by Proposition 2.6.2 it is closed and hence φ is continuous.

Now take an element $x \in A$ such that $\|x\| = 1$, then,

$$|\varphi(x)|^n = |\varphi(x^n)|,$$

by multiplicativity and since φ is continuous,

$$|\varphi(x^n)| \leqslant \|\varphi\|\|x^n\| \leqslant \|\varphi\|,$$

so $|\varphi(x)| \leqslant \|\varphi\|^{1/n}$ and letting n tend to infinity, we get $|\varphi(x)| \leqslant 1$, so $\|\varphi\| = \max_{\|x\|=1} |\varphi(x)| \leqslant 1$.

On the other hand if $\varphi \neq 0$ there is at least an element $x \in A$ such that $\varphi(x) \neq 0$ and by linearity $\varphi(x) = \varphi(1x) = \varphi(1)\varphi(x)$, so $\varphi(1) = 1$ and hence $\|\varphi\| = 1$.

Finally, for each $a \in A$, $\varphi(a) - a \in \mathrm{Ker}\varphi$, which is a proper ideal, so it contains no invertible elements, so $\varphi(a) \in \sigma(a)$. □

Such a non-zero homomorphism from A into $\mathbb{C}$ is called a character and it is denoted by χ. We denote by $\chi(A)$ the characters of A, that is, all continuous homomorphisms from A into $\mathbb{C}$. Of course, $\chi(A) \subset A^*$, where A^* is the dual. We endow A^* with the topology $\sigma(A^*, A)$, this means, the topology of pointwise convergence on elements of A and next we will see that $\chi(A)$ is a w^*-compact subset of A^*.

Proposition 2.6.4. *The set of characters $\chi(A) \subset A^*$ is w^*-compact.*

Proof. It suffices to show that it is closed under the weak star topology since w^*-compactness follows directly by the Banach-Alaoglu Theorem. □

2.6.2 The Gelfand's Transform

We have introduced $\chi(A)$ a subset of the dual A^*, consisting of the continuous homomorphisms from A into $\mathbb{C}$. We endow A^* with the topology defined by means of the seminorms $\|\cdot\|_a$, $a \in A$ which act on A^* as $\|u\|_a = |u(a)|$ for all $u \in A^*$. We say that $u_n \to u$ with respect to this topology if, and only if, $u_n(a) \to u(a)$ for all $a \in A$, that is, the pointwise convergence on elements of A.

Now, we define the Gelfand transform $\widehat{a}$ of an element $a \in A$ as,

$$\mathcal{G} : A \to \mathcal{C}(\chi(A))$$

$$a \mapsto \hat{a}$$

The correpondence $a \mapsto \widehat{a}$ is called the Gelfand's morphism, and we denote the set $\widehat{A} = \{\widehat{a}; a \in A\}$.

This correspondence is continuous. Indeed,

$$\begin{aligned} \|\widehat{a}\|_{\mathcal{C}(\chi(A))} &= \sup_{\varphi \in \chi(A)} |\widehat{a}(\varphi)| \\ &= \sup_{\varphi \in \chi(A)} |\varphi(a)| \\ &\leqslant r(a) \text{ [since by Proposition 2.6.3 } \varphi(x) \in \sigma(a) \text{ for all } a \in A] \\ &\leqslant \|a\|. \end{aligned}$$

So, the Gelfand's morphism is continuous. Next, we show the relation between the Gelfand transform of an element in a Banach algebra A and its spectrum. But, before that, we need an important lemma.

Lemma 2.6.5. *An ideal is maximal if, and only if it is the kernel of some character.*

Proof. We have already shown that the kernel of a character is a maximal ideal, in the proof of Proposition 2.6.3 as a consequence of Proposition 2.6.2.

Conversely, consider M a maximal ideal of a Banach algebra A. Then, A/M is a Banach algebra with no invertible elements: Indeed, let $\pi : A \to A/M$. If A/M had invertible elements, this would mean that A/M contains some proper ideal, in such a case, $\pi^{-1}(I)$ would be a proper ideal, containing M with $\pi^{-1}(I) \neq M$ which is a contradiction since M is maximal.

Thus, every element in A/M has no invertible elements. So, since A/M is a Banach algebra without invertible elements besides 0, by Gelfand-Mazur's Theorem 2.6.1 A/M is isometric to $\mathbb{C}$. Then, if $\psi : A/M \to \mathbb{C}$ is such isometry, we have that $\varphi = \pi \circ \psi$ is a character on A and $\mathrm{Ker}\varphi = M$. □

Theorem 2.6.6. *Let A be a commutative Banach algebra. Then, for every $a \in A$,*

$$\sigma(a) = \{\varphi(a);\ \varphi \in \chi(A)\},$$

and therefore,

$$r(a) = \sup_{\varphi \in \chi(A)} \widehat{a}(\varphi) = \|\widehat{a}\|_{\mathcal{C}(\chi(A))}.$$

Proof. If we have $\lambda \in \sigma(a)$, then $a' = \lambda - a$ is not invertible. Let $I = \{ba';\ b \in A\}$. This is an ideal because A is commutative and it is proper since a' is not invertible, and is contained in some maximal ideal M which, by Lemma 2.6.5, is the kernel of some character. So, $I \subset M$ which is the kernel of some $\varphi \in \chi(A)$. So, $\varphi(a') = \varphi(\lambda - a) = \lambda - \varphi(a) = 0$, hence $\varphi(a) = \lambda$. □

Chapter 3

Wermer's Theorem on invariant subspaces

In this section we show an important result of operators having a non-trivial invariant subspace. The main goal in this section is to prove John Wermer's result which, provides a sufficient condition based on the behaviour of the norms of the iterates of the operator and the spectrum of the operator in order to ensure the existence of non-trivial invariant subspaces.

We will start by giving the hypothesis we will require for the norms of the iterates but in a more general setting. We will present an algebra of sequences with weights and such weights will satisfy the condition we show next. Then, we will show several lemmas and results concerning this algebra of sequences which will apply to the case of Wermer's Theorem.

Definition 3.0.1. *Let $\{\rho_n\}_{n\in\mathbb{Z}}$ be a sequence of positive numbers. We say that the sequence $\{\rho_n\}_{n\in\mathbb{Z}}$ satisfies condition* (1) *if it is majorized by a sequence $\{d_n\}_{n\in\mathbb{Z}}$, that is, $\rho_n \leqslant d_n$ for all n, where $\{d_n\}_{n\in\mathbb{Z}}$ satisfies the following properties:*

(i) It is symmetric, $d_{-n} = d_n$ and $d_n \geqslant 1$ for all n.

(ii) $\sum_{n=0}^{\infty} \frac{\log d_n}{1+n^2} < +\infty$.

(iii) d_n is non-decreasing as $|n| \to +\infty$.

(iv) $\frac{\log d_n}{n}$ decreases as $|n| \to +\infty$.

Remark 3.0.3. We must remark at this point, that if, for instance, $\rho_n = \|T^n\|$ for some bounded invertible operator T, satisfies condition (1), then the spectrum of T is necessarily contained in the unit circle $\mathbb{T}$: Indeed, suppose $\lambda \in \sigma(T)$ and $|\lambda| > 1$, this implies that $r(T) = \lim_n \|T^n\|^{1/n} > 1$ since $\|T^n\| \geqslant 1$ for all n, as we shall see later, so there exsists a number $R > 1$ such that $\|T^n\| > R^n$ for n large enough. Thus, since there is a sequence d_n such that $d_n \geqslant \rho_n$, we have that $\log d_n > n \log R$ for n big enough which would imply that $\sum_{n=0}^{+\infty} \frac{\log d_n}{1+n^2} = +\infty$.

On the other hand, if $\lambda \in \sigma(T)$ and $|\lambda| < 1$, then $1/\lambda \in \sigma(T^1)$ which is again contradictive with condition (1).

Finally, $\|T^n\| \geqslant 1$ for if $\|T^n\| < 1$ for $n > 0$, we have that for all $m > 0$, $\|T^{nm}\| \leqslant \|T^n\|^m$ and therefore the spectral radius $r(T) = \lim_m \|T^{nm}\|^{\frac{1}{nm}} < 1$ which is contradictory by the previous argument. Similar for negative n.

The aim of this chapter is then, to prove the following result.

Theorem 3.0.7. *Let T be a bounded operator on E a Banach space with bounded inverse. Let $\rho_n = \|T^n\|$ satisfy condition* (1)*. If $\sigma(T)$ does not reduce to a single point, then T has a non-trivial invariant subspace in E.*

First of all, we will show that if we assume that Wermer's result is true the assumption on the spectrum having more than a point can be dropped if the norm of the iterates of the operator have polynomial growth. Let us first give the general statement and then, check that if $\|T^n\| = O(|n|^k)$ then we do not need to assume anything on the spectrum of T.

The objective is to prove the forementioned result. This will require some lemmas and concepts that we will discuss later. First, let us see that, if Theorem 3.0.7 is true and the sequence $\|T^n\|$ is of polynomial order, then the requirement on the spectrum having more than one points can be dropped.

Theorem 3.0.8. *Let E be a Banach space and $T : E \to E$ a bounded linear operator with bounded inverse. Then if $\|T^n\| = O(|n|^k)$ for some finite k. Then T has a non-trivial invariant subspace.*

Before we prove Theorem 3.0.8 we need the following result due to Gelfand and Hille [10, pp, 128-129].

Proposition 3.0.9. *Let T be a bounded linear operator acting on a Banach space. If the sequence $\|T^n\| = O(|n|^k)$ and the spectrum of T consists of a single point λ_0, i.e: $\sigma(T) = \{\lambda_0\}$. Then, $(T - \lambda_0 I)^{k+1} = 0$*

We are now in a position to prove Theorem 3.0.8:

Proof. Since $\|T^n\| = O(|n|^k)$, condition (1) is achieved: Indeed, since $\|T^n\| \leqslant C|n|^k$, we take $d_n = C|n|^k$ if $n \neq 0$ and $d_0 = 1$, and $C > 1$ so $d_n = C|n|^k > 1$. It is also symmetric and the series,

$$\sum_{n=1}^{+\infty} \frac{\log(C|n|^k)}{1+n^2} = \sum_{n=1}^{+\infty} \frac{\log C}{1+n^2} + k \sum_{n=1}^{+\infty} \frac{\log|n|}{1+n^2} < +\infty.$$

Moreover, d_n is clearly non-decreasing, and $\log d_{n+1}/(n+1) \leqslant \log d_n/n$ and hence decreasing.

So, by Theorem 3.0.7 we just need to check the case of the spectrum of T having a single point. Now, by the previous proposition, if $\sigma(T) = \{\lambda_0\}$ then $(T - \lambda_0 I)^{k+1} = 0$. So, T has a non trivial invariant subspace for either $\mathrm{Ker}(T - \lambda_0 I)$, or the closure of the image of $T - \lambda_0 I$.

Indeed, define $Y := \mathrm{Ker}(T - \lambda_0 I) \subseteq E$. If $Y = E$,

$$Tx = \lambda_0 x \quad \forall x \in E,$$

that is, T is a multiple of the identity and then it has invariant subspaces.

On the other hand, if $Y = \{0\}$, this means that $(T - \lambda_0 I)x = 0$ if, and only if $x = 0$. We have by Gelfand-Hille that,

$$(T - \lambda_0 I)^{k+1} = 0 \quad \text{for all } x \in E$$

So,

$$(T - \lambda_0 I)(T - \lambda_0 I)^k x = 0 \quad \text{for all } x \in E,$$

and by the previous reasoning, this is equivalent to say that $(T-\lambda_0 I)^k x = 0$ for all $x \in E$. Now, recursively, we arrive to,

$$(T-\lambda_0 I)^2 x = (T-\lambda_0 I)(T-\lambda_0 I)x = 0 \text{ for all } x \in E$$

So, $T(T-\lambda_0 I) = \lambda_0(T-\lambda_0)$ and hence, $\overline{\operatorname{Im}(T-\lambda_0 I)}$ is invariant under T. □

We introduce now a space of sequences that will be useful in the proof of Wermer's result. We consider $\{\rho_n\}_{n\in\mathbb{Z}}$ a sequence of real numbers equal or greater than 1 such that $\rho_{n+m} \leqslant \rho_n\rho_m$ for any $n,m \in \mathbb{Z}$, and such that for every $R > 1$ $\rho_n = o(R^{|n|})$ as $|n| \to +\infty$, that means $\lim_n \frac{\rho_n}{R^{|n|}} = 0$.

Let $L = \ell^1(\rho_n)$ be the space of all complex valued sequences $\{a_n\}_{n\in\mathbb{Z}}$ such that $\sum_n |a_n|\rho_n < +\infty$. The space L endowed with the norm $\|a\| = \sum_n |a_n|\rho_n$ and the convolution operation is a Banach algebra.

Now, the dual L^* of the space L, is the set of bounded linear forms $x^* = (x_n)_{n\in\mathbb{Z}}$ acting as $x^* : L \to \mathbb{C}$, $x^*(a) = \sum_n a_n x_n$. So,

$$\begin{aligned} \|x^*\|_{L^*} &= \sup_{\|a\|=1} |\sum_n a_n x_n| \leqslant \sup_{\|a\|=1} \sum_n |a_n||x_n| = \sup_{\|a\|=1} \sum_n |a_n|\rho_n \frac{|x_n|}{\rho_n} \\ &\leqslant \sup_n |x_n|/\rho_n < +\infty \end{aligned}$$

Actually the above inequalities are equalities: Fixed $x^* \in L^*$, choose $a \in L$, the sequence $a = (a_n)_n$ such that the n-th element of the sequence is $a_n = \frac{\overline{x_n}}{|x_n|^2}\frac{1}{\rho_n}$ and is 0 everywhere else. It is clear that $\sum_n a_n x_n > 0$ for all n and $a \in B_L$ since $\|a\| = \sum_m |a_m|\rho_m = \frac{1}{\rho_n}\rho_n = 1$. Remember that $\rho_n \neq 0$ since it is a sequence of positive numbers. Finally for this a,

$$\sum_n a_n x_n = \sup_n |x_n|/\rho_n.$$

So, L^* is the space of sequences $x^* = (x_n)_n$ such that $\sup_n |x_n|/\rho_n < +\infty$, endowed with the norm $\|x^*\|_{L^*} = \sup_n |x_n|/\rho_n$.

Let us now prove an important property of the homomorphisms on this space L:

Proposition 3.0.10. *Every complex valued homomorphism α acting on L must be of the form:*

$$\begin{aligned} \alpha : \ & L \to \mathbb{C} \\ & a \mapsto \alpha(a) = \sum_n a_n \lambda^n \end{aligned}$$

with $\lambda \in \mathbb{T}$, varying according to α.

Proof. We have to impose the three properties an homomorphism $\alpha : L \to \mathbb{C}$ between two algebras must satisfy:

(i) $\alpha(ka) = k\alpha(a)$ for all $a \in L$ and $k \in \mathbb{C}$,

(ii) $\alpha(a+b) = \alpha(a) + \alpha(b)$ for all $a, b \in L$,

(iii) $\alpha(a * b) = \alpha(a)\alpha(b)$ for all $a, b \in L$.

Let $(e_n^k)_{n\in\mathbb{Z}}$ be the sequence consisting of all zeros and one in the k-th position. Observe that given $a \in L$ we can write:

$$a = \sum_{k=-\infty}^{+\infty} a_k e^k \quad \text{with } a_k \in \mathbb{C} \quad \text{for all} \quad n \in \mathbb{Z},$$

So it is enough to define α by means of the images of the sequences $e^k = (e_n^k)_n$. Then, let us call

$$\lambda = \alpha(e^1),$$

where $\lambda \in \mathbb{C}$ must have modulus 1, since it is an homomorphism of a Banach algebra into $\mathbb{C}$ and by Proposition 2.6.3 α is continuous and $\|\alpha\| = 1$.

Now, we observe that,

$$e_1 * e_1 = \sum_{m\in\mathbb{Z}} e_{n-m}^1 e_m^1 = e^2,$$

then using property (iii) we have,

$$\alpha(e^2) = \alpha(e^1 * e^1) = \alpha(e^1)\alpha(e^1) = \lambda^2.$$

Recursively,

$$\alpha(e^k) = \alpha(e^{k-1} * e^{k-1}) = \alpha(e^{k-1})\alpha(e^{k-1}) = \lambda^k.$$

Finally, given a sequence $a \in L$, $a = (a_n)_n$ using the expression mentioned above, we have,

$$\alpha(a) = \alpha\left(\sum_{k=-\infty}^{+\infty} a_k e^k\right) = \sum_{k=-\infty}^{+\infty} a_k \alpha(e^k) = \sum_{k=-\infty}^{+\infty} a_k \lambda^k,$$

by using both property (i) and (ii). □

Now, given a sequence $x^* \in L^*$, $x^* = (x_n)_n$ it is natural to define the following functions:

$$f_{x^*}^+(\lambda) = \sum_{n=1}^{+\infty} \frac{x_n}{\lambda^n}, \qquad \text{and} \qquad f_{x^*}^-(\lambda) = -\sum_{n=0}^{+\infty} \lambda^n x_{-n}, \tag{3.1}$$

where $f_{x^*}^+$ is defined and analytic for $\lambda \in \mathbb{C}$, $|\lambda| > 1$ and $f_{x^*}^-$ is defined and analytic for $|\lambda| < 1$. Let us check this; since $|x_n|/\rho_n \leqslant \sup_n |x_n|/\rho_n = \|x^*\|_{L^*}$, then $|x_n| \leqslant \|x^*\|_{L^*}\rho_n$ hence $|x_n| = o(R^{|n|})$ for each $R > 1$. This means that $\lim_n \frac{|x_n|}{R^{|n|}} = 0$. So,

$$f_{x^*}^+(\lambda) = \sum_{n=1}^{+\infty} \frac{x_n}{R^n}\left(\frac{R}{\lambda}\right)^n,$$

where $\frac{x_n}{R^n} \to 0$ so $f_{x^*}^+$ will be analytic for all $|\lambda| > |R| > 1$. Similarly,

$$f_{x^*}^-(\lambda) = -\sum_{n=1}^{+\infty} \frac{x_{-n}}{R^n}(\lambda R)^n,$$

where $\frac{x_{-n}}{R^n} \to 0$ so $f_{x^*}^-$ will be analytic for $|\lambda| < \frac{1}{R} < 1$.

Here, we define the concept of spectrum of an element $x^* \in L^*$ using the previous functions $f_{x^*}^+$ and $f_{x^*}^-$.

Definition 3.0.2. *Let $x^* \in L^*$. We define the spectrum of x^*, spec(x^*), as the set of points $\lambda \in \mathbb{T}$ such that the functions $f^+_{x^*}$ and $f^-_{x^*}$ are not analytic continuation one of each other along any arc containing λ. That is,*

$$\textit{spec}(x^*) = \{\lambda \in \mathbb{T};\ f^+_{x^*} \not\#_\gamma f^-_{x^*} \text{ for all arc } \gamma \text{ with } \lambda \in \gamma\}.$$

Next step we want to show that given a closed subset of the unit circle, the set of sequences of L^* whose spectrum lies inside this closed set is a weak*-closed set. In order to do so, we need to invoke a necessary result on subharmonic functions due to Nils Sjöberg. [16, pp, 309-319]

Theorem 3.0.11 (N. Sjöberg)**.** *Let $h(t)$ be an even function, decreasing as $t \to +\infty$, unbounded at the origin and such that, $\int_0^x \log^+ h(t)dt < +\infty$ for all $x > 0$.*

Given a rectangle: $R = \{(x, y) \in \mathbb{R}^2 :\ -a \leqslant x \leqslant a,\ -b \leqslant y \leqslant b\}$, and b' with $0 < b' < b$, then, there exists a constant $M = M(a, b, b', h)$ such that for any subharmonic function $u(x, y)$ on R and $u(x, y) \leqslant e^{h(|x|)}$ we have $u(x, y) \leqslant M$ on R.

Lemma 3.0.12. *Let $\{\rho_n\}_{n\in\mathbb{Z}}$ satisfy condition (1), and $\Lambda \subset \mathbb{T}$ closed. Then the set,*

$$E_\Lambda = \{x^* \in L^* :\quad \textit{spec}(x^*) \subset \Lambda\}$$

is a weak-closed subspace of L^*.*

Proof. Given $x^*, y^* \in E_\Lambda$ and $\alpha \in \mathbb{C}$ a scalar, it is straightforward that $x^* + y^* \in E_\Lambda$ and $\alpha x^* \in E_\Lambda$. It remains to show that given $(x^*_k)_{k\in\mathbb{N}} \subset L^*$ such that $x^*_k \xrightarrow{w*} x^*$ with $x^*_k \in E_\Lambda$ for all k then $x^* \in E_\Lambda$.

Recall that with the weak* topology on L^*, $x^*_k \xrightarrow{w*} x^*$ as $k \to +\infty$ if, and only if $x^*_k(a) \xrightarrow{k\to+\infty} x^*(a)$ for all $a \in L$.

To see that $x^* \in E_\Lambda$, we take an element $\lambda_0 \notin \Lambda$ and see that $\lambda_0 \notin \operatorname{spec}(x^*)$. Now, since Λ is closed, hence $\mathbb{T} \setminus \Lambda$ is open, there is a closed disk $D = D(\lambda_0)$ centered at λ_0 and disjoint from Λ. Now, for each k, since $x^*_k \in E_\Lambda$, its spectrum lies outside of the disk D. So there exists for each k a function f_k which is analytic in the interior of D and by uniqueness $f_k(\lambda) = f^+_{x^*_k}(\lambda)$ for $|\lambda| > 1$ and $f_k(\lambda) = f^-_{x^*_k}(\lambda)$ for $|\lambda| < 1$.

We shall now see that the functions f_k are uniformly bounded in D. To do so, we will use Theorem 3.0.11. So, we will find a function h satisfying the same conditions as in the theorem, such that for all $\lambda \in D$,

$$|f_k(\lambda)| \leqslant e^{h(\log|\lambda|)} \quad \text{for all} \quad |\lambda| > 1,$$

and,

$$|f_k(\lambda)| \leqslant e^{h(\log(1/|\lambda|))} \quad \text{for all} \quad |\lambda| < 1.$$

Consider a sequence d_n satisfying condition (1). We have that $\frac{\log d_n}{n}$ decreases to 0: Indeed, otherwise there would exist a strictly positive constant such that $\frac{\log d_n}{n} \geqslant C$ for all n since $\frac{\log n}{n}$ decreases and $d_n \geqslant 1$ for all n by hypothesis, but then,

$$\sum_{n\geqslant 0} \frac{\log d_n}{1+n^2} = \sum_{n\geqslant 0} \frac{\log d_n}{n}\frac{n}{1+n^2} \geqslant C \sum_{n\geqslant 0} \frac{1}{n+1} = +\infty,$$

which is a contradiction.

Therefore, for a given number $t \in \mathbb{R}$, $t > 0$, there exists a unique first positive integer, let us call it $N = N(t)$, such that, $\log d_n/n \leqslant t/2$ for all $n > N$. Thus,

$$\begin{aligned}\sum_{n=0}^{+\infty} e^{-tn} d_n &= \sum_{n=0}^{+\infty} \exp\left(n\left(-t+\frac{\log d_n}{n}\right)\right) \leqslant \sum_{n=0}^{N-1} \exp\left(n\left(-t+\frac{\log d_n}{n}\right)\right) \\ &+ \sum_{n=N}^{+\infty} \exp\left(n\left(-t+\frac{t}{2}\right)\right) \leqslant \sum_{n=0}^{N-1} d_n + \sum_{n=N}^{+\infty} e^{-(t/2)n} \leqslant N d_N + O\left(\frac{1}{t}\right) \\ &\leqslant N(t)e^{(t/2)N(t)} + O\left(\frac{1}{t}\right).\end{aligned}$$

Here, we used that $d_n \leqslant d_N$ if $n < N$, since d_n is non-decreasing.

We define now, $h(t) = \log \sum_{n=0}^{+\infty} e^{-tn} d_n$ for $t > 0$ and define $h(-t) = h(t)$ for $t < 0$, since h must be even. Then,

$$h(t) = \log \sum_{n=0}^{+\infty} e^{-tn} d_n \leqslant \log\left(N(t)e^{(t/2)N(t)} + O(1/t)\right) \leqslant \log\left(2N(t)e^{(t/2)N(t)} O(1/t)\right)$$

for t between 0 and 1. We use the fact that $a + b \leqslant 2ab$ for $a, b > 1$. So,

$$h(t) \leqslant \log N(t) + \frac{t}{2} N(t) + \log KO(1/t) \leqslant \frac{N(t)}{2} + \frac{N(t)}{2} + \log(1/t) + C = N(t) + \log\frac{1}{t} + C.$$

since $\log x < \frac{x}{2}$ and $0 < t < 1$.

Hence for all $0 < x < 1$,

$$\int_0^x \log^+ h(t)dt < +\infty,$$

as long as,

$$\int_0^x \log N(t)dt < +\infty \quad \text{for all} \quad 0 < x < +\infty,$$

which is true as we shall see later.

Next, we define $k(y)$ piecewise as follows,

$$k(y) = \begin{cases} \frac{\log d_n}{n} = k(n) & \text{if} \quad y = n \in \mathbb{Z}, \quad n > 0, \\ k(y) & \text{linear otherwise.} \end{cases}$$

That is, the linear continuous extension of $\log d_n/n$ to $(0, +\infty)$. So, k decreases to zero as $y \to +\infty$ since $\log d_n/n$ is decreasing and $d_n > 1$ for all n and,

$$\int_1^{+\infty} \frac{k(y)}{y} dy \leqslant \sum_{n=1}^{+\infty} \frac{\log d_n}{n^2} < +\infty.$$

Now, for each $t > 0$, let $z(t)$ be the unique value such that $k(e^{z(t)}) = t$, since k is a dreceasing function and hence injective and therefore piecewise invertible. For each $x > 0$ the integral,

$$\int_0^x z(t)dt,$$

is bounded for all x between 0 and 1. Indeed, if we express this integral in terms of the distribution function of z, which is:

$$\mu_z(s) = m\{\sigma \in [0,1];\ z(\sigma) > s\},$$

where m denotes the Lebesgue measure. Then,

$$\int_0^x z(t)dt = \int_0^{+\infty} \mu_z(s)ds.$$

But,

$$\begin{aligned}\mu_z(s) &= m\{\sigma \in [0,1];\ z(\sigma) > s\} = m\{\sigma \in [0,1];\ e^{z(\sigma)} > e^s\} \\ &= m\{\sigma \in [0,1];\ k\left(e^{z(\sigma)}\right) < k(e^s)\} = m\{\sigma \in [0,1];\ \sigma < k(e^s)\} = k(e^s).\end{aligned}$$

Here, we used the fact that the exponential is a strictly increasing function and k is strictly decreasing, so inequalities change. So,

$$\int_0^x z(t)dt = \int_0^{+\infty} \mu_z(s)ds = \int_0^{+\infty} k(e^s)ds.$$

Now, by a change of variables $y = e^s$,

$$\int_0^{+\infty} k(e^s)ds = \int_0^{+\infty} \frac{k(y)}{y}dy < +\infty.$$

Now, since $N(t) \leqslant e^{z(t/2)} + 1$ we have that,

$$\int_0^x \log N(t)dt < +\infty \quad \text{for all } 0 < x < 1,$$

so

$$\int_0^x \log^+ h(t)dt < +\infty,$$

and $h(t)$ decreases as $|t|$ increases since $\sum_n e^{-tn} d_n$ decreases. Therefore, the function $h(t)$ fulfills the conditions in Theorem 3.0.11.

Since x_k^* converges weakly, $\|x_k^*\|$ is bounded uniformly in k, and without loss of generality we can assume the bound to be one, otherwise we can divide the sequences by the norm.

Choose λ inside α and $|\lambda| > 1$. Then,

$$|f_k(\lambda)| = |f_k^+(\lambda)| \leqslant \sum_{n=1}^{+\infty} \frac{|x_n^k|}{|\lambda|^n} \leqslant \|x_k^*\| \sum_{n=1}^{+\infty} \frac{1}{|\lambda|^n}\rho_n \leqslant \sum_{n=1}^{+\infty} \frac{1}{|\lambda|^n} d_n,$$

since $\rho_n \leqslant d_n$, and for λ inside α and $|\lambda| < 1$,

$$|f_k(\lambda)| = |f_k^-(\lambda)| \leqslant \sum_{n=1}^{+\infty} |x_{-n}^k||\lambda|^n \leqslant \|x_k^*\| \sum_{n=1}^{+\infty} |\lambda|^n \rho_n \leqslant \sum_{n=1}^{+\infty} |\lambda|^n d_n,$$

Therefore for all k, we have for λ inside α and $|\lambda| > 1$,

$$|f_k^+(\lambda)| \leqslant e^{h(\log|\lambda|)},$$

and for λ inside α and $|\lambda| < 1$,

$$|f_k^-(\lambda)| \leqslant e^{h(\log(1/|\lambda|))}.$$

Now, since these inequalities do not depend on the chosen point $\lambda_0 \in D$, we can suppose that $\lambda_0 = 1$ otherwise we can shift λ_0 to 1 via a rotation. Let us now define,

$$g_k(w) = f_k\left(\frac{1-w}{1+w}\right),$$

that is, we apply to λ a conformal transformation that sends the right half-plane to the interior of the unit circle.

If we write $w = t + is$, we can find a rectangle $R = \{t + is;\ |t| \leqslant a, |s| \leqslant b\}$ such that λ lies inside α if $w \in R$. So if $t > 0$ then $|\lambda| < 1$ and if $t < 0$ then $|\lambda| > 1$. So, on the one hand, for $t > 0$, hence for $|\lambda| < 1$,

$$|g_k(w)| = \left|f_k\left(\frac{1-w}{1+w}\right)\right| \leqslant \exp\left(h\left(\log\frac{1+w}{1-w}\right)\right), \tag{3.2}$$

and on the other hand for $t < 0$, and hence $|\lambda| > 1$,

$$|g_k(w)| = \left|f_k\left(\frac{1-w}{1+w}\right)\right| \leqslant \exp\left(h\left(\log\frac{1-w}{1+w}\right)\right).$$

But since for $t > 0$ we can find $c = c(a, b)$ a constant depending on a and b such that $\left|\frac{1+w}{1-w}\right| > e^{t/c}$, and hence, $\log\left|\frac{1+w}{1-w}\right| > t/c$, then, since h is decreasing,

$$h\left(\log\left|\frac{1-w}{1+w}\right|\right) < h(t/c) \quad \text{for all} \quad w \in R,$$

so using the upperbound (3.2) we have,

$$|g_k(w)| \leqslant e^{h(t/c)} \quad \text{if} \quad t > 0.$$

Using the same reasoning for $w \in R$ with $t < 0$, we have that,

$$|g_k(w)| \leqslant e^{h(|t|/c)} \quad \text{if} \quad t < 0.$$

Now, by a change of variables $y = \frac{t}{c}$ we have,

$$\int_0^x \log h(t/c)dt = c\int_0^{x/c} \log h(y)dy.$$

So we found an h satisfying the properties of the Theorem 3.0.11 and such that,

$$|g_k(t + is)| \leqslant e^{h(|t|/c)} \quad \text{for all } |t| < a, \quad |b| < s \quad \text{and} \quad k \in \mathbb{Z},$$

hence, applying the Theorem 3.0.11 we have that the subharmonic functions $|g_k(w)|$ are uniformly bounded on the rectangle R, in particular in some neighbourhood of 0, and hence the functions f_k are uniformly bounded in some neighbourhood of λ_0. Therefore, there exists some subsequence f_{k_j} converging uniformly in some neighbourhood of λ_0 to a function $f(\lambda)$ which is analytic there since the functions f_{k_j} are also analytic there and the convergence is uniform.

Since x_k^* converges weakly to x^*, so $x_n^k \to x_n$ for all n and,

$$|x_n^k| \leqslant \|x^k\|\rho_n \leqslant \rho_n.$$

Therefore, for $|\lambda| > 1$ the functions $f_n(\lambda)$ converge to $f_{x^*}^+(\lambda)$ and for $|\lambda| < 1$ the functions $f_n(\lambda)$ convergence to $f_{x^*}^-(\lambda)$. So the functions f_k converge to a function f which is $f = f^+$ for $|\lambda| > 1$ and $f = f^-$ for $|\lambda| < 1$ and analytic near λ_0.

So λ_0 does not lie in the spectrum of x^*, that is, $x^* \notin E_\Lambda$ as we wanted to see. □

Definition 3.0.3. *Let $a, b \in L$ and $x^* \in L^*$, $x^* = (x_n)_n$. We define the elements:*

$$(a * b)_n = \sum_{m\in\mathbb{Z}} a_{n-m} b_m \qquad \textit{for all } n \in \mathbb{Z}$$

$$(a * x)_n = \sum_{m\in\mathbb{Z}} a_{m-n} x_m \qquad \textit{for all } n \in \mathbb{Z}$$

and finally we define the Fourier series of a as:

$$\widehat{a}(\lambda) = \sum_{n\in\mathbb{Z}} a_n \lambda^n, \qquad |\lambda| = 1.$$

Remark 3.0.4. Observe that with this definition of convolution in L, the algebra L is commutative. Indeed, take $a, b \in L$,

$$(a * b)_n = \sum_{m\in\mathbb{Z}} a_{n-m} b_m = \sum_{i\in\mathbb{Z}} a_i b_{n-i} = (b * a)_n.$$

Lemma 3.0.13. *Let any closed set $\Lambda \subset \mathbb{T}$, and a point $z \notin \Lambda$. Then, there is a sequence $a \in L$ such that*

$$\widehat{a}(\lambda) = 0 \qquad \textit{for all } \lambda \in \Lambda$$

$$\widehat{a}(z) \neq 0.$$

Proof. Consider the sequence $x_w^* = w^n$ with $|w| = 1$. This sequence exists in L^* since $\sup_n \frac{|x_w^*|}{\rho_n} = \sup_n 1/\rho_n < +\infty$ since $\rho_n > 1$ for all $n \in \mathbb{Z}$.

Now, we consider the functions $f_{x_w^*}^+$ and $f_{x_w^*}^-$;

$$f_{x_w^*}^+(\lambda) = \sum_{n=1}^{+\infty} \lambda^n w^{-n} = \frac{-\lambda}{\lambda - w} \quad \text{if } |\lambda| < 1,$$

$$f_{x_w^*}^-(\lambda) = -\sum_{n=0}^{+\infty} \frac{w^n}{\lambda^n} = \frac{-\lambda}{\lambda - w} \quad \text{if } |\lambda| > 1.$$

We observe that the spectrum of x_w^* has the unique point $w \in \mathbb{T}$ since $f_{x_w^*}^+ \not\#_\gamma f_{x_w^*}^-$ for all arc γ containing w. Define,

$$E_\Lambda = \{x^* \in L^* : \quad \text{spec}(x^*) \subset \Lambda\}.$$

We have that $x_z^* \notin E_\Lambda$.

On the other hand, we saw in Lemma 3.0.12 that E_Λ is a weak*-closed subspace of L^*, so by Hahn-Banach there exists an element $a \in L$ such that $x^*(a) = 0$ for $x^* \in E_\Lambda$ and $x_z^*(a) \neq 0$. But each sequence of the type x_w^* with $w \in \Lambda \subset \mathbb{T}$ belongs to E_Λ as we have just seen. Therefore,

$$\widehat{a}(w) = \sum_{n\in\mathbb{Z}} a_n w^n = x_w^*(a) = 0 \qquad \text{if } w \in \Lambda$$

$$\widehat{a}(z) = \sum_{n\in\mathbb{Z}} a_n z^n = x_z^*(a) \neq 0$$

as we wanted to see. □

Lemma 3.0.14. *Given $y^* \in L^*$. If $a * y^* = 0$ for $a \in L$, then $\widehat{a}(\lambda) = 0$ for all $\lambda \in$ spec(y^*).*

Proof. Let $\Gamma = \{\lambda \in \mathbb{T};\ a * y^* = 0 \Rightarrow \widehat{a}(\lambda) = 0; a \in L\}$. We want to prove that spec$(y^*) \subset \Gamma$. First, we claim that the set Γ is closed: Observe that we can write Γ as the intersection of the kernels of the Fourier series of a for those $a \in L$ such that $a * y^* = 0$, that is,

$$\Gamma = \bigcap_{a*y^*=0} \text{Ker}(\widehat{a}),$$

and $\widehat{a} : L \to \mathbb{C}$ is a continuous homomorphism of L into $\mathbb{C}$ since Γ is the intersection of closed sets, it is closed.

Consider now any larger closed subset $\Gamma' \subset \mathbb{T}$ such that Γ is contained in the interior (in the topology of $\mathbb{T}$ induced by the Euclidean topology of $\mathbb{C}$) of Γ', then we consider,

$$E_{\Gamma'} = \{x^* \in L^* : \ \text{spec}(x^*) \subset \Gamma'\}.$$

Now, we claim that $y^* \in E_{\Gamma'}$, i.e. spec$(y^*) \subset \Gamma'$. Indeed, let us suppose the contrary. That is, $y^* \notin E_{\Gamma'}$, then spec$(y^*) \not\subset \Gamma'$ so there exists at least $\lambda_0 \in$ spec(y^*) such that $\lambda_0 \notin \Gamma'$ and this means that $\lambda_0 \notin \Gamma$ so there is $a \in L$ such that $a * y^* = 0$ implies $\widehat{a}(\lambda_0) \neq 0$.

On the other hand since $E_{\Gamma'}$ is weak*-closed by Lemma 3.0.12, and by Hahn-Banach Theorem there exists a sequence $a \in L$ with $y^*(a) = \sum_n a_n y_n \neq 0$ and $x^*(a) = \sum_n a_n x_n = 0$ for all $x^* = (x_n)_n \subset E_{\Gamma'}$. In particular, for $x_\lambda^* = \lambda^n$ we have $x_\lambda(a) = \sum_n a_n \lambda^n = 0$ for all $\lambda \in \Gamma'$ and observe that $x_\lambda^*(a) = \widehat{a}(\lambda)$. So, $\widehat{a}(\lambda) = 0$ for all $\lambda \in \Gamma'$ and we will see next that this implies $a * y^* = 0$ which is a contradiction for the $\lambda_0 \notin \Gamma'$.

Consider the set,

$$I = \{b \in L;\ b * (a * y^*) = 0, a \in L\}.$$

This set is a closed proper ideal of the Banach algebra L. Indeed, given $a' \in L$ and $b' \in I$ then $(a' * b') * (a * y^*) = a' * (b' * a * y^*) = 0$ so, $a' * b' \in I$. It is proper since $a * y^* \neq 0$ and it is closed since I is a maximal ideal, and hence by Lemma 2.6.5 is the kernel of some character, and hence it is closed. Then, since L is an algebra with unit, the ideal I has a zero, that is, there exists $\mu \in \mathbb{T}$ such that $b \in I$ implies that $\widehat{b}(\mu) = 0$.

Moreover, the point μ is in Γ, otherwise we could find an element $a_0 \in L$ such that $a_0 * y^* = 0$ and $\widehat{a_0}(\mu) \neq 0$ by the definition of Γ, and of course $a_0 \in I$, since $a_0 * (a * y^*) = a * (a_0 * y^*) = 0$, so μ could not have been a zero of the ideal I.

Now, we use Lemma 3.0.13 and find an element $a_1 \in L$ such that $\widehat{a_1}(\mu) \neq 0$ and $\widehat{a_1}(\lambda) = 0$ for all λ outside the interior of Γ which is closed. So, we have two elements a and a_1 whose Fourier series vanish on complementay sets, hence $a_1 * a = 0$, since $\widehat{a * a_1} = \widehat{a}\widehat{a_1} = 0$ and this implies $a_1 \in I$, because $a_1 * a = 0$ implies $a_1 * a * y^* = 0$ and therefore $\widehat{a_1}(\mu) = 0$ which is a contradiction. In conclusion, $a * y^* = 0$.

In particular, the zero coordinate of the sequence $(a*y^*)_n = \sum_m a_{m-n} y_m$, which is, $(a*y^*)_0 = \sum_m a_m y_m = y^*(a)$ vanishes and this is a contradiction with the choice of $a \in L$ at the begining of this proof. So, $y^* \in E_{\Gamma'}$, i,e: the spectrum of y^* lies inside Γ'. Since all this reasoning holds for any Γ' whose interior contains Γ, this also holds for Γ and hence the lemma is proved. $\square$

Let us now concentrate on the more general setting of E a Banach space, E^* its dual. Let $T : E \to E$ be a bounded linear operator with bounded inverse $T^{-1} : E \to E$. Let $S = T^*$ be its adjoint operator acting on the dual E^*. As usual, we denote by $\sigma(T)$ its spectrum and R_λ the resolvent operator, $R_\lambda T = (\lambda I - T)^{-1}$ for those $\lambda \notin \sigma(T)$.

We know that, given any continuous linear form $u \in E^*$, and an element $\varphi \in E$, the functions:

$$\begin{aligned} f:\ \ \sigma(T)^c &\to \mathbb{C} \\ \lambda &\mapsto f(\lambda) = u(R_\lambda(\varphi)), \end{aligned}$$

are analytic in any given connected component as we saw in Proposition 2.1.1. Since in our situation $\sigma(T) \subset \mathbb{T}$, we define,

$$f^+_{u,\varphi}(\lambda) = u(R_\lambda(\varphi)) \quad \text{for all} \quad |\lambda| > 1, \qquad f^-_{u,\varphi}(\lambda) = u(R_\lambda(\varphi)) \quad \text{for all} \quad |\lambda| < 1. \tag{3.3}$$

The functions $f^+_{u,\varphi}$ and $f^-_{u,\varphi}$ are analytic. First, recall that, given an operator T whose spectrum lies inside the unit circle, we have the respective resolvent functions defined inside and outside the unit disk,

$$\begin{aligned} R_\lambda &= (\lambda - T)^{-1} = \sum_{n \geqslant 1} \frac{T^{n-1}}{\lambda^n} \quad \text{for all } |\lambda| > 1, \\ R_\lambda &= (\lambda - T)^{-1} = -\sum_{n \geqslant 0} T^{-n-1} \lambda^n \quad \text{for all } |\lambda| < 1, \end{aligned}$$

which converge in operator norm in the given domains by Proposition 2.1.2, item (b). Now, since $u \in E^*$ is a continuous linear form,

$$\begin{aligned} u(R_\lambda \varphi) &= \sum_{n \geqslant 1} \frac{u(T^{n-1}\varphi)}{\lambda^n} \quad \text{for all } |\lambda| > 1, \\ u(R_\lambda \varphi) &= -\sum_{n \geqslant 0} u(T^{-n-1}\varphi)\lambda^n \quad \text{for all } |\lambda| < 1, \end{aligned}$$

but since the sequence $\rho_n = \|T^n\| \leqslant \|T\|^n = o(R^{|n|})$ for $R > 1$, we have that $\lim_n \frac{\|T^n\|}{R^{|n|}} = 0$, this means the sequence $T^n/R^{|n|}$ converges to zero in norm. But convergence in norm implies weak convergence, so $u(T^n\varphi/R^{|n|}) \xrightarrow{n\to+\infty} 0$ for all $\varphi \in E$. So,

$$u(R_\lambda\varphi) = \sum_{n\geqslant 1} u\left(\frac{T^{n-1}\varphi}{R^n}\right)\left(\frac{R}{\lambda}\right)^n \quad \text{for all } |\lambda| > 1,$$
$$u(R_\lambda\varphi) = -\sum_{n\geqslant 0} u\left(\frac{T^{-n-1}\varphi}{R^n}\right)(\lambda R)^n \quad \text{for all } |\lambda| < 1,$$

by the same reason we used in 3.1, these functions $f^+_{u,\varphi}$ and $f^-_{u,\varphi}$ are analytic for $|\lambda| > 1$ and $|\lambda| < 1$ respectively. Notice here the similarity with the functions $f^+_{x^*}$ and $f^-_{x^*}$ for a given $x^* \in L^*$.

The fact that, for a given $u \in E^*$, these functions are analytic inside and outside the unit disk, allow us to define the set of values $\lambda \in \mathbb{T}$ such that the functions $f^+_{u,\varphi}$ and $f^-_{u,\varphi}$ do not continue each other analytically over any arc which contains λ as we did for the functions $f^+_{x^*}$ and $f^-_{x^*}$. Later on, we will see the relationship between both definitions.

Definition 3.0.4. *Let $\sigma(T) \subset \mathbb{T}$. Given $\varphi \in E$, we define;*

$$\Lambda_\varphi = \{p \in \mathbb{T} : \quad f^+_{u,\varphi} \not\#_\gamma f^-_{u,\varphi} \textit{ with } p \in \gamma \textit{ for some } u \in E^*\}$$

Using the notion of analytic continuation along a curve that we saw in the Preliminaries, and given a closed subset $\Gamma \subset \mathbb{T}$, we are in a position to define the set of functions $C_\Lambda \subset E$ which is invariant under T. First, we will find an invariant set for T then we will see later in the proof of Wermer's Theorem, that such set is non-trivial. This set will be defined as the set of $\varphi \in E$ such that the functions $f^+_{u,\varphi}$ and $f^-_{u,\varphi}$ do not continue each other analytically over any arc containing $\lambda \in \Gamma \subset \mathbb{T}$.

Theorem 3.0.15. *Let the sequence $\rho_n = \|T^n\|$ satisfy condition (1). Let $\Lambda \subset \mathbb{T}$ closed. Then the set,*

$$C_\Lambda = \{\varphi \in E : \quad \Lambda_\varphi \subseteq \Lambda\}$$

is a closed subspace of E and is invariant under T and T^{-1}.

Proof. First, since $\rho_{n+m} = \|T^{n+m}\| \leqslant \|T^n\|\|T^m\| = \rho_n\rho_m$, and satisfies condition (1) by hypothesis, in particular, $\rho_n = o(R^{|n|})$ for $R > 1$, we can define the Banach algebra $L = \ell^1(\rho_n)$ with $\rho_n = \|T^n\|$. Recall that $\sigma(T) \subset \mathbb{T}$.

We fix now an element $\varphi \in E$ and an element in the dual $u \in E^*$. We consider the sequence $(x_n)_{n\in\mathbb{Z}} = (u(T^n\varphi))_{n\in\mathbb{Z}}$. Since u and T are continuous, we have,

$$|u(T^n\varphi)| \leqslant \|u\|_{E^*}\|T^n\varphi\|_E \leqslant \|u\|_{E^*}\|\varphi\|_E\rho_n.$$

Therefore,

$$\sup_n |u(T^n\varphi)/\rho_n| \leqslant \|u\|_{E^*}\|\varphi\|_E < +\infty,$$

and hence $x^* = (u(T^n\varphi))_{n\in\mathbb{Z}} \in L^*$.

Suppose we have an element $\lambda_0 \in \Lambda_{T\varphi}$, we want to see that then $\lambda_0 \in \operatorname{spec}(x^*)$ with $x^* = \{u(T^n\varphi)\}_n \subset E^*$.

Using the definition of spectrum we gave for a sequence $x^* \in L^*$ and comparing it to the definition of the set Λ_ψ for $\psi \in E$, then if,

$$\lambda_0 \in \{\lambda \in \mathbb{T} : \ f^+_{u,T\varphi} \not\#_\gamma f^-_{u,T\varphi} \ \text{ for some } \ u \in E^*\},$$

where the functions $f^+_{u,T\varphi}$ and $f^-_{u,T\varphi}$ are defined as before and depend on $T\varphi$, that is,

$$f^+_{u,T\varphi}(\lambda) = u(R_\lambda(T\varphi)) = \sum_{n\geqslant 1} \frac{u\,(T^n\varphi)}{\lambda^n} \quad \text{for all} \quad |\lambda| > 1,$$
$$f^-_{u,T\varphi}(\lambda) = u(R_\lambda(T\varphi)) = -\sum_{n\geqslant 0} u\left(T^{-n}\varphi\right)\lambda^n \quad \text{for all} \quad |\lambda| < 1.$$

Then, there exists some $u \in E^*$ such that $f^+_{u,T\varphi}$ and $f^-_{u,T\varphi}$ do not continue each other analytically over any arc γ containing λ_0, and this is precisely the definition of spectrum of the sequence $x^* = (u(T^n\varphi))_n$, observe the form of $f^+_{u,T\varphi}$ and $f^-_{u,T\varphi}$, so $\lambda_0 \in \operatorname{spec}(u(T^n\varphi))$. So, by the same reason if $\lambda_0 \notin \Lambda_{T\varphi}$ then $\lambda_0 \notin \operatorname{spec}(u(T^n\varphi))$, that is, the functions $f^+_{u,T\varphi}$ and $f^-_{u,T\varphi}$ continue each other analytically over any arc containing λ_0 for all $u \in E^*$.

Let us see that $\Lambda_\varphi = \Lambda_{T\varphi}$. First, since T commutes with the resolvent operator R_λ we have that,

$$u(R_\lambda(T\varphi)) = u(TR_\lambda\varphi) = (u \circ T^{-1})(R_\lambda\varphi).$$

So, if we denote $S = T^*$ the adjoint operator we have,

$$u(R_\lambda(T\varphi)) = \langle u, R_\lambda(T\varphi)\rangle = \langle u, TR_\lambda(\varphi)\rangle = \langle Su, R_\lambda(\varphi)\rangle.$$

Now if $\lambda_0 \notin \Lambda_\varphi$, then $\langle Su, R_\lambda(\varphi)\rangle$ is analytic in a neighbourhood of λ_0 and so is $u(R_\lambda(T\varphi))$ by the identity above. Hence, $\lambda_0 \notin \Lambda_{T\varphi}$.

On the other hand let us suppose $\lambda_0 \notin \Lambda_{T\varphi}$ and pick any $u \in E^*$. As before, we have

$$u(R_\lambda\varphi) = \langle u, R_\lambda\varphi\rangle = \langle u, R_\lambda T^{-1}(T\varphi)\rangle = \langle S^{-1}u, R_\lambda(T\varphi)\rangle,$$

and since $\langle S^{-1}u, R_\lambda(T\varphi)\rangle$ is analytic in a neighbourhood of λ_0 then $\lambda_0 \notin \Lambda_\varphi$ by the identity.

Altogether we have shown that $\Lambda_\varphi = \Lambda_{T\varphi}$, and then $\Lambda_{T^{-1}\varphi} = \Lambda_\varphi$ as well. So this shows that if $\varphi \in C_\Lambda$, then $T\varphi, T^{-1}\varphi \in C_\Lambda$ as well. So the set C_Λ is invariant under T.

Moreover, it is a subspace since given $\varphi_1, \varphi_2 \in C_\Lambda$ and $z, w \in \mathbb{C}$ scalars. We have, by the linearity of the resolvent,

$$\langle u, R_\lambda(z\varphi_1 + w\varphi_2)\rangle = z\langle u, R_\lambda\varphi_1\rangle + w\langle u, R_\lambda\varphi_2\rangle,$$

so, $z\varphi_1 + w\varphi_2 \in C_\Lambda$.

So it remains to show that it is closed. If we have a sequence $(\varphi_m)_m \subset C_\Lambda$ converging to an element $\varphi \in E$, then we must show that $\varphi \in C_\Lambda$. So we must show that $\Lambda_\varphi \subset \Lambda$, that is, $\lambda \notin \Lambda$ implies that $\lambda \notin \Lambda_\varphi$.

We take $\lambda_0 \notin \Lambda$ and let us suppose that $\lambda_0 \in \Lambda_\varphi$. Then, as we have seen, $\lambda_0 \in \Lambda_{T\varphi}$, so we have an element $u \in E^*$ such that $\lambda_0 \in \operatorname{spec}(x^*)$ with $x^* = \{u(T^n\varphi\}_n \in L^*$.

On the other hand, since $\varphi_m \in C_\Lambda$ for all m, then if an element λ is not in Λ then $\lambda \notin \Lambda_{\varphi_m}$ for all m, so $\varphi_m \notin \Lambda_{T\varphi_m}$, and hence $\lambda \notin \mathrm{spec}(x_m^*)$ with $x_m^* = \{u(T^n\varphi_m)\}$. So, again since $\Lambda_{\varphi_m} \subset \Lambda$ for all m, the spectrum of x_m^* is included in Λ for all m. On the other hand, x_m^* converges weakly to $x^* \in L^*$ since $\varphi_m \to \varphi$ and hence $u(T^n\varphi_m) \to u(T^n\varphi)$ for each n and this limit is in L^* since $|u(T^n\varphi)| \lesssim \rho_n$ for each n.

Therefore, by Lemma 3.0.12, the spectrum of x^* is also in Λ and hence $\lambda_0 \notin \mathrm{spec}(x^*)$ which is a contradiction with the fact that $\lambda_0 \in \Lambda_{T\varphi}$. Hence, $\lambda_0 \notin \Lambda_{T\varphi} = \Lambda_\varphi$, so $\Lambda_\varphi \subset \Lambda$ and finally, C_Λ is closed as we wanted to prove.

□

Lemma 3.0.16. *Let T be a linear bounded operator acting on E, with bounded inverse such that $\sigma(T) \subset \mathbb{T}$. Let $\lambda_0 \in \sigma(T)$ and α any circle centered at λ_0. Then, there exists some $\varphi \in E$ and some μ in the interior of α such that $\mu \in \Lambda_\varphi$.*

Proof. Let us argue by contradiction. Suppose that for every $\varphi \in E$, and for every μ in the interior of α then $\mu \notin \Lambda_\varphi$, that is, the function f_u^+ defined on the unit disk is analytic in the interior of α.

Consider another circle β centered at λ_0 with smaller radius than α. Then, since $u(R_\lambda\varphi)$ is continuous on β, and β is a compact set, then $u(R_\lambda\varphi)$ is bounded on β, and since u is a continuous linear form.

Now, R_λ as a function in λ is continuous on β except maybe at the two points where β intersects with the unit circle: Indeed, consider for all $\lambda \in \beta$ the family of operators

$$\begin{array}{cccc} T_\lambda : E \oplus E^* & \longrightarrow & \mathbb{C} \\ (\varphi, u) & \mapsto & \langle u, R_\lambda\varphi\rangle \end{array},$$

then for all $(\varphi, u) \in E \oplus E^*$,

$$\sum_{\lambda\in\beta} |T_\lambda(\varphi, u)| = \sup \lambda \in \beta\langle u, R_\lambda\varphi\rangle < +\infty,$$

so by the Banach-Steinhaus Theorem this implies that

$$\sup_{\lambda\in\beta} \|T_\lambda\| < +\infty.$$

But,

$$\|T_\lambda\| = \sup_{\substack{\|\varphi\|\leqslant 1 \\ \|u\|\leqslant 1}} |\langle u, R_\lambda\varphi\rangle| = \|R_\lambda\|_{L(E)},$$

and hence,

$$\sup_{\lambda\in\beta} \|R_\lambda\| < +\infty.$$

Now, by the Cauchy integral formula,

$$R = \frac{1}{2\pi i}\oint_\beta \frac{R_\xi}{\xi - \lambda_0} d\xi,$$

and upperbounding, for a fixed $\varphi \in E$,

$$\|R\varphi\|_E \leqslant C\frac{1}{2\pi}\|\varphi\|_E \oint_\beta \frac{d\xi}{|\lambda_0 - \xi|} \leqslant \frac{\|\varphi\|_E}{2\pi}\text{long}(\beta) \sup_{\xi\in\beta} \frac{1}{|\xi - \lambda_0|} \leqslant C'\|\varphi\|_E.$$

So, the integral defines a bounded operator $R : E \to E$. Then, multiplying by $(\lambda_0 I - T)$,

$$(\lambda_0 I - T)R = \frac{1}{2\pi i}\oint_\beta \frac{(\xi I - T)R_\xi}{\xi - \lambda_0}d\xi - \frac{1}{2\pi i}\oint_\beta \frac{(\lambda_0 - \xi)R_\xi}{\lambda_0 - \xi}d\xi.$$

But,

$$\oint_\beta u(R_\xi\varphi)d\xi = 0,$$

for all $u \in E^*$ and $\varphi \in E$ by analyticity.

So,

$$\frac{1}{2\pi i}\oint_\beta R_\xi d\xi = 0,$$

and hence $(\lambda_0 I - T)R = I$.

In a similar way, we have that $R(\lambda_0 I - T) = I$. Therefore, $R = (\lambda_0 I - T)^{-1}$ and hence $\mu \notin \sigma(T)$ which is a contradiction. □

Theorem 3.0.17 (J. Wermer)**.** *Let T be a linear and bounded operator acting on a Banach space E, with bounded inverse. Let $\rho_n = \|T\|^n$ satisfy condition* (1)*. If $\sigma(T)$ does not reduce to a single point, then T has a non-trivial invariant subspace in E.*

Proof. The aim is to show that T has a non-trivial invariant subspace, in fact, by Theorem 3.0.15 we know that C_Λ is an invariant subspace of T, so it remains to show that it is not trivial.

Since $\sigma(T)$ does not reduce to a single point, we can find two distinct points $\lambda_1, \lambda_2 \in \sigma(T)$, $\lambda_1 \neq \lambda_2$. Let D_1, D_2 be two disks centered at λ_1 and λ_2 respectively, with radius small enough so that they are disjoint.

By Lemma, 3.0.16 we can find elements $\varphi_1, \varphi_2 \in E$ and points $p_1 \in D_1$ and $p_2 \in D_2$ such that $p_1 \in \Lambda_{\varphi_1}$ and $p_2 \in \Lambda_{\varphi_2}$. Since $D_1 \cap D_2 = \emptyset$, then $p_1 \neq p_2$.

Now, let Λ be a closed arc in $\mathbb{T}$ containing p_1 but not p_2, and a smaller open arc $\Lambda_1 \subset \Lambda$ containing p_1.

Then, by Lemma 3.0.13 there exists a sequence $a \in L$ such that $\widehat{a}(p_1) = 1$ and $\widehat{a}(\lambda) = 0$ for all $\lambda \notin \Lambda_1$.

Define now, $\psi = \sum_{n\in\mathbb{Z}} a_n T^n \varphi_1 \in E$. We should now see that $\psi \neq 0$ and that $\Lambda_\psi \subset \Lambda$ in order to ensure that $C_\Lambda := \{\varphi \in E : \quad \Lambda_\varphi \subseteq \Lambda\} \neq \{0\}$.

Let us suppose that $\psi = 0$. We pick $b \in L$ such that $\widehat{b}(p_1) = 1$. Now, for any $u \in E^*$, we compute the convolution between two elements of L, since $\{u(T^n\psi)\}_n \subset \mathbb{C}$,

$$\begin{aligned}(b * \{u(T^n\psi)\})_m &= \sum_{k\in\mathbb{Z}} b_{m-k} u(T^k\psi) = \sum_{k\in\mathbb{Z}} b_{m-k} u\left(T^k \sum_{l\in\mathbb{Z}} a_l T^l \varphi_1\right) \\ &= \sum_{k\in\mathbb{Z}} b_{m-k} u\left(\sum_{l\in\mathbb{Z}} a_l T^{k+l}\varphi_1\right) = \sum_{k\in\mathbb{Z}} b_{m-k} \sum_{l\in\mathbb{Z}} a_l u\left(T^{k+l}\varphi_1\right)\end{aligned}$$

Now, we set $i = k + l$,

$$\begin{aligned}\sum_{i,k\in\mathbb{Z}} b_{m-k}a_{i-k}u\left(T^i\varphi_1\right) &= \sum_{k\in\mathbb{Z}} b_{m-k}\sum_{i\in\mathbb{Z}} a_{i-k}u\left(T^i\varphi_1\right) = \sum_{k\in\mathbb{Z}} b_{m-k}(a*\{(T^n\varphi_1\})_k \\ &= (b*(a*\{u(T^n\varphi_1)\}))_m = (a*b*\{T^n\varphi_1\})_m.\end{aligned}$$

Since $\psi = 0$, then $T^n\psi = 0$ and $u(T^n\psi) = 0$ thereafter $b * u(T^n\psi) = 0$. So,

$$(b*\{u(T^n\psi)\})_m = ((a*b)*\{u(T^n\varphi_1)\})_m = 0$$

Now we can apply Lemma 3.0.14 and then the Fourier series of $a * b$, i.e:

$$\widehat{(a*b)}(\lambda) = \sum_{n\in\mathbb{Z}}(a*b)_n\lambda^n \quad \text{for} \quad |\lambda| = 1,$$

vanishes for all λ in the spectrum of T, in particular at p_1. So,

$$\widehat{(a*b)}(p_1) = \widehat{a}(p_1)\widehat{b}(p_1) = 0.$$

But by the choice of a and b in L, we have $\widehat{a}(p_1) = \widehat{b}(p_1) = 1$ which is a contradiction, hence $\psi \neq 0$.

In order to show that $\Lambda_\psi \subseteq \Lambda$, we will show that given $q \notin \Lambda$ then $q \notin \Lambda_\psi$.

Suppose the contrary, $q \in \Lambda_\psi$. Then, there exists an element in the dual $u_1 \in E^*$ with q in the spectrum of the sequence $x_n = (u_1(T^n\psi))_n$. Indeed, recall that,

$$\Lambda_\varphi = \{p \in \mathbb{T} : \ f^+_{u,\varphi} \not\#_\gamma f^-_{u,\varphi} \text{ with } p \in \gamma \text{ for some } u \in E^*\}$$

$$\text{spec}(x^*) = \{\lambda \in \mathbb{T} : \quad f^+_{x^*} \not\#_\gamma f^-_{x^*} \quad \forall\gamma \supset \{\lambda\}\}$$

and observe that,

$$f^+_{x^*}(\lambda) = \sum_{n\geqslant 1}\frac{u_1(T^n\psi)}{\lambda^n} = u_1\left(\sum_{n\geqslant 0}\frac{T^n\psi}{\lambda^n}\right) = u_1\left(R_\lambda(\psi)\right) \quad \text{for } |\lambda| > 1$$

$$f^-_{x^*}(\lambda) = -\sum_{n\geqslant 0}\lambda^n u_1(T^{-n}\psi) = u_1\left(R_\lambda(\psi)\right) \quad \text{for } |\lambda| < 1$$

So if $q \in \mathbb{T}$ is such that $u_1(R_\lambda)$ does not continue analytically over any arc that contains q, then f^+ and f^- do not continue each other analytically over any arc containing q either, and hence $q \in \text{spec}(u_1(T^n\varphi_1))$. Therefore, if $b \in L$ and $(b * x^*)_n = 0$ where $x^* = (x_n)_n = \{u_1(T^n\varphi_1)\}_n$, by Lemma 3.0.14 we have that $\widehat{b}(q) = 0$, but we can also choose $b \in L$ with $\widehat{b}$ vanishing on Λ by Lemma 3.0.13 applied to the set Λ, and $\widehat{b}(q) \neq 0$. Then, $b * a = 0$, since the Fourier series of a and b vanish on complementary sets.

As before, since $(a * b) * u_1(T^n\varphi_1) = b * u_1(T^n\psi)$, we have $b * u_1(T^n\psi) = 0$, and hence, as before, by lemma 3.0.14 since $q \in \text{spec}(u(T^n\varphi_1))$, $\widehat{b}(q) = 0$, which is a contradiction. So, $q \notin \Lambda_\psi$.

Moreover, we have just seen that there is a function $\psi \neq 0$, $\psi \in C_\varphi$. On the other hand, since $p_2 \in \Lambda_{\varphi_2}$, then $\Lambda_{\varphi_2} \not\subseteq \Lambda$. So $\varphi_2 \notin C_\Lambda$ which shows $C_\Lambda \neq E$. □

We have then seen sufficient conditions for the norms of the iterates of a linear bounded operator with bounded inverse in order to assure that T has a non-trivial invariant subspace in E. Then, at this level questions arise, such as, are these conditions necessary? or the optimal ones? For instance, we required the sequence $\{\rho_n\}_{n\in\mathbb{Z}}$ of positive numbers to be majorized by a sequence d_n satisfying, among other conditions, that

$$\sum_{n\geqslant 0} \frac{\log d_n}{1+n^2} < +\infty,$$

and this property was only used in the proof of the Lemma 3.0.12 when we wanted to prove that, given a closed subset $\Lambda \subset \mathbb{T}$, the set of elements in L^* such that their spectrum lies inside Λ was weak*-closed. And this lemma was precisely important in the proofs of the Lemmas 3.0.13 and 3.0.14 in order to invoke the Hahn-Banach theorem which we used to separate points in Λ and in the Theorem 3.0.15 to show that the set C_Λ was closed. This tool was essencial in the proof of Wermer's result and that is why we require the fact that the spectrum has to have at least two points, so that we can ensure that the set C_Λ is not the whole space.

So, the next question arise. Is it necessary to require the assumption on the spectrum having at least two points? All these questions are still to be answered.

Chapter 4

Further results related to Wermer's Theorem

As we mentioned in the introduction, the property of a subspace being hyperinvariant is a strong one. Recall that $L(E)$ is the set of linear bounded operator acting from E to E. We say that $Y \subset E$, closed and $Y \neq \{0\}, E$, is invariant under $T \in L(E)$ if for all $x \in Y$ then $Tx \in Y$. If we then denote by $C(T, E)$ the set of operators which commute with T, we say that Y is an hyperinvariant subspace for T if it is invariant for every operator $S \in C(T, E)$.

We define now a Beurling sequence, which is a sequence satisfying some of the properties of condition (1) mentioned in chapter 3. So, we ask the sequence to satisfy less conditions.

Definition 4.0.5 (Beurling sequence)**.** *A sequence of real numbers $\{\rho_n\}_{n\in\mathbb{Z}}$ such that $\rho_0 = 1$ and $\rho_n \geqslant 1$, for all $n \in \mathbb{Z}$ is said to be a Beurling sequence if the following conditions are fulfilled:*

(i) $\rho_{n+m} \leqslant \rho_n \rho_m$, *for all* $n, m \in \mathbb{Z}$.

(ii)

$$\sum_{n\in\mathbb{Z}} \frac{\log \rho_n}{1+n^2} < +\infty.$$

If we replace $\mathbb{Z}$ by $\mathbb{N}$ the sequence is said to be, one sided Beurling sequence $\{\rho_n\}_{n\in\mathbb{N}}$.

The main result of Aharon Atzmon work is the following Theorem, which gives sufficient conditions to have hyperinvariant subspaces.

Theorem 4.0.18 (A. Atzmon)**.** *Let E be a complex Banach space and T an operator in $L(E)$. Assume there exist $\{x_n\}_{n\in\mathbb{Z}} \subset E$ and $\{x_n^*\}_{n\in\mathbb{Z}} \subset E^*$ with $x_0, x_0^* \neq 0$ such that*

$$Tx_n = x_{n+1} \quad \textit{and} \quad T^*x_n^* = x_{n+1}^* \quad \textit{for all} \quad n \in \mathbb{Z}, \tag{4.1}$$

where T^* denotes the adjoint operator of T acting on the dual E^*.

Then, the following sufficient conditions imply that either T is a multiple of the identity operator or T has a non-trivial hyperinvariant subspace:

(i) The sequence $\{\|x_n^*\|\}_{n\in\mathbb{Z}}$ is dominated by a Beurling sequence and $\|x_n\| = O(|n|^k)$ as $|n| \to +\infty$ for some integer $k \geqslant 0$.

(ii) *The sequence* $\{\|x_n\|\}_{n\in\mathbb{Z}}$ *is dominated by a Beurling sequence and* $\|x_n^*\| = O(|n|^k)$ *as* $|n| \to +\infty$ *for some integer* $k \geqslant 0$.

(iii) *Both* $\{\|x_n\|\}_{n\in\mathbb{Z}}$ *and* $\{\|x_n^*\|\}_{n\in\mathbb{Z}}$ *are dominated by Beurling sequences and the functions defined on* $\mathbb{C} \setminus \mathbb{T}$ *by:*

$$f_x(\lambda) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} x_{-n}\lambda^{n-1}, & |z| < 1, \\ \displaystyle -\sum_{n=-\infty}^{0} x_{-n}\lambda^{n-1}, & |z| > 1 \end{cases} \tag{4.2}$$

and,

$$f_x^*(\lambda) = \begin{cases} \displaystyle\sum_{n=1}^{+\infty} x_{-n}^*\lambda^{n-1}, & |z| < 1, \\ \displaystyle -\sum_{n=-\infty}^{0} x_{-n}^*\lambda^{n-1}, & |z| > 1 \end{cases} \tag{4.3}$$

hav more than one singularity on $\mathbb{T}$.

(iv) *The elements* x_0 *and* y_0 *are not contained in the respective sets in* E *and* E^* *defined as:* $\overline{span}\{x_n :\ n \in \mathbb{Z},\ n \neq 0\}$ *and* $\overline{span}\{x_n^* :\ n \in \mathbb{Z},\ n \neq 0\}$ *and,*

$$\sum_{n\in\mathbb{Z}} \frac{(\log^+ \|x_n\| + \log^+ \|x_n^*\|)}{1+n^2} < +\infty, \tag{4.4}$$

and for some constant $C > 0$,

$$\|x_n\| \leqslant C\|x_{n+1}\| \quad \text{and} \quad \|x_n^*\| \leqslant C\|x_{n+1}^*\|, \quad \text{for all} \quad n \in \mathbb{Z}. \tag{4.5}$$

(v) *For ome integer* j,

$$\inf_{n\in\mathbb{Z}} \|x_{n+j}\|\|x_{-n}^*\| = 0. \tag{4.6}$$

Observe that condition (*iii*) corresponds to the definition we gave in 3.1, which we already saw that such functions are analytic on the given domain of definition, and that the spectrum of such sequences are precisely the singular points of these functions which are those $\lambda \in \mathbb{T}$ such that no analytic continuation over any arc containing the point is possible.

Definition 4.0.6 (Single valued extension property)**.** *We will say that an operator* $T \in L(E)$ *has the single valued extension property (s.v.e.p.) if for any analytic function* $f : \Omega \to E$ *defined on an open set* $\Omega \subset \mathbb{C}$*, with* $(T - \lambda I)f(\lambda) \equiv 0$*, it results* $f(\lambda) \equiv 0$*.*

Remark 4.0.5. For any operator $T \in L(E)$ having the s.v.e.p. and $f \in E$, we can consider the set of $\lambda_0 \in \Omega$ such that there exists $\lambda \to f(\lambda)$ analytic continuation in a neighbourhood of λ_0 with the property that $(T - \lambda I)f(\lambda) \equiv f$.

Following the previous definition and remark we observe that, from condition (iii) in Theorem 4.0.18 and the hypothesis 4.1 from the same Theorem, we have that, for all $|\lambda| \neq 1$,

$$\begin{aligned}(T-\lambda I)f_x(\lambda) &= Tf_x(\lambda)-\lambda f_x(\lambda)=\sum_{n=1}^{+\infty} Tx_{-n}\lambda^{n-1}-\sum_{n=1}^{+\infty} x_{-n}\lambda^{n}\\ &= \sum_{n=1}^{+\infty} x_{-n+1}\lambda^{n-1}-\sum_{n=1}^{+\infty} x_{-n}\lambda^{n}=x_0+\sum_{n=2} x_{-n+1}\lambda^{n-1}-\sum_{n=1}^{+\infty} x_{-n}\lambda^{n}=x_0.\end{aligned}$$

And therefore, if T has the s.v.e.p. we conclude that the singularity set of f_x coincides with the spectrum of x_0, $\sigma_T(x_0)\subset\mathbb{T}$. Exactly the same, if T^* has the s.v.e.p. we have that the singularity set of f_{x^*} coincides with the set $\sigma_{T^*}(x_0^*)$.

Now, assuming that T and T^* have the s.v.e.p. we can formulate condition (iii) from Theorem 4.0.18 in the following way:

(iii) The sequences $\{\|x_n\|\}_{n\in\mathbb{Z}}$ and $\{\|x_n^*\|\}_{n\in\mathbb{Z}}$ are dominated by Beurling sequences and the set $\sigma_T(x_0)\cup\sigma_{T^*}(x_0^*)$ has more than one point.

Thus, Theorem 4.0.18 implies clearly, the following theorem:

Theorem 4.0.19. *Let T be an invertible operator in $L(E)$ and $x_0\in E$ and $x_0^*\in E^*$ non zero elements. If the sequences $\{T^n x_0\}_{n\in\mathbb{Z}}$ and $\{(T^*)^n x_0^*\}_{n\in\mathbb{Z}}$ satisfy one of the forementioned hypothesis stated in Theorem 4.0.18, then either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.*

We will see at this point, that Wermer's Theorem 3.0.17, from Chapter 3 can be weakened since Theorem 4.0.19 implies Wermer's result from Chapter 3 as we shall see now.

Theorem 4.0.20. *Let T be an invertible operator in $L(E)$ such that*

$$\sum_{n\in\mathbb{Z}}\frac{\log\|T^n\|}{1+n^2}<+\infty, \tag{4.7}$$

and the spectrum of T contains more than one point then either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.

Proof. Let us consider the sequence $\rho_n=\|T^n\|$ as we did in Chapter 3. We observe immediately that $\{\rho_n\}_{n\in\mathbb{Z}}$ is a Beurling sequence since $\|T^{n+m}\|\leqslant\|T^n\|\|T^m\|$. Since we have that,

$$\|T^n x\|\leqslant\|T^n\|\|x\|=\rho_n\|x\|,\quad\text{and}\quad\|(T^*)^n x^*\|\leqslant\|(T^*)^n\|\|x^*\|=\rho_n\|x^*\|\quad\text{for all}\quad n\in\mathbb{Z}.$$

Let $\sigma(T)$ be the spectrum of T and consider,

$$R_\lambda=(T-\lambda I)^{-1},\quad \lambda\in\sigma(T)^c$$

the resolvent operator. Recall that $\sigma(T)\subset\mathbb{T}$ for being T such that $\|T^n\|$ is Beurling, as we saw in 3.0.3. So, as we did in Chapter 3 we have,

$$R_\lambda T=\sum_{n=1}^{+\infty} T^{-n}\lambda^{n-1},\quad |\lambda|<1$$

and,

$$R_\lambda T = -\sum_{n=-\infty}^{0} T^{-n}\lambda^{n-1}, \quad |\lambda| > 1.$$

Therefore, if $x \in E$, and $f_{T^n x}$ the function associated to the sequence $\{T^n x\}_{n\in\mathbb{Z}}$ by 4.2, we have immediately,

$$f_{T^n x}(\lambda) = R_\lambda T x \quad \text{for all} \quad \lambda \in \mathbb{C} \setminus \mathbb{T}.$$

Assume now that $\lambda_0 \in \sigma(T)$, by a Theorem by Helson [9, Thm 3], there exists a non-zero element $x_0 \in E$ such that

$$\begin{array}{ccc} \mathbb{D} & \longrightarrow & E \\ \lambda & \mapsto & R_\lambda T x_0 \end{array}$$

has no analytic continuation to any neighbourhood of λ_0. So, λ_0 is a singular point of $f_{T^n x}$.

Since, by hypothesis $\sigma(T)$ has more than one point, there exists $\lambda_1 \in \sigma(T)$, $\lambda_1 \neq \lambda_0$, and since $\sigma(T) = \sigma(T^*)$ following exactly the same argument with the function $f_{(T^*)^n x^*}$ associated to the sequence $\{(T^*)^n x^*\}_{n\in\mathbb{Z}}$, we will have that λ_1 is a singularity of $f_{(T^*)^n x^*}$. Thus, we are in the assumptions of Theorem 4.0.19 and the proof is complete. □

Another related result as a consequence of Theorem 4.0.18 is the following one, an extension of the result given by B. Sz.-Nagy and B. Foias [14, p. 74] which is shown in [5, p. 134].

Theorem 4.0.21 (Colojoara-Foias)**.** *Let E be a reflexive Banach space, and $\{\rho_n\}_{n\in\mathbb{Z}}$ an increasing sequence of positive numbers satisfying*

$$\limsup_{m\to+\infty} \frac{\rho_{m+n}}{\rho_m} \leqslant Cn^k \quad \textit{for all} \quad n \in \mathbb{N} \tag{4.8}$$

for some constant $C > 0$ and integer $k \leqslant 0$. Let T be an operator in $L(E)$ such that

$$\|T^n\| = O(\rho_n), \quad \textit{as} \quad n \to \infty, \tag{4.9}$$

and assume there exists elements $x \in E$ and $x^* \in E^*$ such that

$$\limsup_{n\to\infty} \|\rho_n^{-1} T^n x\| > 0 \tag{4.10}$$

and,

$$\limsup_{n\to\infty} \|\rho_n^{-1} (T^*)^n x^*\| > 0. \tag{4.11}$$

Then, either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.

Before the proof of the above result, we need the following proposition and a corollary of it whose proof can be seen in [3, p. 22].

Proposition 4.0.22. *Let T be an injective bounded linear operator acting on a Banach space E, and assume there is a sequence $\{x_n^*\}_{n\in\mathbb{Z}} \subset E^*$ and an element $x \in E$ such that*

$$\sup_{\substack{m,n\in\mathbb{N} \\ m\leqslant n}} \|(T^*)^m x_n^*\| < +\infty, \tag{4.12}$$

and

$$\limsup_{n\to+\infty} |\langle T^n x, x_n^*\rangle| > 0. \tag{4.13}$$

Then there exists a norm bounded sequence $\{y_n^*\}_{n\in\mathbb{Z}} \subset E^*$ *with* $y_0^* \neq 0$ *such that*

$$T^* y_n^* = y_{n-1}^* \quad \textit{for all} \quad n \in \mathbb{N}. \tag{4.14}$$

Moreover, if there is a sequence of positive numbers $\{a_n\}_{n\in\mathbb{N}}$ *such that*

$$\limsup_{n\to+\infty} \|(T^*)^{m+n} x_n^*\| \leqslant a_n, \quad \textit{for all} \quad m \in \mathbb{N}, \tag{4.15}$$

then,

$$\|(T^*)^m y_0^*\| \leqslant a_n, \quad \textit{for all} \quad m \in \mathbb{N}. \tag{4.16}$$

Corollary 4.0.23. *Let* T *be an injective operator in* $L(E)$ *and assume there is an incresing sequence* $\{\rho_n\}_{n\in\mathbb{Z}}$ *of positive numbers and* $x \in E$ *such that*

$$\|T^n\| = O(\rho_n), \quad \textit{as} \quad n \to \infty,$$

and

$$\limsup_{n\to\infty} \|\rho_n^{-1} T^n x\| > 0.$$

Then, there exists a norm bounded sequence $\{y_n^*\}_{n\in\mathbb{Z}} \subset E^*$ *with* $y_0^* \neq 0$ *such that,*

$$T^* y_n^* = y_{n-1}^* \quad \textit{for all} \quad n \in \mathbb{N}. \tag{4.17}$$

Moreover, if there also exists a sequence $\{a_n\}_{n\in\mathbb{Z}}$ *of positive numbers such that*

$$\limsup_{n\to+\infty} \frac{\rho_{m+n}}{\rho_n} \leqslant a_m \quad \textit{for all} \quad m \in \mathbb{N}, \tag{4.18}$$

then

$$\|(T^*)^m y_0^*\| \leqslant a_m, \quad \textit{for all} \quad m \in \mathbb{N}.$$

Now, we are in a position to prove Theorem 4.0.21:

Proof. First of all, we observe that if $T \neq 0$ and T is not injective then its kernel is a non-trivial hyperinvariant subspace for T: Indeed, if $x \in \mathrm{Ker}(T) \neq \{0\}$ then for all $S \in C(T, E)$ we have that, $TSx = STx = 0$ and hence $Sx \in \mathrm{Ker}(T)$, thus $\mathrm{Ker}(T)$ is invariant under any S which commutes with T. Also, if T^* is not injective, then the closure of the image of T is a non-trivial hyperinvariant subspace for T. This is again clear, since as a general fact T^* injective implies that $\overline{\mathrm{Im}T} = E$ and the other way around. So, if $y = Tx \in \mathrm{Im}(T)$ then, for any $S \in C(T, E)$ we have that $Sy = STx = TSx$ and therefore $Sx \in \mathrm{Im}(T)$ and since T^* is not injective, the closure of $\mathrm{Im}(T)$ is a proper subspace of E.

So, following the above mentioned reasoning we shall assume that T and T^* are both injective. This, together with the condition 4.1 and the hypothesis that $x_0, x_0^* \neq 0$ implies that $x_n, x_n^* \neq 0$ for all $n \in \mathbb{Z}$.

Now, from the previous Corollary 4.0.23 and hypothesis 4.25, 4.26 and 4.27 we obtain that there exists norm bounded a sequence $\{y_n^*\}_{n\in\mathbb{Z}} \subset E^*$ with $y_0^* \neq 0$ such that 4.17 holds and

$$(T^*)^n y_n^* = O(|n|^k), \quad \text{as} \quad n \to \infty.$$

Consider now the sequence $\{z_n^*\}_{n\in\mathbb{Z}} \subset E^*$ defined by means of $\{y_n^*\}_{n\in\mathbb{Z}}$ as:

$$z_n = y_{-n}^* \quad \text{for} \quad n < 0, \quad \text{and} \quad z_n = (T^*)^n y_n^* \quad \text{for} \quad n \geqslant 0.$$

Now, it follows from properties of the sequence $\{y_n^*\}_{n\in\mathbb{Z}}$ and the fact that $(T^*)^n y_n^* = O(|n|^k)$, as $n \to \infty$, that condition 4.1 and condition (ii) from Theorem 4.0.18 hold for the sequence $\{z_n^*\}_{n\in\mathbb{Z}}$.

Now, due to the fact that E is reflexive, and by hypothesis 4.28 we obtain that there exists a sequence $\{x_n\}_{n\in\mathbb{Z}} \subset E$ with $x_0 \neq 0$ such that 4.1 and condition (i) from Theorem 4.0.18 hold and hence the conclusion of Theorem 4.0.18 is attained. □

Finally, we show a more restrictive result by Beuzamy which asks the norm of T to be one and it is also a consequence of Theorem 4.0.18.

Theorem 4.0.24 (Beauzamy). *Let T be a bounded linear operator acting on a Banach space E such that $\|T\| = 1$, and assume that for some $x \in E$ we have,*

$$\limsup_{n\to\infty} \|T^n x\| > 0.$$

Moreover, suppose there exists a sequence $\{x_n\}_{n\in\mathbb{Z}} \subset E$ with $x_0 \neq 0$ such that the sequence $\{\|x_n\|\}_{n\in\mathbb{N}}$ is dominated by a one-sided Beurling sequence and such that,

$$Tx_n = x_{n-1} \quad \textit{for all} \quad n \in \mathbb{N}.$$

Then, either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.

This Theorem is consequence of the former ones, let us give an extension of Beauzamy's result by means of Theorem 4.0.21 and 4.0.24.

Theorem 4.0.25. *Let T be a bounded linear operator and $\{\rho_n\}_{n\in\mathbb{Z}}$ an increasing Beurling sequence satisfying conditions 4.25 and 4.26 from Theorem 4.0.21 and assume there is a vector $x \in E$ such that 4.27 from 4.0.21 holds too. Finally, assume there is a sequence $\{x_n\}_{n\in\mathbb{Z}} \subset E$ which satisfies the hypothesis from Beauzamy's Theorem 4.0.24. Then either T is a multiple of the identity or T has a non-trivial invariant subspace.*

Proof. Again, using Corollary 4.0.23, from hypothesis 4.25, 4.26 and 4.27 we conclude that there exists a norm bounded sequence $\{x_n^*\}_{n\in\mathbb{Z}} \subset E^*$ satisfying 4.14 and 4.16 from Proposition 4.0.22 with $a_n = n^k$ for all $n \in \mathbb{N}$.

Now, consider the sequences $\{y_n\}_{n\in\mathbb{Z}} \subset E$ and $\{y_n^*\}_{n\in\mathbb{Z}} \subset E^*$ defined by:

$$y_n = x_{-n} \quad \text{for} \quad n < 0, \quad \text{and} \quad y_n = T^n x_0 \quad \text{for} \quad n \geqslant 0,$$

and,

$$y_n^* = x_{-n}^* \quad \text{for} \quad n < 0, \quad \text{and} \quad y_n^* = (T^*)^n x_0^* \quad \text{for} \quad n \geqslant 0.$$

Now, we have that

$$\|y_n\| = \|x_{-n}\|, \quad \text{for all} \quad n < 0,$$

and

$$\|y_{-n}\| = \|T^{-n}x_0\| = \|T^{-n+1}T^{-1}x_0\| = \|T^{-n+1}x_1\| = \cdots = \|x_n\| \quad \text{for all} \quad n \geqslant 0.$$

so the sequence $\{\|y_n\|\}_{n\in\mathbb{Z}}$ is dominated by a Beurling sequence, since $\{\|x_n\|\}_{n\in\mathbb{N}}$ is dominated by a one-sided Beurling sequence and finally, by the same reasoning since $\|(T^*)^n x_0\| \leqslant n^k$, we have that $\{\|y_n^*\|\}_{n\in\mathbb{Z}}$ is bounded by n^k so $\|y_n^*\| = O(|n|^k)$ as $n \to +\infty$.

So Theorem 4.0.18 item (ii) applies to the sequences $\{y_n\}_{n\in\mathbb{Z}}$ and $\{y_n^*\}_{n\in\mathbb{Z}}$ and this comple tes the proof. □

We will now see some results which are slight modifications from Colojoaras and Foias result seen in Theorem 4.0.21, and also improvements on the hypothesis of this Theorem. We will see that the condition on the growth of the norm of the iterates of T need not be polynomial, in fact, it can be exponential.

First, we see that in Theorem 4.0.21, the condition 4.28 and the fact that E must be a reflexive Banach space, can be modified as follows:

Theorem 4.0.26. *Let E be a complex Banach space and T a bounded linear operator acting on E to itself. Let $\sigma(T)$ be the spectrum. Assume there is an increasing sequence $\{\rho_n\}_{n\in\mathbb{Z}}$ of positive numbers and an element $x \in E$ satisfying conditions 4.25, 4.26 and 4.27 from Theorem 4.0.21.*

Then if $\sigma(T) \cap \mathbb{T}$ is countable, T^* has an eigenvalue, and hence either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.

Observe that if T^* has an eigenvalue, that is, $\lambda \in \mathbb{T}$ such that $T^*x^* = \lambda x^*$ for some $x^* \in E^*$, and $T \neq \lambda I$, then $\overline{\text{Im}(T-\lambda I)}$is a non-trivial hyperinvariant subspace for T. Indeed, if $y \in \overline{\text{Im}(T-\lambda I)} = \{(T-\lambda I)x \in E;\ x \in E\}$, then for all $S \in C(T,E)$ we have, $Sy = S(T-\lambda I)x = STx - S\lambda x = TSx - \lambda Sx = (T-\lambda I)Sx$ and hence $Sy \in \overline{\text{Im}(T-\lambda I)}$. It is non-trival. It is non-zero since $T \neq \lambda I$ and $\overline{\text{Im}(T-\lambda I)}$ is not dense because $\text{Ker}(T^*-\lambda I) \neq 0$.

Another remark which is important to point to is that the condition on $\sigma(T) \cap \mathbb{T}$ being countable can be replaced by a weaker hypothesis demanding that the set $\sigma(T) \cap \mathbb{T}$ is a set of null-measure with respect to the Lebesgue measure on $\mathbb{T}$.

The following result will lead us to the improvement of Colojoaras and Foias result seen in Theorem 4.0.21.

Theorem 4.0.27. *Let E be a complex Banach space and let $T \in L(E)$. Assume we have sequences $\{x_n\}_{n\in\mathbb{Z}} \subset E$ and $\{x_n^*\}_{n\in\mathbb{Z}} \subset E^*$ with $x_0 \neq 0$ and $x_0^* \neq 0$ such that*

$$Tx_n = x_{n+1} \quad \text{and} \quad T^*x_n^* = x_{n+1}^*.$$

Suppose also that $\{\|x_n^*\|\}_{n\in\mathbb{Z}}$ is dominated by a Beurling sequence and that for some integer $k \geqslant 0$ and constant $C > 0$,

$$\|x_n\| + \|x_n^*\| = O(n^k), \quad \text{as} \quad n \to +\infty, \tag{4.19}$$

and,

$$\|x_{-n}\| = O(e^{Cn^{1/2}}), \quad as \quad n \to +\infty. \tag{4.20}$$

Then, either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.

Remark 4.0.6. This Theorem also holds if we replace the conditions 4.19 and 4.20 by the next ones:

$$\|x_{-n}\| + \|x^*_{-n}\| = O(n^k), \quad \text{as} \quad n \to +\infty,$$

and,

$$\|x_n\| = O(e^{Cn^{1/2}}), \quad \text{as} \quad n \to +\infty.$$

The above mentioned Theorem 4.0.27 has the following consequences:

Corollary 4.0.28. *Let T be an invertible operator in $L(E)$ and suppose there are non-zero elements $x_0 \in E$ and $x^*_0 \in E^*$ such that*

$$\|T^n x_0\| + \|(T^*)^n x^*_0\| = O(n^k), \quad as \quad n \to +\infty, \tag{4.21}$$

and,

$$\|T^{-n} x_0\| + \|(T^*)^{-n} x^*_0\| = O(e^{Cn^{1/2}}), \quad as \quad n \to +\infty, \tag{4.22}$$

for some integer $k \geqslant 0$ and constant $C > 0$.

Then either T is a multiple fo the identity or T has an hyperinvariant subspace.

Proof. It is clear that from items 4.21 and 4.22 we have that the sequences,

$$x_n = T^n x_0 \quad \text{and} \quad x^*_n = (T^*)^n x^*_0, \quad n \in \mathbb{Z}$$

satisfy the hypothesis of Theorem 4.0.27 □

Corollary 4.0.29. *Let T be an invertible operator in $L(E)$ and suppose that*

$$\|T^n\| = O(n^k), \quad as \quad n \to +\infty, \tag{4.23}$$

and,

$$\|T^n\| = O(e^{Cn^{1/2}}), \quad as \quad n \to -\infty, \tag{4.24}$$

for some integer $k \geqslant 0$ and constant $C > 0$.

Then either T is a multiple fo the identity or T has an hyperinvariant subspace.

Observe that Corollary 4.0.29 is a particular case of Wermer's Theorem seen in Chapter 3.

Finally, we obtain the following result as a consequence of Theorem 4.0.27, which is Colojoaras and Foias Theorem replacing the first hypothesis by a weaker one.

Theorem 4.0.30. *Let E be a reflexive Banach space, and $\{\rho_n\}_{n\in\mathbb{Z}}$ an increasing sequence of positive numbers satisfying*

$$\limsup_{m\to+\infty} \frac{\rho_{m+n}}{\rho_m} =\leqslant Ke^{Cn^{1/2}} \quad \textit{for all} \quad n \in \mathbb{N} \tag{4.25}$$

for some constant $C > 0$ and integer $k \leqslant 0$. Let T be an operator in $L(E)$ such that

$$\|T^n\| = O(\rho_n), \quad \textit{as} \quad n \to \infty, \tag{4.26}$$

and assume there exists elements $x \in E$ and $x^* \in E^*$ such that

$$\limsup_{n\to\infty} \|\rho_n^{-1} T^n x\| > 0 \tag{4.27}$$

and,

$$\limsup_{n\to\infty} \|\rho_n^{-1} (T^*)^n x^*\| > 0. \tag{4.28}$$

Then, either T is a multiple of the identity or T has a non-trivial hyperinvariant subspace.

Proof. We have just to apply Corollary 4.0.29 taking the sequence $a_n = Ke^{Cn^{1/2}}$, for all $n \in \mathbb{N}$ and from the fact that

$$\limsup_{m\to+\infty} \frac{\rho_{m+n}}{\rho_m} \leqslant Ke^{Cn^{1/2}} \quad \text{for all} \quad n \in \mathbb{N}$$

and following exactly the same proof as in Theorem 4.0.21 bearing in mind that the sequence $\{y_n^*\}_{n\in\mathbb{Z}}$ satisfies that,

$$\|(T^*)^m y_0^*\| \leqslant a_m = Ke^{Cm^{1/2}}, \quad \text{for all} \quad m \in \mathbb{N}.$$

The same for the sequence $\{x_n\}_{n\in\mathbb{Z}} \subset E$ as in the proof of Theorem 4.0.21. □

The later work of K. Kellay gives an extension of the work by A. Atzmon in the sense that it stablishes the existence of hyperinvariant subspaces based on a weaker condition than the one given by A. Atzmon. Let us first state the Theorem by K. Kellay and then compare it to the results seen until now.

Theorem 4.0.31 (K. Kellay)**.** *Let T be an operator in $L(E)$ such that $T \neq \lambda I$. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be an incresing Beurling sequence such that*

$$\limsup_{m\to+\infty} \frac{\rho_{m+n}}{\rho_m} = O(e^{\varepsilon n^{1/2}}) \quad \textit{for all} \quad \varepsilon > 0, \quad \textit{as} \quad n \to +\infty$$

and such that

$$\|T^n\| = O(\rho_n) \quad \textit{as} \quad n \to +\infty.$$

Assume there is an element $x \in E$ such that,

$$\limsup_{n\to+\infty} \frac{\|T^n x\|}{\rho_n} > 0,$$

and a sequence $\{x_n\}_{n\in\mathbb{N}} \subset E$ *such that*

$$Tx_{n+1} = x_n \quad \textit{for all} \quad n \geqslant 0,$$

$$\sum_{n\geqslant 0} \frac{\log^+ \|x_n\|}{1+n^2} < +\infty, \tag{4.29}$$

$$\textit{and } \|x_{n+1}\| \leqslant C\|x_n\| \quad \textit{for all} \quad n \geqslant 0 \quad \textit{and} \quad C > 0. \tag{4.30}$$

Then, T *has an hyperinvariant subspace.*

Observe that condition 4.29 is weaker than the fact that a sequence $\{\|x_n\|\}_{n\in\mathbb{N}} \subset E$ is dominated by a Beurling sequence. Indeed, if $\{x_n\}_{n\in\mathbb{N}} \subset E$ is such that $\|x_n\| \leqslant C\rho_n$ for some constant $C > 0$ and $\{\rho_n\}_{n\in\mathbb{N}}$ a Beurling sequence, then

$$\sum_{n\geqslant 0} \frac{\log^+ \|x_n\|}{1+n^2} \leqslant C \sum_{n\geqslant 0} \frac{\log^+ \rho_n}{1+n^2} < +\infty,$$

and since,

$$\|x_{n+1}\| \leqslant C\rho_{n+1} \quad \text{and} \quad \|x_n\| \leqslant C\rho_n,$$

hence,

$$\frac{\|x_{n+1}\|}{\|x_n\|} \leqslant \rho_1$$

and we obtain what we wanted.

Chapter 5

An example of an invertible operator without invariant subspaces

This Chapter has the main goal to define a bounded and linear operator with bounded inverse with no invariant subspaces. The operator will be defined on a Banach space that will be an infinite-dimensional direct sum of James p-spaces. We will present the James p-spaces which are Banach spaces of sequences satisfying a certain boundedness variation property depending on p, (p>1). We will find a basis for E and define the operator as a right-shift on these sequences.

5.1 James p-spaces and strictly singular operators

Before we define the notion of *strictly singular operator* we need to give meaning to *norm increasing operators.*

Definition 5.1.1 (Norm increasing). *Given E and F two normed spaces and $T : E \to F$ a bounded linear operator. We say that T is norm increasing if there exists $\varepsilon > 0$ such that*

$$\|Tx\|_F \geqslant \varepsilon \|x\|_E \quad \textit{for all} \quad x \in E.$$

Using this definition we can define strictly singular operators as those which such property is not fulfilled in any infinite-dimensional subspace of E.

Definition 5.1.2 (Strictly singular operator). *Let E and F be complex Banach spaces, and $T : E \to F$ a bounded linear operator. We say that T is strictly singular if there is no subspace $W \subset E$ of infinite dimension such such that the restriction $T|_W$ is norm increasing*

A compact operator is always strictly singular. Nevertheless, the converse is not true, and an evidence of this is that Read shows a strictly singular operator without invariant subspaces, so it can not be compact since it would contradict Lomonosov's Theorem which in particular states that every compact operator has invariant subspaces.

It is easy to check that compact implies strictly singular. Indeed: Suppose that an operator T is not strictly singular, that is, there is a subspace $F \subset E$ of infinite dimension such that $\|Tx\| \geqslant \varepsilon\|x\|$ for all $x \in F$. Take $y = Tx$ with $y \in F$ and define an operator R such that $R(y) = x$. R is bounded: $\|Ry\| = \|x\| \leqslant \frac{1}{\varepsilon}\|Tx\| = \frac{1}{\varepsilon}\|y\|$. Hence, T is invertible on F and the image of B_F the unit ball of F is isomorphic to B_F, that is $TB_F \equiv B_F$ but T compact implies that B_F is compact but F has infinite dimension so we get a contradiction.

Now, we will consider the space of sequences in $c_0 = \{\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C} : \lim_n a_n = 0\}$ and we define the James p-space for $1 < p < +\infty$ as:

$$J_p = \{a \in c_0 : \|a\| = \left(\sup_{\substack{i_1<\cdots<i_n \\ n\in\mathbb{N}}} \sum_{j=2}^{n} |a_{i_j} - a_{i_{j-1}}|^p\right)^{1/p} < +\infty\}$$

We will construct an operator with no invariant subspaces acting on a Banach space defined by means of an ℓ_2-direct sum of J_p spaces. Consider $\{p_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ an increasing sequence, all of them strictly greater than 2. Define,

$$E = \ell_2 \oplus \bigoplus_{n=1}^{+\infty} J_{p_n}.$$

The construction of such operator will be by means of an increasing sequence which tends to infinity sufficiently fast. Let $\{q_n\}_{n=1}^{+\infty}$ be an increasing sequence of positive integers. Set,

$$a_0 = 0 \quad a_n = q_{2n-1} \quad \text{and} \quad b_n = q_{2n} \quad \text{for all} \quad n \geqslant 1.$$

Thus, we have $a_1 < b_1 < a_2 < b_2 < \ldots$. Define now, a sequence $\{c_n\}_{n=0}^{+\infty}$ by means of a and b as follows,

$$c_0 = 0 \quad c_n = n(a_n + b_n) \quad \text{for all} \quad n \geqslant 1.$$

Finally, we define the following partial sums of v_n,

$$s_n = 1 + \sum_{k=0}^{n-1}(1 + v_k) \quad \text{and} \quad s_0 = 1.$$

We know that every Banach space contains an infinite-dimensional subspace that has a Schauder basis. Let $F \subset E$ be the infinite-dimensional subspace of E spanned by the unit vectors, $\{f_{ij}\}_{i,j\in\mathbb{Z}^+}$, where fixed i, the element $\{f_{ij}\}$ is the Schauder basis for the J_{p_i} space and for $i = 0$ we denote $\{f_{0j}\}_{j\geqslant 0}$ the basis for the ℓ^2 space.

Now,given $I \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ a set of indices, we will denote by $F_I \subset F$ the set spanned by the unit vectors $\{f_{ij}; (i,j) \in I\}$, and $\pi_I : F \to F_I$ will denote the natural projection from F onto F_I, acting on F as $\pi_I(f_{ij}) = f_{ij}$ whenever $(i,j) \in I$ or $\pi_I(f_{ij}) = 0$ otherwise. Such π_I is continuous for certain choices of I, but we will only be concerned in the choices of I for which π_I is continuous.

5.2 Construction of the operator

The first step will be to define a new sequence $\{e_n\}_{n=0}^{+\infty}$ such that its linear span is dense in $F \subset E$.

To do so, first we will need to reorder the elements of $\{f_{ij}, i,j \geqslant 0\}$ by means of a bijection $\varphi : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$, in such a way that f_{ij} is equal to the element $f_{\varphi(i,j)}$. We will denote F_n

the subspace generated by the linear span of $\{f_1, \ldots, f_n\}$ where $F_n = F_I$ for a choice of I such that it is the unique set of indices with $I = \varphi^{-1}([0,n]) \subset \mathbb{Z}^+ \times \mathbb{Z}^+$.

We need now to define the following set of positive integers:

$$\Omega = \bigcup_{n=1}^{+\infty} \bigcup_{k=1}^{n} [ka_n, ka_n + c_{n-k}],$$

and observe,

$$\Omega = \bigcup_{k=0}^{+\infty} \Omega_k,$$

where,

$$\Omega_k = \bigcup_{n=k+1}^{+\infty} [(n-k)a_n, (n-k)a_n + c_k].$$

As long as the sequence $\{q_n\}_n$ increases sufficiently fast, it can be checked that the set Ω is a disjoint union, and that both Ω and $\mathbb{Z}^+ \setminus \Omega$ are infinite sets. So, from now on we will take for granted that $\{q_n\}_n$ increases sufficiently rapidly so that this condition is fulfilled.

Let now, for each $k \geqslant 0$, σ_k be the natural bijection defined as follows:

$$\begin{array}{rccc} \sigma_k : & \Omega_k & \longrightarrow & [0, c_k] \times \mathbb{Z}^+ \\ & (n-k)a_n + m & \mapsto & (m, n-k-1) \end{array}, \text{ with } \quad m \in [0, c_k].$$

From σ_k we define the functions $\Sigma_k(m) = \sigma_k(m) + (d_k, 0)$ in such a way that Σ_k acts on the same domain as σ_k and the image becomes $[d_k, d_k + c_k] \times \mathbb{Z}^+ = [d_k, d_{k+1}) \times \mathbb{Z}^+$ since $d_{k+1} = d_k + c_k + 1$.

Finally, define Σ as the unique map acting on Ω onto $\mathbb{Z}^+ \times \mathbb{Z}^+$ such that $\Sigma|_{\Omega_k} = \Sigma_k$, that is, the restriction of Σ on the set Ω_k coincides with the functions Σ_k defined before.

Now, Σ is a bijection from Ω onto $[d_0, +\infty] \times \mathbb{Z}^+ = \mathbb{N} \times \mathbb{Z}^+$. So unsing Σ we may get a bijection between $\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+$ by mapping the set $\mathbb{Z}^+ \setminus \Omega$ onto $\{0\} \times \mathbb{Z}^+$.

So, if we define,

$$\begin{array}{rccc} \Sigma : & \mathbb{Z}^+ & \longrightarrow & \mathbb{Z}^+ \times \mathbb{Z}^+ \\ & m & \mapsto & \Sigma(m) = (0, \psi^{-1}(m)) \end{array},$$

where, ψ is the unique increasing bijection between the sets $\mathbb{Z}^+ \setminus \Omega$ and $\mathbb{Z}^+$.

The next step will be to construct the basis $\{e_n\}_{n=0}^{+\infty}$ of F and the operator without invariant subspaces will be a "right shift" of the basis elements of F.

Assuming that the sequence $\{q_n\}_n$ increases sufficiently fast in such a way that the function Σ, and hence Σ^{-1} are well-defined. We will construct for each k, the element $f_k = f_{i,j}$ where $(i,j) = \Sigma(k)$.

Now, we claim that there exists a sequence $\{e_n\}_{n=0}^{+\infty} \subset F$ such that,

$$f_0 = e_0, \tag{5.1}$$

and, given r, n, k such that $0 < r \leqslant n$ and $ra_n \leqslant k \leqslant ra_n + c_{n-r}$ then,

$$f_k = (n-r+1)^{k-ra_n} a_{n-r}((1+n)^{ra_n} e_k - n^{(r-1)a_n - 1} e_{k-ra_n+(r-1)a_{n-1}}). \tag{5.2}$$

Moreover, if $0 < r < n$ and $ra_n + c_{n-r} < k < (r+1)a_n$ then,

$$f_k = (1+n)^k e^{\left((r+\frac{1}{2})a_n - k\right)/\sqrt{a_n}} e_k. \tag{5.3}$$

on the other hand if $n \geqslant 1$ and $c_{n-1} < k < a_n$ then

$$f_k = (1+n)^k e^{\left(\frac{1}{2}a_n - k\right)/\sqrt{a_n}} e_k. \tag{5.4}$$

If r, n, k satisfy that $0 < r \leqslant n$ and $[r(a_n + b_n) \leqslant k \leqslant na_n + rb_n$ then,

$$f_k = (1+n)^k e_k - b_n(1+n)^{i-b_n} e_{i-b_n}. \tag{5.5}$$

Fianlly, if the integers r, n and k are such that $0 \leqslant r < n$ and $na_n + rb_n < k < (r+1)(a_n + b_n)$ then,

$$f_k = (1+n)^k 2^{\left((r+\frac{1}{2})b_n - k\right)/\sqrt{b_n}} e_k. \tag{5.6}$$

Recall that the e_k's can be isolated from the f_k's since,

$$f_k = \sum_{n=0}^{k} \lambda_{k,n} e_k,$$

with $\lambda_{k,n} \neq 0$ for all $k \in \mathbb{Z}^+$ and the matrix $(\lambda_{k,n})_{k,n}$ is a lowe triangular matrix with diagonal elements different from zero and hence invertible. Thus, we can define an operator by means of the elements $\{e_n\}_{n=0}^{+\infty}$ and the next Theorem gives us the main result of this Chapter.

Theorem 5.2.1 (Read)**.** *Assuming we have an sequence $\{q_n\}_{n=0}^{+\infty}$ that increases sufficiently rapidly. Then, there exists a unique sequence $\{e_n\}_{n=0}^{+\infty}$ satisfying the conditions 5.1, 5.2, 5.3, 5.4, 5.5 and 5.6 which is a basis for the space F and there is a unique continuous linear operator $T : F \to F$ and strictly singular such that $Te_k = e_{k+1}$ for each k and T has no invariant subspaces.*

The last step is to contruct an invertible operator without invariant subspaces using a strictly singular operator with no invariant subspace. To do so, we will use the following result about strictly singular operator whose proof can be seen with detail in [2, pp, 278–279].

Theorem 5.2.2. *Let T be a strictly singular operator in $L(E)$ with E an infinite dimensional Banach spacer, then the spectrum of T is countable, $0 \in \sigma(T)$ and it is the only possible accumulation point, and all the non-zero elements of $\sigma(T)$ are eigenvalues.*

Now, by Read's Theorem, we have a strictly singular operator T with no invariant subspaces, so $\sigma(T) = \{0\}$. Indeed, if $\sigma(T) \neq \{0\}$ then by Theorem 5.2.2 T would have an eigenvalue but this means that T would have an invariant subspace and this can not happen.

Finally, let us consider the operator $S = I - T$. S is an invertible operator with no invariant subspaces. It is indeed invertible since $\sigma(T) = \{0\}$ so all operators of the form $\lambda I - T$ for all $\lambda \in \mathbb{C} \setminus \{0\}$ are invertible. Moreover, if $Y \subset E$ and $Y \neq \{0\}, E$ is invariant under S, then:

$$TY = (I + S)Y = Y + SY \subset Y,$$

and this implies that S has no invariant subspaces either.

Therefore, we have constructed an invertible operator with no invariant subspaces which shows that, indeed, not every bounded and linear operator on an infinite-dimensional and separable Banach space necessarily has non-trivial invariant subspaces.

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