# POSITIVE SCHUR PROPERTIES IN SPACES OF REGULAR OPERATORS

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Abstract. Properties of Schur type for Banach lattices of regular operators and tensor products are analyzed. It is shown that the dual positive Schur property behaves well with respect to Fremlin's projective tensor product, which allows us to construct new examples of spaces with this property. Similar results concerning the positive Grothendieck property are also presented.

## 1. Introduction

A particular feature of the space $\ell_1$ of summable sequences is that every sequence $(x_n) \subset \ell_1$ which converges in the weak topology is also norm convergent. In other words, sequential weak-convergence and norm-convergence are equivalent in $\ell_1$. A Banach space in which this equivalence holds is said to have the *Schur property*, and this is intimately related to the ubiquity of subspaces isomorphic to $\ell_1$. In the context of Banach lattices, a weaker notion has been considered when the above equivalence is required only for positive elements. Namely, a Banach lattice $E$ has the *positive Schur property* (PSP) when every weakly null sequence of positive elements $(x_n)_n$ in $E_+$ is convergent to zero in norm. The simplest example of a Banach lattice with the PSP but without the Schur property is the space $L_1$ over a nonatomic measure space.

In the study of the duality for the PSP, the *dual positive Schur property* (DPSP) has been recently introduced in [2]. Namely, a Banach lattice $E$ is said to have the DPSP when every sequence of positive functionals $(x_n^*)_n$ in $E_+^*$ which is weakly-$*$ convergent to zero, is necessarily convergent to zero in norm. If a Banach lattice $E$ has the DPSP, then its dual $E^*$ has the PSP, but the converse is not necessarily true. A detailed study of the DPSP can be found in [17].

Our aim in this paper is to construct new examples of Banach lattices with these properties, as well as studying similar properties in Banach lattices of regular operators.

A starting point for our work is the following theorem from [15]:

**Theorem 1.** *Let $E$, $F$ be Banach lattices. The following are equivalent:*

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(1) *$E^*$ and $F$ have PSP.*
(2) *$\mathcal{L}_r(E, F)$ has PSP.*

Note that this theorem relates, in a certain sense, the weak topology in the space $\mathcal{L}_r(E, F)$, which is in general not very well understood, with that of $E$ and $F$. Moreover, in spaces of operators there is another relevant topology which can be considered in this respect: the weak operator topology (in short, *wot*). We will introduce a version of the positive Schur property for spaces of operators that corresponds to the wot. In addition, a characterization of this property in terms of the DPSP and PSP of the spaces $E$ and $F$ will be provided (see Theorem 2).

As one may expect, similar results can also be obtained for the weak-$*$ operator topology in $\mathcal{L}_r(E, F^*)$. Moreover, in this case, our analysis yields also a characterization of the DPSP of Fremlin's projective tensor product of two Banach lattices $E \otimes_{|\pi|} F$ (see Theorem 3).

Finally, we also study another property closely related to PSP and DPSP: the *positive Grothendieck property* (PGP). Recall that a Banach space $E$ has the Grothendieck property when every weak-$*$ null sequence in $E^*$ is also weakly null. In a similar way, a Banach lattice $E$ is said to have the PGP if every sequence of positive elements $(x_n^*)_n$ in $E_+^*$ which is weak-$*$ null is also weakly null.

It can be easily seen that a Banach lattice has DPSP if and only if $E^*$ has PSP and $E$ has PGP. This connection motivates the study of the same kind of results for spaces of operators and Fremlin's tensor product of Banach lattices (see Theorem 5).

## 2. Notation and preliminaries

We follow standard terminology concerning Banach lattices and positive operators as can be found in the monographs [1, 11]. Let us start recalling the basic definitions of this work.

A Banach lattice $E$ has the dual positive Schur property (DPSP) if every sequence of positive functionals $(x_n^*) \subset E_+^*$ which is weak-$*$ convergent to zero satisfies $\|x_n^*\| \to 0$ ([2], [17]). A Banach lattice $E$ is said to have the positive Grothendieck property (PGP) if every sequence of positive elements $(x_n^*)_n$ in $E_+^*$ which is weak-$*$ null is also weakly null.

Since every operator $T : E \to c_0$ can be identified with a weak-$*$ null sequence $x_n^*$ in $E^*$, it follows that a Banach lattice $E$ has DPSP if and only if every positive operator $T : E \to c_0$ is compact; while, $E$ has PGP if and only if every positive operator $T : E \to c_0$ is weakly compact. It is immediate to see that $E$ has DPSP if and only if $E$ has PGP and $E^*$ has PSP.

It is also clear that if $E$ is a Banach lattice with the DPSP (respectively PGP) and $\mathcal{I} \subset E$ is a closed ideal, then $E \backslash \mathcal{I}$ has DPSP (resp. PGP).

Recall that a Banach lattice has the PSP if every normalized disjoint sequence has a subsequence equivalent to the unit vector basis of $\ell_1$ (cf. [5, 16]). Similarly, a Banach lattice $E$ has the DPSP if and only if every sequence of positive disjoint elements in $E^*_+$ which is weak-$*$ convergent to zero is norm null [17, Proposition 2.3].

It is straightforward to check that $C(K)$ spaces have the DPSP for every compact Hausdorff space $K$. More generally, for an $AM$-space $E$ it follows that DPSP and PGP are equivalent properties, and these hold if and only if there is no positively complemented order copy of $c_0$ in $E$ [17, Proposition 4.1]. A criterion for Musielak-Orlicz spaces to have DPSP can be also found in [17].

For Banach lattices $E$ and $F$, an operator $T : E \to F$ is positive when $Tx \in F_+$ whenever $x \in E_+$. We denote the set of positive operators between $E$ and $F$ by $\mathcal{L}_+(E, F)$, and the space of regular operators $\mathcal{L}_r(E, F)$ is the linear space generated by $\mathcal{L}_+(E, F)$. Recall that given a Dedekind-complete Banach lattice $F$, for every Banach lattice $E$, the space of regular operators $\mathcal{L}_r(E, F)$ is again a Banach lattice with the norm

$$\|T\|_r = \inf\{\|S\| : S \in \mathcal{L}_+(E, F),\ |Tx| \leq S|x|, \forall x \in E\}.$$

Note that every dual Banach lattice $F^*$ is Dedekind-complete, hence $\mathcal{L}_r(E, F^*)$ is always a Banach lattice. In order to avoid cumbersome repetition, everytime we deal with the space $\mathcal{L}_r(E, F)$, we will assume that $F$ is Dedekind-complete.

Recall that given two vector spaces $E, F$, we can consider the algebraic tensor product $E \otimes F$ as the space of linear combinations of elements of the form $e \otimes f$ with $e \in E$ and $f \in F$. The Fremlin tensor product ([6], [7]) of two Banach lattices $E$ and $F$ is the completion of $E \otimes F$ with respect to the norm

$$\|u\|_{E\otimes_{|\pi|}F} = \inf\{\sum_{i=1}^{n} \|x_i\|_E \|y_i\|_F :\ x_i \in E_+,\ y_i \in F_+,\ |u| \leq \sum_{i=1}^{n} x_i \otimes y_i\}.$$

This space, with the order given by the closure of $E_+ \otimes F_+$ with respect to $\|\cdot\|_{|\pi|}$, becomes a Banach lattice.

Moreover, it turns out that the mapping

$$\begin{array}{rccc} T: & (E\otimes_{|\pi|} F)^* & \longrightarrow & \mathcal{L}_r(E, F^*) \\ & \phi & \longmapsto & T_\phi \end{array}$$

where $T_\phi$ is given by $\langle T_\phi x, y\rangle = \phi(x \otimes y)$ for every $x \in E$, $y \in F$, defines a surjective Riesz isometry (cf. [6, 9]).

## 3. Positive Schur properties in $\mathcal{L}_r(E, F)$

Recall that a sequence of operators $(T_n) \subset \mathcal{L}(E, F)$ converges to zero in the weak operator topology ($wot$) if for every $x \in E$ and $y^* \in F^*$ we have

$$\langle y^*, T_n x\rangle \to 0.$$

Similarly, given a sequence $(T_n) \subset \mathcal{L}(E, F^*)$, we say that this sequence converges to zero in the weak$-*$ operator topology ($w^*ot$) if for every $x \in E$ and $y \in F$ we have

$$\langle T_n x, y \rangle \to 0.$$

The following notions are related to the positive Schur property for Banach lattices of regular operators.

**Definition 1.** *$\mathcal{L}_r(E, F)$ has the wot-positive Schur property ($\mathcal{L}_r(E, F) \in wot-PSP$) if for every sequence of positive operators $(T_n) \subset \mathcal{L}_+(E, F)$ with $T_n \to 0$ in the weak operator topology, then $\|T_n\| \to 0$.*

**Definition 2.** *$\mathcal{L}_r(E, F^*)$ has the $w^*ot$-positive Schur property ($\mathcal{L}_r(E, F^*) \in w^*ot-PSP$) if for every sequence of positive operators $(T_n) \subset \mathcal{L}_+(E, F^*)$ with $T_n \to 0$ in the weak-$*$ operator topology, then $\|T_n\| \to 0$.*

Observe that for Banach lattices $E$ and $F$ we have:

$$\mathcal{L}_r(E, F) \in PSP \quad \Rightarrow \quad \mathcal{L}_r(E, F) \in wot - PSP,$$

$$\mathcal{L}_r(E, F^*) \in PSP \quad \Rightarrow \quad \mathcal{L}_r(E, F^*) \in wot{-}PSP \quad \Rightarrow \quad \mathcal{L}_r(E, F^*) \in w^*ot{-}PSP.$$

**Theorem 2.** *Let $E$, $F$ be Banach lattices. The following are equivalent:*

1. *$E$ has DPSP and $F$ has PSP.*
2. *$\mathcal{L}_r(E, F)$ has wot-PSP.*

*Proof.* (1) $\Rightarrow$ (2): We will proceed by contradiction. Let $(T_n) \subset \mathcal{L}_+(E, F)$ be a sequence of positive operators such that for every $x \in E$ and $y^* \in F^*$

$$\langle y^*, T_n x \rangle \to 0$$

but for some $\alpha > 0$ we have $\|T_n\| > \alpha$ for every $n \in \mathbb{N}$.

Since $T_n \geq 0$, for every $n \in \mathbb{N}$ there is $x_n \in E_+$ with $\|x_n\| \leq 1$ such that

$$\|T_n x_n\| \geq \alpha.$$

Notice first that for any $y^* \in F^*$ and $x \in E$, by hypothesis, we have that

$$\langle T_n^* y^*, x \rangle = \langle y^*, T_n x \rangle \to 0,$$

which means that $(T_n^* y^*)$ is weak-$*$ convergent to zero in $E^*$. In particular, since $T_n \geq 0$ and $E$ has DPSP, it follows that

$$\|T_n^*(y^*)\| \leq \|T_n^*(y_+^*)\| + \|T_n^*(y_-^*)\| \to 0.$$

Thus, we have that for every $y^* \in F^*$

$$|\langle y^*, T_n x_n \rangle| = |\langle T_n^* y^*, x_n \rangle| \leq \|T_n^* y^*\| \|x_n\| \to 0.$$

Therefore, $(T_n x_n)$ is a positive weakly null sequence in $F$, and by the PSP it follows that $\|T_n x_n\| \to 0$ in contradiction with $\|T_n x_n\| > \alpha$.

(2) $\Rightarrow$ (1): Let $(x_n) \subset E^*_+$ be a weak-$*$ null sequence. Fix $y \in F_+\backslash\{0\}$ and let us define $T_n : E \to F$ by

$$T_n(x) = \langle x^*_n, x\rangle y.$$

Clearly, $(T_n)$ is a sequence of positive operators which converge to zero in the *wot*. Therefore, since $\mathcal{L}_r(E, F)$ has *wot*-PSP it follows that $\|T_n\| \to 0$. This implies that

$$\|x^*_n\| = \frac{\|T_n\|}{\|y\|} \to 0$$

showing that $E$ has DPSP.

Let now $(y_n) \subset F_+$ be a weakly null sequence. Given $x^* \in E^*_+\backslash\{0\}$, we can define $T_n : E \to F$ by

$$T_n(x) = \langle x^*, x\rangle y_n.$$

Clearly, $(T_n)$ is a sequence of positive operators which converge to zero in the *wot*. Therefore, since $\mathcal{L}_r(E, F)$ has *wot*-PSP it follows that $\|T_n\| \to 0$. This implies that

$$\|y_n\| = \frac{\|T_n\|}{\|x^*\|} \to 0$$

showing that $F$ has PSP. □

Concerning $w^*ot$-PSP we have the following:

**Theorem 3.** *Let $E$, $F$ be Banach lattices. The following are equivalent:*

1. *$E$ and $F$ have DPSP.*
2. *$\mathcal{L}_r(E, F^*)$ has $w^*ot$-PSP.*
3. *$E \otimes_{|\pi|} F$ has DPSP.*

*Proof.* (1) $\Rightarrow$ (2): As in the proof of Theorem 2, we will proceed by contradiction. Let $(T_n) \subset \mathcal{L}_+(E, F^*)$ be a sequence of positive operators such that for every $x \in E$ and $y \in F$

$$\langle T_n x, y\rangle \to 0$$

but for some $\alpha > 0$ we have $\|T_n\| > \alpha$ for every $n \in \mathbb{N}$.

Since $T_n \geq 0$, for every $n \in \mathbb{N}$ there is $x_n \in E_+$ with $\|x_n\| \leq 1$ such that

$$\|T_n x_n\| \geq \alpha.$$

Notice first that for any $y \in F$ and $x \in E$, by hypothesis, we have that

$$\langle T^*_n y, x\rangle = \langle y, T_n x\rangle \to 0,$$

which means that $(T^*_n y)$ is weak-$*$ convergent to zero in $E^*$. In particular, since $E$ has DPSP, for any $y \in F$ it follows that

$$\|T^*_n(y)\| \leq \|T^*_n(y_+)\| + \|T^*_n(y_-)\| \to 0.$$

Thus, we have that for every $y \in F$

$$|\langle y, T_n x_n\rangle| = |\langle T^*_n y, x_n\rangle| \leq \|T^*_n y\|\|x_n\| \to 0.$$

Therefore, $(T_n x_n)$ is a positive weakly-$*$ null sequence in $F^*$, and by the DPSP of $F$ it follows that $\|T_n x_n\| \to 0$ in contradiction with $\|T_n x_n\| > \alpha$.

(2) $\Rightarrow$ (3): Let $(\phi_n) \subset (E \otimes_{|\pi|} F)^*_+$ be a weak-$*$ null sequence. Using the identification $(E \otimes_{|\pi|} F)^* = \mathcal{L}_r(E, F^*)$ given by $\phi \mapsto T_\phi$ where

$$\langle T_\phi x, y\rangle = \phi(x \otimes y)$$

for every $x \in E$, $y \in F$, it follows that $T_{\phi_n} \to 0$ in the $w^*ot$. Now, since $\mathcal{L}_r(E, F^*)$ has $w^*ot$-PSP, it follows that $\|T_{\phi_n}\| \to 0$. Since $\|\phi_n\| = \|T_{\phi_n}\|$, $E \otimes_{|\pi|} F$ has DPSP.

(3) $\Rightarrow$ (1): Let $(x_n^*) \subset E_+^*$ be a weak-$*$ null sequence. Pick an arbitrary $y_0^* \in F_+^* \backslash \{0\}$, and let us define $\phi_n(x \otimes y) = x_n^*(x) y_0^*(y)$. Clearly, $\phi_n$ extends to an element in $(E \otimes_{|\pi|} F)^*_+$. Moreover, for each $u \in E \otimes_{|\pi|} F$, let $x_i \in E_+$, $y_i \in F_+$ for $i = 1, \dots, k$ be such that

$$|u| \leq \sum_{i=1}^k x_i \otimes y_i \quad \text{and} \quad \|u\|_{E \otimes_{|\pi|} F} \approx \sum_{i=1}^k \|x_i\| \|y_i\|.$$

We have that

$$|\phi_n(u)| \leq \phi_n\Big(\sum_{i=1}^k x_i \otimes y_i\Big) = x_n^*\Big(\sum_{i=1}^k y_0^*(y_i) x_i\Big) \to 0,$$

since $(x_n^*)$ is weak-$*$ null. Thus, by the DPSP of $E \otimes_{|\pi|} F$, it follows that $\|\phi_n\| \to 0$. This implies that $\|x_n^*\| \to 0$. In fact, let $\widetilde{x_n} \in E_+$ and $\widetilde{y_0} \in F_+$ be such that $\|\widetilde{x_n}\| = \|\widetilde{y_0}\| = 1$ satisfying

$$\|x_n^*\| \approx \langle x_n^*, \widetilde{x_n}\rangle \quad \text{and} \quad \|y_0^*\| \approx \langle y_0^*, \widetilde{y_0}\rangle.$$

It follows that

$$\|\phi_n\| = \sup\{\phi_n\Big(\sum_i |x_i| \otimes |y_i|\Big) : \sum_i \|x_i\|\|y_i\| \leq 1\} \geq \phi_n(\widetilde{x_n} \otimes \widetilde{y_0}) \approx \|x_n^*\| \|y_0^*\|,$$

which shows that $\|x_n^*\| \to 0$. Thus, $E$ has DPSP. The proof for $F$ follows by symmetry.

□

Recall that given Banach lattices $E, F$ the conjugation mapping defines a positive isometry from $\mathcal{L}_r(E, F)$ into $\mathcal{L}_r(F^*, E^*)$. From the previous Theorems 2 and 3, it follows that $\mathcal{L}_r(E, F)$ has $wot$-PSP provided that $\mathcal{L}_r(F^*, E^*)$ has $w^*ot$-PSP.

Let us show now an application of Theorem 3. Recently, it has been shown in [3] that the 2-concavification of a Banach lattice can be identified with its diagonal in the $|\pi|$-tensor square. Recall that given a Banach lattice $E$ (or more generally a vector lattice), and $p > 0$ we can introduce the operations $x \oplus y = (x^p + y^p)^{\frac{1}{p}}$ and $\lambda \odot x = \lambda^{\frac{1}{p}} x$ for $x, y \in E$, $\lambda \in \mathbb{R}$ (see [10, Section 1.d]). The set $E$ endowed with the operators $\oplus, \odot$ becomes a vector lattice denoted $E_{(p)}$. Following the notation of [3], given a Banach lattice $E$, and $p > 0$ the space $E_{[p]}$ denotes the completion of the normed lattice $E_{(p)}/\ker\|\cdot\|_{(p)}$, where

$$\|x\|_{(p)} = \inf\Big\{\sum_{i=1}^{n}\|v_i\|^p : |x| \le \Big(\sum_{i=1}^{n} v_i^p\Big)^{1/p},\ v_i \in E_+\Big\}.$$

If $p > 1$ and $E$ is $p$-convex, then $\|\cdot\|_{(p)}$ is a norm in $E_{(p)}$ and $E_{[p]} = E_{(p)}$ which coincides with the $p$-concavification process given in [10, Section 1.d]. While if $0 < p < 1$, then $E_{[p]} = E^{(\frac{1}{p})}$ is the $\frac{1}{p}$-convexification of $E$.

**Corollary 1.** *If a Banach lattice $E$ has DPSP, then $E_{[2]}$ also has DPSP.*

*Proof.* Since $E$ has DPSP, by Theorem 3 it follows that $E \otimes_{|\pi|} E$ also has DPSP. Now, by [3, Theorem 11], there is a positive quotient from $E \otimes_{|\pi|} E$ to $E_{[2]}$. Thus, $E_{[2]}$ also has DPSP. □

Iterating this result, we get that if $E$ has DPSP, then $E_{[n]}$ also has DPSP for every $n \in \mathbb{N}\backslash\{0\}$: Indeed, by [12], $E_{[n]}$ is a positive quotient of $E \otimes_{|\pi|} E \otimes_{|\pi|} \cdots \otimes_{|\pi|} E$, and the result follows from Theorem 3.

Let us focus now on understanding under which conditions $\mathcal{L}_r(E, F)$ has DPSP.

**Definition 3.** *A couple of Banach lattices $(E, F)$ satisfy condition $(*)$ if for every positive operator $T : E \to F$ with $\|T\| \le 1$, and every $\varepsilon > 0$, there is $u \in (E^* \otimes_{|\pi|} F)_+$, $\|u\|_{E^*\otimes_{|\pi|}F} \le 1$ and $S \in \mathcal{L}_r(E, F)$ with $\|S\| \le \varepsilon$ such that*

$$T \le u + S.$$

Note that condition $(*)$ is equivalent to the following density property in $\mathcal{L}_r(E, F)$:

$$\overline{B_{(E^*\otimes_{|\pi|}F)_+}}^{\mathcal{L}_r(E,F)} = B_{\mathcal{L}_+(E,F)}.$$

**Example 1.** *It is easy to see that the couple $(L_1(\Omega, \mu), L_\infty(\Omega', \nu))$ satisfy condition $(*)$ for any measure spaces $(\Omega, \mu), (\Omega', \nu)$. In fact, every positive operator $T : L_1(\Omega, \mu) \to L_\infty(\Omega', \nu)$ with $\|T\| \le 1$ satisfies*

$$T(f) \le \int_\Omega f d\mu \mathbb{1}_{\Omega'} = \int_\Omega d\mu \otimes \mathbb{1}_{\Omega'}\,(f)$$

*for every positive $f$.*

**Theorem 4.** *Let $E$, $F$ be Banach lattices. Consider the following statements.*

(1) *$\mathcal{L}_r(E, F)$ has DPSP.*
(2) *$E^*$ and $F$ have DPSP.*

*In general, we have that (1) ⇒ (2). If $(E, F)$ satisfy condition $(*)$, then we also have (2) ⇒ (1).*

*Proof.* (1) ⇒ (2): Let us see first that $E^*$ has DPSP. Let $(x_n^{**}) \subset E_+^{**}$ be a weak-$*$ null sequence, i.e. $\langle x_n^{**}, x^*\rangle \to 0$ for every $x^* \in E^*$. Pick an arbitrary $y_0^* \in F^*$ with $\|y_0^*\| = 1$, and let us define $\phi_n^{y_0^*} \in (\mathcal{L}_r(E, F))_+^*$ by

$$\phi_n^{y_0^*}(T) = \langle x_n^{**}, T^* y_0^*\rangle.$$

Observe that for a fixed $T \in \mathcal{L}_r(E,F)$, we have that $\phi_n^{y_0^*}(T) \to 0$, so by the DPSP of the space $\mathcal{L}_r(E,F)$ it follows that $\|\phi_n^{y_0^*}\| \to 0$.

Now, for each $n \in \mathbb{N}$, let us consider $\widetilde{x_n^*} \in E_+^*$ with $\|\widetilde{x_n^*}\| \leq 1$ such that

$$\langle x_n^{**}, \widetilde{x_n^*}\rangle \approx \|x_n^{**}\|,$$

and take also $y_0 \in F_+$ with $\|y_0\| \leq 1$ such that $\langle y_0^*, y_0\rangle = 1$. This allows us to define the operators $T_n : E \to F$ given by

$$T_n(x) = \langle \widetilde{x_n^*}, x\rangle y_0.$$

Notice that $\|T_n\| \leq 1$, and that $T_n^*(y_0^*) = \widetilde{x_n^*}$

Hence, we have that

$$\begin{aligned}\|\phi_n^{y_0^*}\| &= \sup\{\langle x_n^{**}, T^* y_0^*\rangle : T \in \mathcal{L}_r(E,F),\ \|T\| \leq 1\}\\ &\geq \langle x_n^{**}, T_n^* y_0^*\rangle = \langle x_n^{**}, \widetilde{x_n^*}\rangle \approx \|x_n^{**}\|,\end{aligned}$$

which shows that $\|x_n^{**}\| \to 0$. Thus $E^*$ has DPSP.

Let us now show that $F$ has DPSP. To this end, let $(y_n^*) \subset F_+^*$ be a weak-$*$ null sequence, i.e. $\langle y_n^*, y\rangle \to 0$ for every $y \in F$. Pick an arbitrary $x_0 \in E$ with $\|x_0\| = 1$, and let us define $\psi_n^{x_0} \in (\mathcal{L}_r(E,F))_+^*$ by

$$\psi_n^{x_0}(T) = \langle T x_0, y_n^*\rangle.$$

Observe that for a fixed $T \in \mathcal{L}_r(E,F)$, we have that $\psi_n^{x_0}(T) \to 0$, and since $\mathcal{L}_r(E,F)$ has DPSP, it follows that $\|\psi_n^{x_0}\| \to 0$.

Now, for each $n \in \mathbb{N}$, let us consider $\widetilde{y_n} \in E_+$ with $\|\widetilde{y_n}\| \leq 1$ such that

$$\langle y_n^*, \widetilde{y_n}\rangle \approx \|y_n^*\|,$$

and take also $x_0^* \in E_+^*$ with $\|x_0^*\| \leq 1$ such that $\langle x_0^*, x_0\rangle \approx 1$. This allows us to define the operators $T_n : E \to F$ given by

$$T_n(x) = \langle x_0^*, x\rangle \widetilde{y_n}.$$

Hence, we have that

$$\begin{aligned}\|\psi_n^{x_0}\| &= \sup\{\langle T x_0, y_n^*\rangle : T \in \mathcal{L}_r(E,F),\ \|T\| \leq 1\}\\ &\geq \langle T_n x_0, y_n^*\rangle = \langle x_0^*, x_0\rangle\langle y_n^*, \widetilde{y_n}\rangle \approx \|y_n^*\|,\end{aligned}$$

which shows that $\|y_n^*\| \to 0$. Thus $F$ has DPSP.

(2) $\Rightarrow$ (1): Suppose now that $E^*$ and $F$ have DPSP. Let us consider a positive operator

$$R : \mathcal{L}_r(E,F) \to c_0\,.$$

For each $x^* \in E^*$ and each $y \in F$, consider $x^* \otimes y \in \mathcal{L}_r(E,F)$ given by

$$x^* \otimes y(x) = \langle x^*, x\rangle y.$$

Let us denote

$$R(x^*\otimes y)=\big(a_n(x^*,y)\big).$$

This allows us to define a sequence of positive operators $T_n : E^* \to F^*$ by means of the identity

$$\langle T_n x^*, y\rangle = a_n(x^*,y).$$

Now, for every $x^*\in E^*$ and every $y\in F$, since $R$ takes values in $c_0$ we have that

$$\langle T_n x^*, y\rangle = a_n(x^*,y)\to 0.$$

Moreover, since $E^*$ and $F$ have DPSP, by Theorem 3, it follows that $\mathcal{L}_r(E^*,F^*)$ has $w^*ot$-PSP. Therefore, we have that $\|T_n\|\to 0$. Now, if $(E,F)$ satisfy condition $(*)$, then for every positive operator $T: E\to F$ with $\|T\|\le 1$ and every $\varepsilon$ there is $u\in E^*\otimes_{|\pi|}F$, $\|u\|_{E^*\otimes_{|\pi|}F}\le 1$, and $S\in\mathcal{L}_r(E,F)$ with $\|S\|\le\varepsilon$ such that

$$T\le |u|+S.$$

Therefore, for every $\varepsilon>0$ we have

$$0\le R(T)\le R(|u|)+R(S)\le (\|T_n\|+\|R\|\varepsilon).$$

This implies that for every $\varepsilon>0$ we have

$$R(B_{\mathcal{L}_r(E,F)})\subset [0,(\|T_n\|)]+\varepsilon B_{c_0}.$$

Since the order interval $[0,(\|T_n\|)]$ is compact in $c_0$, it follows that $R$ is a compact operator. Thus, $\mathcal{L}_r(E,F)$ has DPSP.

□

As a consequence, taking Example 1 into consideration, we get that $\mathcal{L}_r(L_1(\mu),L_\infty(\nu))$ has DPSP. Notice that in this case, $\mathcal{L}_r(L_1(\mu),L_\infty(\nu))$ is an $AM$-space (see [13]).

Theorem 4 was partially motivated by the following question: If $E$ has DPSP, must $E^{**}$ have also DPSP? Note that for the class of AM-spaces this has affirmative answer. Recall also that there are examples of spaces with PSP (or even Schur property) whose bidual fails to have PSP:

**Example 2.** It is easy to check that the space $E=\big(\bigoplus_{n\in\mathbb{N}}\ell_2^n\big)_{\ell_1}$ has the Schur property. The bidual $E^{**}$ would have the PSP if and only if every positive weakly compact operator on $E^*$ is $M$-weakly compact (cf. [16]). However, the space $E^*=\big(\bigoplus_{n\in\mathbb{N}}\ell_2^n\big)_{\ell_\infty}$ contains a positively complemented copy of $\ell_2$ [4]. The corresponding projection $P: E^*\to\ell_2$ provides a weakly compact operator which cannot be compact (equivalently, $M$-weakly compact), hence $E^{**}$ fails to have PSP.

## 4. Positive Grothendieck properties in $\mathcal{L}_r(E,F)$

Recall that a Banach lattice $E$ has the positive Grothendieck property (PGP) if weak-$*$ null sequences of positive elements in $E^*$ are also weakly null. The PGP coincides with the Grothendieck property for Banach lattices with the interpolation property [11, Theorem 5.3.13]. Recall there is a close connection between PSP and DPSP: a Banach lattice $E$ has DPSP if and only if $E^*$ has PSP and $E$ has PGP.

In analogy with the above defined properties $wot$-PSP and $w^*ot$-PSP for $\mathcal{L}_r(E,F^*)$ we also have a version of PGP.

**Definition 4.** *Given Banach lattices $E$ and $F$, we say that $\mathcal{L}_r(E,F^*)$ has $w^*ot$-positive Grothendieck property ($w^*ot$-PGP) if for every sequence of positive operators $(T_n)\subset\mathcal{L}_+(E,F^*)$ with $T_n\to 0$ in the $w^*ot$, we have $T_n\to 0$ weakly.*

Recall that we can identify the space $(E\otimes_{|\pi|}F)^*$ with $\mathcal{L}_r(E,F^*)$ as follows: for any $\phi\in(E\otimes_{|\pi|}F)^*$, we consider the operator $T_\phi:E\to F^*$ given by

$$\langle T_\phi x,y\rangle=\phi(x\otimes y)$$

for every $x\in E$, $y\in F$.

**Theorem 5.** *Given Banach lattices $E$ and $F$, the following are equivalent:*

(1) *$E$ and $F$ have PGP.*
(2) *$\mathcal{L}_r(E,F^*)$ has $w^*ot$-PGP.*
(3) *$E\otimes_{|\pi|}F$ has PGP.*

*Proof.* (1) $\Rightarrow$ (2) For an operator $T\in\mathcal{L}_r(E,F^*)$ let us consider $\Phi(T)\in C(B_{E_+^{**}}\times B_{F_+^{**}})$ given by

$$\Phi(T)(x^{**},y^{**})=\langle T^{**}x^{**},y^{**}\rangle$$

for each $x^{**}\in B_{E_+^{**}}$ and $y^{**}\in B_{F_+^{**}}$. It is clear that $\Phi$ defines a positive linear mapping between $\mathcal{L}_r(E,F^*)$ and $C(B_{E_+^{**}}\times B_{F_+^{**}})$ with

$$\|\Phi(T)\|_{C(B_{E^{**}}\times B_{F^{**}})}=\|T^{**}\|=\|T\|.$$

Now, if $(T_n)\subset\mathcal{L}_+(E,F^*)$ is a sequence converging to 0 in the $w^*ot$, this means that for every $x\in E$ and $y\in F$ we have that

$$\langle T_nx,y\rangle\to 0.$$

In particular, for a fixed $x\in E_+$ we have that $T_nx\xrightarrow{w^*}0$ in $F^*$, which by the PGP of $F$ implies that

$$\langle T_nx,y^{**}\rangle\to 0$$

for every $y^{**}\in F^{**}$. Therefore, for a fixed $y^{**}\in F_+^{**}$, we have that $T_n^*y^{**}\xrightarrow{w^*}0$ in $E^*$, and since $E$ has PGP this yields that

$$\langle T_n^{**}x^{**},y^{**}\rangle=\langle x^{**},T_n^*y^{**}\rangle\to 0$$

for every $x^{**}\in B_{E_+^{**}}$ and $y^{**}\in B_{F_+^{**}}$.

Hence $(\Phi(T_n))$ converges to 0 weakly in $C(B_{E_+^{**}}\times B_{F_+^{**}})$, thus so does $(T_n)$ in $\mathcal{L}_r(E,F^*)$. Therefore, $\mathcal{L}_r(E,F^*)$ has $w^*ot$-PGP.

(2) ⇒ (3) Suppose that $\mathcal{L}_r(E, F^*)$ has $w^*ot$-PGP, and let us use the identification $(E \otimes_{|\pi|} F)^* = \mathcal{L}_r(E, F^*)$ used before: for any $\phi \in (E \otimes_{|\pi|} F)^*$, we consider the operator $T_\phi : E \to F^*$ given by

$$\langle T_\phi x, y\rangle = \phi(x \otimes y)$$

for every $x \in E$, $y \in F$.

Now, for any weak-$*$ null sequence $(\phi_n)_n \subset (E \otimes_{|\pi|} F)^*_+$ we have a sequence of operators $(T_{\phi_n})_n$ in $\mathcal{L}_+(E, F^*)$ tending to 0 in the $w^*ot$. Therefore, if $\mathcal{L}_r(E, F^*)$ has $w^*ot$-PGP, it follows that $T_{\phi_n} \to 0$ weakly, and so does the corresponding sequence $(\phi_n)_n$. This proves that $E \otimes_{|\pi|} F$ has PGP.

(3) ⇒ (1) Let $(x_n^*) \subset E^*_+$ with $x_n \to 0$ in the weak-$*$ topology. For a fixed $y_0^* \in F^*_+$, we can consider the sequence $\varphi_n \in (E \otimes_{|\pi|} F)^*$ given by

$$\langle \varphi_n, x \otimes y\rangle = \langle x_n^*, x\rangle\langle y_0^*, y\rangle.$$

It is clear that $(\varphi_n)$ is a positive weak-$*$ null sequence in $(E \otimes_{|\pi|} F)^*$, and by hypothesis we have that $\varphi_n \to 0$ weakly. Since $E^{**} \otimes_{|\pi|} F^{**}$ can be identified with a subspace of $(E \otimes_{|\pi|} F)^{**}$, in particular we have that for every $x^{**} \in E^{**}$ and some $y_0^{**}$ with $\langle y_0^{**}, y_0^*\rangle \neq 0$ that $\varphi_n(x^{**} \otimes y_0^{**}) \to 0$. This yields that $\langle x_n^*, x^{**}\rangle \to 0$ for every $x^{**} \in E^{**}$. Thus, $E$ has PSP. By symmetry the same holds for $F$. □

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