POSITIVELY NORMING SETS IN BANACH FUNCTION SPACES

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Abstract. The notion of positively norming set, a specific definition of norming type sets for Banach lattices, is analyzed. We show that the size of positively norming sets (in terms of compactness and order boundedness) is directly related to the existence of lattice copies of L^1-spaces. As an application, we provide a version of Kadec-Pelczynski's dichotomy for order continuous Banach function spaces. A general description of positively norming sets using vector measure integration is also given.

1. Introduction

Norming sets in Banach spaces have shown to be useful tools for studying several geometric and topological properties and have produced an important number of results in the literature ([24, 25, 26]). Recall that a subset B of the dual E^* of a Banach space E is said to be norming (or weak*-norming) if $\inf_{x\in S_E}\sup_{x^*\in B}|\langle x, x^*\rangle|>0$, where S_E is the unit sphere of E. In this paper we are interested in a specific definition of norming type set for Banach lattices, and in particular for Banach function spaces.

Definition 1.1. Given a Banach function space $X(\mu)$ a set $N\subset B_{X'}^+$ will be called *positively α-norming* (for $0<\alpha\leq 1$) if

$$\inf_{\|f\|_X=1}\ \sup_{g\in N}\int |f|g\,d\mu=\alpha.$$

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We will say that N is positively norming if it is positively α-norming for some $\alpha \in (0, 1]$.

Our motivation in this work stems mainly from two facts. On the one hand, there are several recent results relating the weak* closure of norming sets with embedding isomorphic copies of ℓ^1 ([5, 15]). In this direction, in this paper we will show that in fact the ℓ^1-structure of a Banach lattice is closely related to the compactness of its positively norming sets (see Theorem 3.2).

On the other hand, the notion of positively norming set has shown to be useful in recent applications of the representation of order continuous Banach lattices as spaces of integrable functions with respect to vector measures. This tool provides a characterization of those subspaces of Banach function spaces that are strongly embedded in L^1-spaces (see [3]). Actually, this technique has shown that in the setting of spaces of integrable functions the notion of positively norming set seems to be more natural and useful than that of norming set. Thus, the aim of this paper is to study positively norming sets in Banach function spaces and give some applications to the lattice structure of the spaces —mainly its ℓ^1-structure— using vector measure representations.

The paper is organized in five sections. After the introductory Sections 1 and 2, Section 3 is devoted to present our main results regarding characterizations by means of positively norming sets of geometric and topological properties of Banach function spaces. In Section 4, a version of Kadec-Pelczynski's disjointification method in terms of the measure of non-compactness of positively norming sets is given. Finally, in Section 5 we show how integration with respect to vector measures can be used to obtain concrete representations of positively norming sets for order continuous Banach lattices. These results are related to the recent developments on the spaces of integrable functions with respect to a vector measure given in [3, 11].

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2. Basic facts concerning positively norming sets

Let us start by recalling some fundamental results on norming sets for Banach spaces. A particular case of norming sets are the so called James boundaries, or simply boundaries, that are defined by the following property: for every $x \in E$, there is $x^* \in B$ such that $\|x\| = \langle x, x^* \rangle$. These sets have shown to be useful for describing the weak topology of the space. Rainwater's Theorem and Simons further developments (see [25, 26]) establish that a bounded sequence $(x_n)_{n=1}^{\infty}$ in E converges weakly to x if and only if $(\langle x_n, x^* \rangle)_{n=1}^{\infty}$ converges to x for every $x^* \in B$. The general version of this result is due to Pfitzner, and states that a bounded set of E is weakly compact if and only if it is compact for the topology $\sigma(B, E)$ ([24]).

In our approach, the measure of non-compactness of a positively norming set has turned out a useful tool for our purposes. The measure of non-compactness (for operators) on Banach lattices have been widely used to quantify the relation between the compactness properties and the order properties of operators on Banach lattices (see [9], [27]).

Let us recall now some definitions concerning Banach lattices and Banach function spaces. If F is a Banach lattice, we write F^+ for its positive cone as usual. Let (Ω, Σ, μ) be a σ-finite measure space. A Banach function space $X(\mu)$ over μ (X for short) is a Banach ideal of classes of μ-a.e. equal locally integrable functions endowed with a lattice norm that contains all characteristic functions of sets of finite measure. If $f, g \in X(\mu)$ we write $f \vee g$ for the pointwise supremum of both functions. A Banach function space is said to be order continuous if for each decreasing sequence of positive functions $(f_n)_{n=1}^{\infty}$, $f_n \downarrow 0$ implies $\lim_n \|f_n\| = 0$. If $X(\mu)$ is order continuous, its dual space X^* can be represented as the function space $X'(\mu)$, that is called the Köthe dual of $X(\mu)$ —i.e. the elements of the dual can be represented as integrals—, and the converse is also true. A weak unit is a function $f > 0$ of $X(\mu)$. Throughout the paper, B_X will denote the unit ball of X, and we will write $B_{X'_+}$ for the set $B_{X'} \cap (X')^+$. For a general Banach lattice, we will use also the symbol $B_{X^*_+}$ for the set $B_{X^*} \cap (X^*)^+$. For unexplained terminology on Banach lattices, Banach function spaces and vector measure integration we refer to [19, 20, 23].

For a Banach space E and a vector valued countably additive measure $m : \Sigma \to E$ we write $\langle m, x' \rangle$ for the measure defined by composing m with $x' \in E^*$, and $|\langle m, x' \rangle|$ for its variation. The function $\|m\| : \Sigma \to \mathbb{R}^+$ given by $\|m\|(A) := \sup_{x' \in B_{E^*}} |\langle m, x' \rangle|(A)$ is called the semi-variation of $\|m\|$ in A, and the total variation of $\|m\|$ is defined as $\|m\|(\Omega)$. A Rybakov measure for m is a positive scalar measure arising as $|\langle m, x_0' \rangle|$ for some element $x_0' \in E^*$, which moreover "controls" m, that is $|\langle m, x_0' \rangle|(A) = 0$ if and only if $\|m\|(A) = 0$. It is well known that such a measure always exists (see [10, Ch.IX,2]).

Recall also, that a Σ-measurable function is said to be weakly integrable with respect to m if it is integrable with respect to all the scalar measures $|\langle m, x' \rangle|$, $x' \in E^*$. Besides, if for each $A \in \Sigma$ there is an element $\int_A f dm$ in E such that $\langle \int_A f dm, x' \rangle = \int_A f d\langle m, x' \rangle$ for all $x' \in X^*$, then we say that f is integrable with respect to m. When classes of $\|m\|$-a.e. equal functions are considered as elements of a linear space, the Banach function spaces $L^1_w(m)$ —weakly integrable functions— and $L^1(m)$ —integrable functions— arise. These are Banach function spaces over any Rybakov measure $|\langle m, x_0' \rangle|$ with the a.e. order and the norm $\|f\| := \sup_{x' \in B_{E^*}} \int |f| d|\langle m, x' \rangle|$. For $1 < p < \infty$ the spaces $L^p(m)$ and $L^p_w(m)$ are defined as in the scalar measure case as the p-convexifications of the spaces $L^1(m)$ and $L^1_w(m)$, respectively (see [23, Chapter 3]).

Recall that an important class of norming sets are w*-thick sets (introduced by Fonf in [13]) as those that cannot be written as increasing countable unions of non-norming sets (see [1, 11]). This class has been recently applied in the context of the vector valued integration of measurable functions in [1] and [11]. In the first one, a characterization of weak*-thick sets by means of integrability properties of a certain class of functions defined by the set is given (see [1]). This result has its roots in classical facts about vector measures involving total sets (see [13, Theorem 1], and [10, p.16, p.54]). In the second one, it is shown that the set of weakly integrable functions with respect to a vector measure m coincides with the corresponding set of weakly integrable functions when only the elements of a weak*-thick set are considered for defining the set of scalar measures associated to m ([11, Theorem 2.2]).

Let us briefly introduce now the relation between vector measures and positively norming sets. Recall that an order continuous Banach lattice with weak unit can always be represented isometrically as an $L^1(m)$ space of a Banach space valued vector measure $m : \Sigma \to E$ (see [23, Chapter 3, Proposition 3.9] and the references therein). For instance, if E is a Banach function space, the vector measure m that provides the isometry between E and $L^1(m)$ is given by $A \rightsquigarrow m(A) := \chi_A \in E$, $A \in \Sigma$. These results suggest that in order to study norming type sets for a Banach lattice it is enough to consider norming sets for spaces $L^1(m)$. The advantage of this approach is that these sets have a canonical description as follows. Let μ be a Rybakov measure for a vector measure m. The set of "Radon-Nikodym derivatives"

$$R(m) = \{h \in (L^1(m))' : h = \frac{d|\langle m, x^* \rangle|}{d\mu},\ x^* \in B_{E^*}\}$$

is positively 1-norming for $L^1(m)$ and has some special properties. For example, in the case mentioned above ($m(A) := \chi_A$), a direct computation shows that the set $\mathcal{R}(m)$ is simply the positive cone of the dual space E'. We will show in Section 5 that in fact, all positively norming sets of an order continuous Banach lattice with a weak unit can be written essentially as such a set $R(m)$ up to convex hulls. In fact, the definition of positively norming set appears originally in this setting with the aim of characterizing subspaces of p-convex order continuous Banach function spaces that are strongly embedded in an L^1-space: it is known (see Theorem 3.2 in [3]) that the norm of every such subspace S of $L^p(m)$ can be computed (equivalently) by means of $\|f\|_\nu = \int_{R(m)} \langle |f|, h \rangle \, d\nu(h)$, where ν is a regular Borel measure on a subset $R(m)$ for m defined as above.

Let us start now with some basic facts regarding positively norming sets. Consider S, the subset of $B_{L^\infty(\mu)}$ given by all the functions as $S := \{\chi_A - \chi_{A^c} : A \in \Sigma\}$, where A^c denotes the complement $\Omega \setminus A$ of $A \in \Sigma$. The symbol $M \cdot H$ denotes the pointwise product of the sets of functions M and H. The proof of the following result is straightforward.

Lemma 2.1. *Let $X(\mu)$ be an order continuous Banach function space. For a subset $N \subseteq B^+_{X^*}$, the following statements are equivalent:*

(1) *N is positively norming.*

(2) *$S \cdot N$ is norming.*
(3) *The w^*-closed convex hull of $S \cdot N$ contains a ball.*

The next lemma provides the first instance of the relation between Banach lattice properties of positively norming sets and those of the corresponding space.

Lemma 2.2. *Let $X(\mu)$ be a Banach function space over a finite measure μ. $X(\mu)$ is order continuous if and only if every positively norming set N for $X(\mu)$ is uniformly integrable (or equivalently, it is relatively weakly compact as a subset of $L^1(\mu)$).*

Proof. Since μ is finite, we have the inclusion

$$i : L^\infty(\mu) \hookrightarrow X(\mu).$$

If $X(\mu)$ is order continuous, the interval $[-\chi_\Omega, \chi_\Omega] \subseteq X(\mu)$ is weakly compact ([20, Theorem 2.4.2]) and so the inclusion i is weakly compact. By the order continuity of $X(\mu)$ we have $X(\mu)^* = X(\mu)'$, and so by duality

$$i^* : X(\mu)' \hookrightarrow L^1(\mu)$$

is also weakly compact. Since $N \subseteq B_{X(\mu)'}$ for each positively norming set, we have the direct implication.

For the converse, notice that the positively norming set $B_{X'_+}$ is included in $L^1(\mu)$, and so all the elements of $X(\mu)^*$ can be identified with integrable functions, i.e., the Köthe dual coincides with the topological dual. This implies order continuity of $X(\mu)$ (see [19, p.29]). □

3. Geometry of Banach lattices and positively norming sets

In this section we analyze the relation between Banach lattice geometry and compactness properties of positively norming sets. In particular, we study the connection between L^1-subspaces and compact subsets of a positively norming set, order properties, disjointness and equi-integrability.

3.1. **Norm compactness and AL-spaces.** The size of norming sets for Banach spaces has shown to be directly related to the L^1-structure of their subspaces. The reader can find in the references [5, 14, 15, 16] a deep analysis regarding the existence of isomorphic copies of ℓ^1 in Banach spaces with small norming sets. In the case of positively norming sets, this relation is even more direct. It is clear that for $L^1(\mu)$ over a σ-finite measure space (Ω, Σ, μ) we can consider the positively norming set $\{\chi_\Omega\} \subset L^\infty(\mu)$, since $\|f\|_{L^1(\mu)} = \int |f|\chi_\Omega d\mu$, $f \in L^1(\mu)$.

Therefore, in this case we can take a positively norming set consisting of a single element. Notice that a norming set on an infinite dimensional Banach space must always be an infinite set, so there might be a big contrast between the size of positively norming sets and norming sets on a Banach lattice. Our next result provides a characterization of AL-spaces in terms of compactness properties of positively norming sets. Let us introduce some helpful notation before.

Definition 3.1. Given a Banach function space $X(\mu)$, for each positively norming set $N \subset B_{X'_+}$ we define the measure of non-compactness of N in X' as

$$\kappa(N, X') = \inf\{\varepsilon > 0 : \exists x'_1, \dots, x'_k \in X', \text{ with } N \subset \bigcup_{j=1}^{k} B(x'_k, \varepsilon)\},$$

where $B(x', \varepsilon)$ is the ball of center x' and radius ε in X'.

It is clear from the definition that a positively norming set N is relatively compact in $X'(\mu)$ if and only if $\kappa(N, X') = 0$.

Theorem 3.2. *Let $X(\mu)$ be an order continuous Banach lattice over a finite measure μ. The following statements are equivalent:*

(1) *The norm of $X(\mu)$ is equivalent to the norm of $L^1(\mu)$.*
(2) *There exists $g \in B_{X'_+}$ such that $\{g\}$ is positively norming.*
(3) *There exists a norm-compact positively norming set in $B_{X'_+}$.*
(4) *There exist $\alpha \in (0, 1]$ and a positively α-norming set $N \subset B_{X'_+}$ with $\kappa(N, X') < \alpha$.*
(5) *There exists an order bounded positively norming set in $B_{X'_+}$.*

Proof. (1) $\Rightarrow$ (2) As was mentioned above, we can take $g = \chi_\Omega$. It is clear that

$$\int_\Omega |f| d\mu = \|f\|_{L^1} \geq \alpha \|f\|_X$$

for some $\alpha \in (0,1]$. Since $X(\mu) \subset L^1(\mu)$ —recall that all the functions in X are locally integrable and μ is finite—, the converse inequality always holds.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (5) Let $N \subset B_{X'_+}$ be a positively α-norming set with $\kappa(N, X') < \alpha$, which means that there exist $\delta \in (0, \alpha)$ and $x'_1, \ldots, x'_k \in B_{X'}$ such that

$$N \subset \bigcup_{j=1}^{k} B(x'_j, \delta).$$

Now, for every $f \in X$ and $\varepsilon > 0$ there is $g \in N$ such that

$$(1-\varepsilon)\alpha \|f\|_X \leq \int |f| g d\mu.$$

Let us pick x'_j such that $\|g - x'_j\|_{X'} < \delta$. It follows that

$$\begin{aligned}(1-\varepsilon)\alpha\|f\|_X &\leq \textstyle\int |f|gd\mu \leq \int |f||g-x'_j|d\mu + \int |f||x'_j|d\mu \\ &\leq \delta\|f\|_X + \textstyle\int |f| \bigvee_{j=1}^{k} |x'_j| d\mu.\end{aligned}$$

Hence, for every $f \in X$

$$((1-\varepsilon)\alpha - \delta)\|f\|_X \leq \int |f| \bigvee_{j=1}^{k} |x'_j| d\mu.$$

Taking ε small enough, this means that $\| \bigvee_j |x'_j| \|^{-1} \bigvee_j |x'_j|$ is positively norming and order bounded.

(5) $\Rightarrow$ (1) Let $N \subset B_{X'_+}$ be an order bounded positively norming set for X. Let $\alpha \in (0,1]$ such that

$$\alpha \|f\|_X \leq \sup_{g \in N} \int |f| g d\mu$$

Since N is order bounded, there is $h \in X'$ such that $|g| \leq h$ for every $g \in N$. Moreover, since $L^\infty(\mu) \subset X(\mu) \subset L^1(\mu)$, we have that the inclusion $L^\infty(\mu) \subset X'(\mu)$ has dense image. In particular, we can take $h_0 \in L^\infty(\mu)$ such that

$$\|h - h_0\|_{X'} \leq \frac{\alpha}{2}.$$

Now, we have

$$\alpha \|f\|_X \leq \sup_{g \in N} \int |f| g d\mu \leq \int |f| h \leq \int |f||h - h_0| d\mu + \int |f||h_0| d\mu.$$

We can therefore conclude that

$$\frac{\alpha}{2\|h_0\|_{L^\infty}}\|f\|_X \leq \|f\|_{L^1}.$$

Since there is a constant $k > 0$ such that the inequality $\|f\|_{L^1} \leq k\|f\|_X$ holds for every $f \in X$, this proves (1). □

3.2. **Disjoint elements in positively norming sets.** Notice that in the particular case when $X = c_0$ we can consider a positively norming set for X, $N = \{e_n^* : n \in \mathbb{N}\}$, given by the unit vector basis of $c_0^* = \ell^1$, since for every $x = (x_n) \in c_0$:

$$\|x\|_{c_0} = \sup_n |x_n| = \sup_n \langle |x|, e_n^* \rangle.$$

Therefore, in this case we can take a positively norming set which consists of disjoint terms. More generally we have:

Theorem 3.3. *Let $X(\mu)$ be an order continuous Banach function space. The following are equivalent:*

1. *There is a positively norming set $N = \{g_\gamma : \gamma \in \Gamma\}$ which consists of disjoint elements.*
2. *$X(\mu)$ is lattice isomorphic to a c_0 (disjoint) sum of L^1-spaces:*

$$\Big(\oplus_{\gamma \in \Gamma} L^1(\mu_\gamma) \Big)_{c_0}$$

(where the spaces $L^1(\mu_\gamma)$ might be finite dimensional, i.e. ℓ_n^1.)

Proof. Let us see first that (1) ⇒ (2). Recall that an order continuous Banach function space is order complete. Let $N = \{g_\gamma : \gamma \in \Gamma\}$ be a positively norming set for X with $g_\gamma \wedge g_\beta = 0$ for any $g_\gamma \neq g_\beta$ in N. For each $\gamma \in \Gamma$ let $\Omega_\gamma = \{\omega \in \Omega : g_\gamma(\omega) \neq 0\}$, which are disjoint sets in Ω; otherwise, we would find a non trivial element f in X not supported in the union of the sets Ω_γ. Notice that since N is positively norming for X, $\Omega = \bigcup_{\gamma \in \Gamma} \Omega_\gamma$. Let us denote

$$X_\gamma = \{f \in X : \, f = f\chi_{\Omega_\gamma}\}.$$

For each $\gamma \in \Gamma$, we can endow X_γ with the norm

$$\|f\|_\gamma = \int |f| g_\gamma d\mu$$

which makes this space isomorphic to a space of the form $L^1(\mu_\gamma)$ for some scalar measure μ_γ. Now, given any $f \in X$ we can consider $P_\gamma(f) = f\chi_{\Omega_\gamma} \in X_\gamma$ and since N is positively norming we have

$$\|f\|_X \approx \sup_{\gamma\in\Gamma} \int |f| g_\gamma d\mu = \sup_{\gamma\in\Gamma} \int \bigvee_{\beta\in\Gamma} P_\beta(|f|) g_\gamma d\mu = \sup_{\gamma\in\Gamma} \|P_\gamma(f)\|_{L^1(\mu_\gamma)}.$$

This shows that X is isomorphic to $\Big(\oplus_{\gamma\in\Gamma} L^1(\mu_\gamma)\Big)_{c_0}$ as claimed.

To see that (2) $\Rightarrow$ (1), let

$$T : X(\mu) \to \Big(\oplus_{\gamma\in\Gamma} L^1(\mu_\gamma)\Big)_{c_0}$$

be a lattice isomorphism. Without loss of generality we can suppose that $\|T^{-1}\| \geq 1$. Let us denote the support of each μ_γ by Ω_γ, and let us consider

$$N = \{T^*(\chi_{\Omega_\gamma}) : \gamma \in \Gamma\} \subset B_{X'_+}.$$

Since Ω_γ are disjoint measure spaces, $T^*(\chi_{\Omega_\gamma})$ are also disjoint. Let us see that N is positively α-norming for $\alpha = 1/\|T^{-1}\|$. Indeed, for every $f \in X$ we have

$$\begin{aligned} \frac{1}{\|T^{-1}\|}\|f\|_X &\leq \|T(f)\|_{\Big(\oplus_{\gamma\in\Gamma} L^1(\mu_\gamma)\Big)_{c_0}} = \sup_{\gamma\in\Gamma} \int_{\Omega_\gamma} |T(f)| d\mu \\ &= \sup_{\gamma\in\Gamma} \int T(|f|)\chi_{\Omega_\gamma} d\mu = \sup_{\gamma\in\Gamma} \int |f| T^*(\chi_{\Omega_\gamma}) d\mu. \end{aligned}$$

□

Notice that σ-finiteness of the measure is not essential in the proof above.

Let us finish this section by analyzing the natural ℓ^p-version of the previous theorem. The result above motivates the following definition, that will be used also in Section 5. Let $1 < p < \infty$. Given a sequence of positive measurable functions $N = (f_n)_{n=1}^\infty$, we define the p-convex cover $co_p(N)$ of N as the set of all (measurable) functions $\sum_{n=1}^\infty a_n f_n$ (defined pointwise), where $\sum_{n=1}^\infty a_n^p \leq 1$.

Definition 3.4. Given a Banach function space $X(\mu)$, a sequence $N = (x'_n)_{n=1}^{\infty}$ of pairwise disjoint functions in $B_{X'_+}$ is a (p)-convex 1-norming set for $X(\mu)$ if for all $f \in X$,

$$\|f\|_{X(\mu)} = \sup_{g \in co_{p'}(N)} \int |f| g d\mu,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proposition 3.5. *Let $1 < p < \infty$. Let $X(\mu)$ be an order continuous Banach function space and consider a disjoint sequence $N = (x'_n)_{n=1}^{\infty}$ of $B_{X'_+}$. Then N is a (p)-convex 1-norming set for $X(\mu)$ if and only if the norm $\|\cdot\|_{X(\mu)}$ coincides with the norm $|||f|||_{N,p} := \Big(\sum_{i=1}^{\infty} \Big(\int |f| x'_i d\mu \Big)^p \Big)^{1/p}$, $f \in X(\mu)$.*

Proof. Take a (p)-convex 1-norming disjoint sequence $\{x_n\}_{n=1}^{\infty}$ and write Ω_n for the support of each x'_n. The following calculation for the norm shows the "only if" part of the proof.

$$\begin{aligned} \|f\| &= \sup_{g \in co_{p'}(N)} \int |f| g d\mu = \sup_{(a_n) \in B_{\ell^{p'}}} \int |f| (\sum_{n=1}^{\infty} a_n x'_n) d\mu \\ &= \sup_{(a_n) \in B_{\ell^{p'}}} \sum_{n=1}^{\infty} a_n \int |f| x'_n d\mu = \Big(\sum_{n=1}^{\infty} \Big(\int |f| x'_n d\mu \Big)^p \Big)^{1/p} \end{aligned}$$

for all $f \in X(\mu)$. The proof of the converse follows the same lines.

□

The disjointness of (p)-convex norming sets allows us to prove the following theorem that is the natural generalization of the result for c_0 given in Theorem 3.3.

Theorem 3.6. *Given an order continuous Banach function space $X(\mu)$ the following are equivalent:*

(1) *There is a (p)-convex 1-norming disjoint sequence $N = \{x_n\}_{n=1}^{\infty}$ for $X(\mu)$.*
(2) *$X(\mu)$ is lattice isometric to an ℓ^p-sum of L^1 spaces as*

$$\Big(\oplus_{n=1}^{\infty} L^1(\mu_n) \Big)_{\ell^p},$$

where the spaces $L^1(\mu_n)$ might be finite dimensional, i.e. ℓ^1_k.

The proof is a consequence of the same kind of arguments that proves Theorem 3.3. Finally, let us provide a simple application of this result that characterizes ℓ^p-type spaces.

Corollary 3.7. *Let $1 < p < \infty$. Let $X(\mu)$ be a Banach function space. Assume that there is a (p)-convex 1-norming set for $X(\mu)$. Then $X(\mu)$ is p-concave (and, in particular, order continuous). Moreover, $X(\mu)$ is p-convex if and only if $X(\mu)$ is isomorphic to $L^p(\mu)$.*

Proof. By Proposition 3.5, $|||\cdot|||_{N,p}$ equals the norm of $X(\mu)$. Take a finite set of functions $f_1, ..., f_n \in X(\mu)$. Then

$$\begin{aligned}
(\sum_{j=1}^n \|f_j\|_X^p)^{\frac{1}{p}} &= \Big(\sum_{j=1}^n \Big[\Big(\sum_{i=1}^\infty \Big|\int |f_j| x_i' d\mu\Big|^p\Big)^{\frac{1}{p}}\Big]^p\Big)^{\frac{1}{p}} \\
&= \Big(\sum_{i=1}^\infty \sum_{j=1}^n \Big|\int |f_j| x_i' d\mu\Big|^p\Big)^{\frac{1}{p}} \\
&\leq \Big(\sum_{i=1}^\infty \sum_{j=1}^n \Big|\int |f_j|^p x_i' d\mu\Big| \Big[\int x_i' d\mu\Big]^{\frac{p}{p'}}\Big)^{\frac{1}{p}} \\
&\leq K\Big(\sum_{i=1}^\infty \Big|\int \Big[(\sum_{j=1}^n |f_j|^p)^{\frac{1}{p}}\Big]^p x_i' d\mu\Big|\Big)^{\frac{1}{p}} = K\|(\sum_{j=1}^n |f_j|^p)^{\frac{1}{p}}\|.
\end{aligned}$$

This proves the first statement. The proof of the second part is a direct application of the Maurey-Rosenthal Theorem (see [7, Cor.5]). □

3.3. **Non-reflexive Banach lattices.** It is well-known that a Banach lattice is reflexive if and only if it does not contain a subspace isomorphic to ℓ^1 or c_0, or equivalently when it does not contain a sublattice isomorphic to ℓ^1 or c_0 ([19, Theorem 1.c.5], [21]).

Recall that a set M in a Banach function space $X(\mu)$ is called equi-integrable if

$$\lim_{\mu(A)\to 0} \sup_{f\in M} \|f\chi_A\|_{X(\mu)} = 0.$$

Remark 3.8. If there is an equi-integrable positively norming set N for X, then X cannot contain a subspace isomorphic to c_0. In order to see this, let N be a positively norming set for X. If X contains a subspace isomorphic to c_0, then we can find $x_n \in B_X^+$ disjoint such that

$$\|\sum_n a_n x_n\| \approx \sup_n |a_n|.$$

Let $y_n \in N$ such that

$$\int y_n x_n \geq C > 0.$$

If A_n denotes the support of x_n, then $y_n \chi_{A_n}$ are disjoint vectors with $y_n \chi_{A_n} \leq y_n$ such that $\|y_n \chi_{A_n}\| \geq C$. Thus, N cannot be equi-integrable.

Remark 3.9. Let us comment that for positive norming sets of the form $R(m)$ that were considered in [3] (see Section 5) equi-integrability is related to an important property of the integration operator. Recall that for a vector measure $m : \Sigma \to E$ the integration operator $I_m : L^1(m) \to E$ is given by

$$I_m(f) = \int f dm.$$

If $X(\mu)$ is identified with $L^1(m)$, then the set

$$R(m) = \{\frac{d|\langle m, x^* \rangle|}{d\mu} \,:\, x^* \in B_{E^*}\}$$

is equi-integrable precisely when the adjoint of the integration operator I_m is L-weakly compact [20, 3.6]. This is equivalent to I_m being M-weakly compact [20, Prop. 3.6.11], i.e. every norm bounded disjoint sequence (x_n) in $L^1(m)$ must satisfy $\|I_m(x_n)\|_E \to 0$.

When a Banach lattice X contains a sublattice isomorphic to ℓ^1 (which is always complemented [20, Proposition 2.3.11]), it is easy to see that one can find a positively norming set whose size is essentially smaller than the whole $B_{X^*_+}$. Namely,

$$\overline{co}^{w^*}(N) \subsetneq B_{X^*_+}.$$

In the case of reflexive function spaces such as $L^p(\mu)$ $(1 < p < \infty)$, the only positively norming sets one can easily consider correspond to dense subsets of $B_{(L_{p'})_+}$. In this case we cannot hope any strict inclusion as the given above. Even for c_0, for which a positively norming set like $N = \{e_n^* : n \in \mathbb{N}\}$ exists (where e_n^* denotes the n-th unit vector of the basis of $\ell^1 = c_0^*$), it also holds that

$$\overline{co}^{w^*}(N) = B_{\ell_{1+}}.$$

These examples and the philosophy of the results in [5], [14], [15], [16] might lead to the thinking that if for some positively norming set N, $\overline{co}^{w^*}(N)$ is small compared to $B_{X_+^*}$ then X must contain ℓ^1. The following example shows that this is simply not the case:

Example 3.10. Let us consider the Schreier space which consists of sequences of scalars $(a_n) \in c_{00}$ such that

$$\|(a_n)\|_X = \sup\{\sum_{i\in S} |a_i| : S = \{n_1 < n_2 < \ldots < n_k\}, \text{with}\, k \leq n_1\} < \infty.$$

This expression defines a norm which turns its completion into a Banach lattice with the order given by the unit vector basis which does not contain a copy of ℓ^1 (cf. [4]). Let us denote this space by X.

It is easy to check that the set

$$N = \{\mathbb{1}_S : S = \{n_1 < n_2 < \ldots < n_k\}\,\text{with}\, k = n_1\}$$

is positively norming for X (where $\mathbb{1}_S$ denotes the element of X^* given by $\langle \mathbb{1}_S, (a_n)\rangle = \sum_{n\in S} a_n$).

However, $B_{X_+^*} \backslash \overline{co}^{w*}(N)$ is still very big. In particular, it is not a relatively compact set. In fact, the sequence $(\mathbb{1}_{\{2n,2n+1\}})_{n\geq 2}$ belongs to this set and has no convergent subsequence.

Notice that a similar fact also holds for the space $(\bigoplus \ell_n^1)_{c_0}$.

In order to clarify the embedding of ℓ^1 sublattices in our context, let us introduce first some notation. Given $g \in B_{X'_+(\mu)}$ and $\varepsilon > 0$, let us define the sets

$$X_{g,\varepsilon} = \{f \in X^+(\mu) : \|f\|_X \leq (1+\varepsilon)\langle g, f\rangle\}.$$

It is easy to check that the sets $X_{g,\varepsilon}$ are closed and convex in $X(\mu)$. Moreover, a set $N \subset B_{X'_+(\mu)}$ is positively 1-norming for $X(\mu)$ if and only if

$$X^+(\mu) = \bigcap_{\varepsilon>0} \bigcup_{g\in N} X_{g,\varepsilon}.$$

In particular, if a positively norming set N contains only an element g, we have $X^+ = \cap_{\varepsilon>0} X_{g,\varepsilon}$. This is essentially the case we have in Theorem 3.11

(1) below. We will also define the *lattice dimension* of a set $A \subset X(\mu)$ as follows:

$$\dim_{\mathcal{L}}(A) = \sup\{n \in \mathbb{N} : \exists\, x_1, \dots, x_n \in A,\, |x_i| \wedge |x_j| = 0\}.$$

Theorem 3.11. *Let $X(\mu)$ be an order continuous Banach function space over a probability measure μ. It holds that:*

1. *$X(\mu)$ contains a sublattice isomorphic to ℓ^1 if and only if for every $\varepsilon > 0$ there is $g \in B_{X'_+}$ such that $\dim_{\mathcal{L}}(X_{g,\varepsilon}) = \infty$.*
2. *$X(\mu)$ has trivial type if and only if for every $\varepsilon > 0$ it holds that $\sup_{g \in B_{X'_+}} \dim_{\mathcal{L}}(X_{g,\varepsilon}) = \infty$.*

Proof. (1) Suppose first that $X(\mu)$ contains a sublattice isomorphic to ℓ^1. By a classical result of James ([17], see also [22]), for every ε we can find a further sublattice $(1+\varepsilon)$-isomorphic to ℓ^1. Let us denote this sublattice by $U = [u_n]$, where $(u_n)_{n=1}^\infty$ denotes the sequence of positive normalized disjoint elements that correspond to the image of the unit vector basis of ℓ^1 under the above mentioned isomorphism. By [20, Proposition 2.3.11], there is a positive projection $P : X(\mu) \to U \simeq \ell^1$. It is not hard to see from the proof that we can take this projection with $\|P\| \le 1 + \varepsilon$.

Let us consider now $e = (1, 1, \dots) \in \ell^\infty = (\ell^1)^*$, and let $g = P^*(e)/\|P\| \in B_{X'_+}$. It clearly holds that

$$\begin{aligned} \langle g, \big|\sum_{n=1}^\infty a_n u_n\big|\rangle &= \langle g, \sum_{n=1}^\infty |a_n| u_n\rangle = \frac{1}{\|P\|}\langle P^*(e), \sum_{n=1}^\infty |a_n| u_n\rangle \\ &= \frac{1}{\|P\|}\langle e, P\Big(\sum_{n=1}^\infty |a_n| u_n\Big)\rangle = \frac{1}{\|P\|}\sum_{n=1}^\infty |a_n| \ge \frac{1}{1+\varepsilon}\Big\|\sum_{n=1}^\infty a_n u_n\Big\|. \end{aligned}$$

Therefore, $(u_n)_{n=1}^\infty \subset X_{g,\varepsilon}$ which implies that $\dim_{\mathcal{L}}(X_{g,\varepsilon}) = \infty$.

Conversely, suppose that for every $\varepsilon > 0$ there is $g \in B_{X'_+}$ such that $\dim_{\mathcal{L}}(X_{g,\varepsilon}) = \infty$. Let $\varepsilon = 1$, then there is $g_1 \in B_{X'_+}$ such that for every $n \in \mathbb{N}$ we can find positive disjoint elements $x_1^n, \dots, x_n^n$ with

$$\|x_i^n\| = 1, \quad \langle g_1, x_i^n\rangle \ge \frac{1}{2}$$

for $i = 1, \dots, n$. Applying Kadec-Pelczynski's dichotomy ([18], see also Section 4) to the double sequence $(x_i^n)_{1 \le i \le n,\, n \in \mathbb{N}}$ we either have:

(a) there is an almost disjoint subsequence $(x_{i_k}^{n_k})_{k=1}^{\infty}$; or,
(b) there are positive constants k_1, k_2 such that $k_1\|x_i^n\|_{L^1(\mu)} \leq \|x_i^n\|_{X(\mu)} \leq k_2\|x_i^n\|_{L^1(\mu)}$.

Case (a) yields that for any sequence of scalars $(a_k)_{k=1}^{\infty}$ we have

$$\begin{aligned}\sum_{k=1}^{\infty}|a_k| &\geq \Big\|\sum_{k=1}^{\infty}a_k x_{i_k}^{n_k}\Big\| \approx \Big\|\sum_{k=1}^{\infty}|a_k|x_{i_k}^{n_k}\Big\| \geq \langle g_1, \sum_{k=1}^{\infty}|a_k|x_{i_k}^{n_k}\rangle \\ &= \sum_{k=1}^{\infty}|a_k|\langle g_1, x_{i_k}^{n_k}\rangle \geq \frac{1}{2}\sum_{k=1}^{\infty}|a_k|\end{aligned}$$

Now, if case (b) holds, let us see that $(x_i^n)_{1\leq i\leq n,\, n\in\mathbb{N}}$ cannot be equi-integrable in $L^1(\mu)$. In fact, if A_k^n denotes the support of (x_i^n), then since μ is a probability measure, for every $n\in\mathbb{N}$, there is i_n such that $\mu(A_{i_n}^n) \leq \frac{1}{n}$. Moreover, for every $n\in\mathbb{N}$ we have

$$\sup_{1\leq j\leq k,\, k\in\mathbb{N}}\int_{A_{i_n}^n}|x_j^k|d\mu \geq \int_{A_{i_n}^n}|x_{i_n}^n|d\mu = \|x_{i_n}^n\|_{L^1(\mu)} \geq \frac{1}{k_2}.$$

Hence,

$$\lim_{n\to\infty}\sup_{1\leq j\leq k,\, k\in\mathbb{N}}\int_{A_{i_n}^n}|x_j^k|d\mu \geq \frac{1}{k_2}$$

although $\mu(A_{i_n}^n)\to 0$ as $n\to\infty$. This shows as claimed that $(x_i^n)_{1\leq i\leq n,\, n\in\mathbb{N}}$ is not equi-integrable and by [2, Theorem 5.2.9], there is a subsequence $(x_{i_k}^{n_k})_{k\in\mathbb{N}}$ equivalent to the unit vector basis of ℓ^1 in the norm of $L^1(\mu)$ (with constant as close to one as desired). Therefore, we have

$$\sum_{k=1}^{\infty}|a_k| \approx \Big\|\sum_{k=1}^{\infty}a_k x_{i_k}^{n_k}\Big\|_{L^1(\mu)} \leq \Big\|\sum_{k=1}^{\infty}a_k x_{i_k}^{n_k}\Big\|_{X(\mu)} \leq \sum_{k=1}^{\infty}|a_k|.$$

(2) For the second statement, if $X(\mu)$ has trivial type, then for every $\varepsilon>0$ and every $n\in\mathbb{N}$ there exist $x_1,\ldots,x_n$ disjoint in $X(\mu)$ such that

$$\frac{1}{1+\varepsilon}\sum_{i=1}^{n}|a_i| \leq \Big\|\sum_{i=1}^{n}a_i x_i\Big\| \leq \sum_{i=1}^{n}|a_i|,$$

(cf. [19, Theorem 1.f.12]). Let $U=\operatorname{span}\{|x_1|,\ldots,|x_n|\}$. It is clear that U is a sublattice of X, and we can consider biorthogonal normalized positive elements $y_1^*, \ldots, y_n^* \in U^*$ satisfying

$$\max_{j=1,\ldots,n} |b_j| \leq \Big\| \sum_{j=1}^n b_j y_j^* \Big\| \leq (1+\varepsilon) \max_{j=1,\ldots,n} |b_j|.$$

Let $y^* = \frac{1}{1+\varepsilon} \sum_{j=1}^n y_j^*$. By [20, Proposition 1.5.7] y^* extends to a positive functional $g \in B_{X'_+}$, and we have

$$\langle g, |x_i| \rangle = \frac{1}{1+\varepsilon} \|x_i\|$$

which shows that $\{|x_1|, \ldots, |x_n|\} \subset X_{g,\varepsilon}$. Therefore, $\dim_{\mathcal{L}}(X_{g,\varepsilon}) \geq n$, and since for every $n \in \mathbb{N}$ there is such a g, the result follows.

Conversely, if for every $\varepsilon > 0$

$$\sup_{g \in B_{X'_+}} \dim_{\mathcal{L}}(X_{g,\varepsilon}) = \infty,$$

this means that for every $\varepsilon > 0$ and $n \in \mathbb{N}$ there is $g \in B_{X'_+}$, and $x_1, \ldots, x_n$ positive disjoint satisfying

$$\|x_i\| = 1 \quad \langle g, x_i \rangle \geq \frac{1}{1+\varepsilon}.$$

Therefore, it follows that

$$\sum_{i=1}^n |a_i| \geq \Big\| \sum_{i=1}^n a_i x_i \Big\| \geq \langle g, \sum_{i=1}^n |a_i| x_i \rangle = \sum_{i=1}^n |a_i| \langle g, x_i \rangle \geq \frac{1}{1+\varepsilon} \sum_{i=1}^n |a_i|.$$

This shows that $X(\mu)$ contains ℓ_n^1 uniformly, so it has trivial type [19, 1.f.12]. □

It is clear that the spaces given in Example 3.10 have trivial type but do not contain sublattices isomorphic to ℓ_1. Therefore, these spaces satisfy condition (2), but not condition (1) in Theorem 3.11.

Positively norming sets also characterize when a sublattice having an ℓ^1-structure is a band, as the following result shows.

Theorem 3.12. *An order continuous Banach function space $X(\mu)$ contains a band isomorphic to a space $L^1(\nu)$ —where ν is absolutely continuous with respect to μ— if and only if there exists $A \in \Sigma$ with $\mu(A) > 0$ and a positively norming set $N \subset B_{X'_+}$ whose elements restricted to A are order bounded, i.e. $\chi_A \cdot N$ is an order bounded set.*

Proof. Let us suppose first that there is a positively norming set $N \subset B_{X'_+}$, a measurable set A with $\mu(A) > 0$ and $h \in X'_+(\mu)$ such that for every $g \in N$ we have $\chi_A g \le \chi_A h$. Let Y be the band of $X(\mu)$ generated by χ_A, that is

$$Y = \{f\chi_A : f \in X(\mu)\}.$$

For every $y \in Y$ we have that

$$\begin{aligned} \alpha\|y\|_{X(\mu)} &\le \sup_{g\in N} \int g|y|d\mu = \sup_{g\in N} \int_A g|y|d\mu \\ &\le \int_A h|y|d\mu \le \|h\chi_A\|_{X'(\mu)}\|y\|_{X(\mu)}. \end{aligned}$$

Therefore, the band Y is isomorphic to the space $L^1(\nu)$ (where $d\nu = \chi_A h d\mu$).

Conversely, if $Z \subset X(\mu)$ is a band isomorphic to $L^1(\nu)$, then Z is of the form

$$Z = \{f\chi_A : f \in X(\mu)\},$$

for some measurable set A with $\mu(A) > 0$. Moreover, the orthogonal band $Z^\perp$ corresponds to

$$Z^\perp = \{f\chi_{\Omega\setminus A} : f \in X(\mu)\},$$

and we have positive band projections $P : X(\mu) \to Z$ and $P^\perp : X(\mu) \to Z^\perp$ corresponding to the multiplication by χ_A and $\chi_{\Omega\setminus A}$ respectively.

Let $f^* \in Z^*$ be the element corresponding to the constant one function of the space where $L^1(\nu)$ is defined, i.e. $f^* = P^*(\chi_A)$, which gives a positively norming set for $L^1(\nu)$). Now, if $N_0 \subset B_{X'_+}$ is any positively norming set for $Z^\perp$ which is supported outside of A (for instance, $(P^\perp)^*(B_{(Z^\perp)^*_+})$ will work), then

$$N = \{P^*(f^*) + g : g \in N_0\}$$

is positively norming for $X(\mu)$ and satisfies the required conditions. □

4. Kadec-Pelczynski's dichotomy and positively norming sets

In what follows we assume that μ is finite. The motivation of this section is to find a version of the classical Kadec-Pelczynski dichotomy for subspaces of order continuous Banach lattices in terms of positively norming sets. Recall that a simpler version of this dichotomy states that a bounded sequence (x_n) in an order continuous Banach lattice X either has a subsequence equivalent to a disjoint sequence in X, or there is a constant $C > 0$ satisfying

$$\|x_n\|_{L_1} \leq \|x_n\|_X \leq C\|x_n\|_{L_1}$$

for every $n \in \mathbb{N}$ (see [12] and [18]).

Let us introduce first the notion of positively norming set for a subspace of a Banach lattice.

Definition 4.1. Let $X(\mu)$ be an order continuous Banach lattice with weak unit. Given a (closed) subspace $Y \subset X(\mu)$, we say that $N \subset B^+_{X'(\mu)}$ is positively α-norming for Y if

$$\inf_{f\in Y, \|f\|_X=1} \sup_{g\in N} \int |f| g d\mu = \alpha.$$

The proof of next result follows the lines of Theorem 3.2.

Lemma 4.2. *Let Y be a subspace of $X(\mu)$. If there exists some $\alpha \in (0, 1]$ and a positively α-norming set for Y, $N \subset B_{X'_+}$, such that $\kappa(N, X') < \alpha$, then Y is strongly embedded in $L^1(\mu)$ (i.e $\|f\|_X \approx \|f\|_{L^1}$ for $f \in Y$).*

Proof. Let $N \subset B_{X'_+}$ be positively α-norming for Y with $\kappa(N, X') < \alpha$. Thus, there exist $\delta \in (0, \alpha)$ and $x'_1, \ldots, x'_k \in B_{X'}$ such that

$$N \subset \bigcup_{j=1}^{k} B(x'_j, \delta).$$

Since we have a continuous inclusion $X(\mu) \subset L^1(\mu)$, it follows that the adjoint inclusion $L^\infty(\mu) \subset X'(\mu)$ has dense image. Therefore, we can take $y_\delta \in L^\infty(\mu)$ such that

$$\| \max_{j=1,\ldots,k} |x'_j| - y_\delta\|_{X'} \leq \frac{\alpha - \delta}{2}.$$

Now, since N is positively α-norming, for every $f \in Y$ and any $\varepsilon > 0$ there is $g \in N$ such that

$$\alpha\|f\|_X \leq \int |f| g d\mu + \varepsilon.$$

Let us pick x'_j such that $\|g - x'_j\| < \delta$. It follows that

$$\begin{aligned}
\alpha\|f\|_X &\leq \int |f| g d\mu + \varepsilon \leq \int |f||g - x'_j| d\mu + \int |f||x'_j| d\mu + \varepsilon \\
&\leq \delta\|f\|_X + \int |f| \max_{j=1,\dots,k} |x'_j| d\mu + \varepsilon \\
&\leq \delta\|f\|_X + \int |f| (\max_{j=1,\dots,k} |x'_j| - y_\delta) d\mu + \int |f| y_\delta d\mu + \varepsilon \\
&\leq (\delta + \frac{\alpha - \delta}{2})\|f\|_X + \|f\|_{L^1}\|y_\delta\|_{L^\infty} + \varepsilon
\end{aligned}$$

and since this holds for every $\varepsilon > 0$, we have that

$$\frac{\alpha - \delta}{2} \frac{1}{\|y_\delta\|_{L^\infty}} \|f\|_X \leq \|f\|_{L^1},$$

for every $f \in Y$. Since the converse inequality $\|f\|_{L^1} \leq k\|f\|_X$ always holds for some constant $k > 0$, it follows that Y is strongly embedded in $L^1(\mu)$. □

Recall that a sequence $(y_n)_{n=1}^\infty$ of a Banach lattice X is called almost disjoint if there is another sequence $(h_n)_{n=1}^\infty$ in X with $|h_n| \wedge |h_m| = 0$, for $n \neq m$, such that $\|y_n - h_n\|_X \to 0$.

Lemma 4.3. *Let Y be a subspace of $X(\mu)$. If for every $\alpha \in (0,1]$ and every positively α-norming set N for Y, it holds that $\kappa(N, X') \geq \alpha$, then Y contains an almost disjoint normalized sequence.*

Proof. The hypothesis implies in particular that for every $\alpha \in (0,1]$, the set $\{\chi_\Omega\}$ is not positively α-norming for Y. This means that for every $\alpha \in (0,1]$ there is $y \in Y$ such that

$$\|y\|_X = 1 \quad \text{and} \quad \int_\Omega |y| d\mu < \alpha.$$

Observe that such an element y satisfies that

$$\mu(\{t \in \Omega : |y(t)| \geq \sqrt{\alpha}\}) < \sqrt{\alpha}.$$

Actually, otherwise we would have that

$$\alpha > \int_\Omega |y| d\mu \geq \int_{\{t : |y(t)| \geq \sqrt{\alpha}\}} |y(t)| d\mu(t) \geq \sqrt{\alpha}\mu(\{t \in \Omega : |y(t)| \geq \sqrt{\alpha}\}) \geq \alpha$$

which is a contradiction.

Therefore for every $\varepsilon \in (0,1)$ we can find $y_\varepsilon \in Y$ such that $\|y_\varepsilon\|_X = 1$ and

$$\mu(\{t \in \Omega : |y_\varepsilon(t)| \geq \varepsilon\}) < \varepsilon.$$

From this fact, in combination with the order continuity of the space $X(\mu)$ it is not hard to find a sequence of normalized elements y_n in Y and a disjoint sequence h_n in $X(\mu)$ such that $\|y_n - h_n\|_X \to 0$ (see for instance [12, Th.4.1] for details). □

The last two lemmas together provide the following version of Kadec-Pelczynski's dichotomy for a subspace of an order continuous Banach lattice in terms of the measure of non-compactness of its positively norming sets.

Theorem 4.4. *Let $X(\mu)$ be an order continuous Banach lattice with weak unit, and $Y \subset X(\mu)$ be a closed subspace. Then we have that*

(1) *either there exist $\alpha \in (0,1]$ and a positively α-norming set $N \subset B_{X'}$ for Y with $\kappa(N, X') < \alpha$: in this case Y is a strongly embedded subspace of $L^1(\mu)$, i.e $\|f\|_X \approx \|f\|_{L^1}$ for $f \in Y$;*
(2) *or for every $\alpha \in (0,1]$ and every positively α-norming set $N \subset B_{X'}$ for Y it holds that $\kappa(N, X') \geq \alpha$: in this case Y contains an almost disjoint sequence.*

5. Representation of positively norming sets and vector measures

Let $X(\mu)$ be an order continuous Banach function space over a finite measure μ. Hence, $X(\mu)$ is representable as $L^1(m)$ for some measure m, i.e. the identity map is an order isometry between $X(\mu)$ and $L^1(m)$ (see [23, Chapter 3]). As we mentioned in Section 2, in [3], the authors considered the set

$$R(m) := \{\frac{d|\langle m, x^* \rangle|}{d\mu} : x^* \in B_{E^*}\},$$

where m is an E-valued vector measure that represents X in the sense explained above. This set is positively norming for $L^1(m) = X(\mu)$ (see the third example of positively norming set in Section 1 of [3]). Our interest is to show that this class of sets provides a standard procedure for defining positively norming sets. In particular, we will see that every positively norming set for $X(\mu)$ arises essentially in this way, i.e. it can be described as $\mathcal{R}(m)$ for a vector measure m representing order isometrically $X(\mu)$ (see Theorem 5.3). In fact all the representations of order continuous Banach function spaces $X(\mu)$, over a finite measure μ, as $L^1(m)$ for some vector measure m can be considered essentially as representations for some $\ell^\infty(N)$-valued vector measure defined by a norming set N.

An easy example of the type of sets we are considering is the following. If $1 < p < \infty$ and μ is as above, the vector measure $m : \Sigma \to L^p(\mu)$, $m(A) := \chi_A$, $A \in \Sigma$, defines the space $L^1(m) = L^p(\mu)$. The set

$$R(m) = \{g \in L^{p'}(\mu) : g = \frac{d|\langle m, h\rangle|}{d\mu} = |h|,\ h \in B_{L^{p'}(\mu)}\}$$

equals the intersection of the positive cone of $L^{p'}$ and $B_{L^{p'}(\mu)}$ and so it is positively norming.

Lemma 5.1. *Let N be a bounded set of positive functions in $L^1(\mu)$, and let us consider the mapping $m_N : \Sigma \to \ell^\infty(N)$ given by $m_N(A) = (\int_A g d\mu)_{g\in N}$. The following statements hold.*

1. *If N is a relatively weakly compact subset of $L^1(\mu)$, then m_N is a σ-additive vector measure.*
2. *If N is so that m_N is σ-additive and the space $L^1(m_N)$ of integrable functions with respect to the vector measure m_N associated to a set N is included in $L^1(\mu)$, then N is relatively weakly compact in $L^1(\mu)$.*

Proof. Since N is a relatively weakly compact set in $L^1(\mu)$, then by [2, Theorem 5.2.9] it is uniformly integrable. Therefore, if we consider a sequence $(A_i)_{i=1}^\infty$ of disjoint measurable sets in Σ, we have

$$\lim_{n\to\infty} m_N(\bigcup_{i=n}^\infty A_i) = \lim_{n\to\infty} \sup_{g\in N} \int_{\cup_n^\infty A_i} g d\mu = 0.$$

This is equivalent to $m_N(\bigcup_{i=1}^\infty A_i) = \sum_{i=1}^\infty m_N(A_i)$ for every sequence $(A_i)_{i=1}^\infty$ of disjoint sets, so m_N is a σ-additive vector measure.

Conversely, if m_N is σ-additive we can define the space $L^1(m_N)$ of integrable functions with respect to m_N, that is order continuous. The set N, that can be written as $N = \{\frac{d|\langle m_N, e_g\rangle|}{d\mu} = g \,|g \in N\}$ (where e_g is the element of $\ell^1(N)$ corresponding to the coordinate given by g) is norming for $L^1(m_N)$. Recall that by assumption $L^1(m_N)$ is a Banach function space over μ. Then by Lemma 2.2, we have that N is weakly compact in $L^1(\mu)$. □

Notice that the requirement of (2) in the result above is satisfied if there is a function $f \in N$ such that $f, 1/f \in L^\infty(\mu)$, (for instance if $\chi_\Omega \in N$). In this case there are $0 < \delta_1, \delta_2$ such that $\delta_1 < f(w) < \delta_2$, $w \in \Omega$, and then a direct computation shows that $L^1(m_N) \hookrightarrow L^1(\mu)$.

Proposition 5.2. *Let $m : \Sigma \to E$ be a σ-additive vector measure, let μ be a Rybakov measure for m and consider the positively norming set N for $L^1(m)$ given by $N := \{d|\langle m, x'\rangle|/d\mu : x' \in B_{E^*}\}$. Take the vector measure $m_N : \Sigma \to \ell^\infty(N)$ defined as in Lemma 5.1. Then $L^1(m_N) = L^1(m)$ and $L^1_w(m_N) = L^1_w(m)$.*

Proof. First by Lemma 2.2 and Lemma 5.1(1), m_N is σ-additive and so $L^1(m_N)$ is an order continuous Banach function space with a weak unit. Since μ is a Rybakov measure for m, $\chi_\Omega \in N$, i.e. $L^1(m)$ and $L^1(m_N)$ are included in $L^1(\mu)$. Also, μ is a Rybakov measure for m_N, since $\mu = |\langle m, x_0'\rangle|$ for a certain x_0' in the unit sphere of E^*, and so it can be written as $|\langle m_N, e_{x_0'}\rangle|$, where $e_{x_0'}$ is the coordinate functional in $\ell^\infty(N)$ defined by the element $d|\langle m, x_0'\rangle|/d\mu$ of N . Notice that m_N is a positive vector measure, and so $\|f\|_{L^1(m_N)} = \|\int |f| dm_N\|_{\ell^\infty(N)}$ for all the integrable functions f (see [23, Lemma 3.13], [8]). Take a simple function f. Then

$$\|f\|_{L^1(m)} = \sup_{x' \in B_{E^*}} \int |f| \, \frac{d|\langle m, x'\rangle|}{d\mu} \, d\mu = \| \int |f| \, dm_N\|_{\ell^\infty(N)} = \|f\|_{L^1(m_N)}.$$

Since simple functions are dense in both spaces, we obtain $L^1(m_N) = L^1(m)$. The other equality is given by the fact that the Köthe bidual of $L^1(m)$ coincides with $L^1_w(m)$ (see [6, Prop.2.4]) and so $L^1_w(m_N) = (L^1(m_N))'' = (L^1(m))'' = L^1_w(m)$. □

Integration with respect to vector measures defined on ℓ^∞-spaces has been studied recently in [11, Ex.2.13]. The authors analyze several w*-thick sets for ℓ^∞ for the aim of studying when integration with respect to the measures defined by these sets coincide with the weak integrability with respect to m.

Using the results in this paper, in what follows we will show how integration with respect to m_N can be related to the scalar integrals with respect to m, when the positively norming set N is given by $\mathcal{R}(m)$ and it is countable.

We need to introduce some notation. If $m : \Sigma \to E$ is a countably additive vector measure and $\Lambda \subseteq E^*$ is a norming set, following [11] we write $\mathcal{L}^1_\Lambda(m)$ for the set of all measurable functions such that $f \in \mathcal{L}^1(\langle m, x^* \rangle)$ for every $x^* \in \Lambda$, and $\mathcal{L}^{1,s}_\Lambda$ for the space of all the measurable functions for which there is a vector integral satisfying the barycentric equality $\langle \int f dm, x' \rangle = \int f d\langle m, x' \rangle$ for all the elements of Λ. By the arguments in Example 2.13 and Theorem 2.10 in [11], we obtain for the corresponding spaces of classes of $\|m\|$-equal functions the following equalities

$$\mathcal{L}^1_w(m_N) = \mathcal{L}^1_{\ell^1}(m_N) = \mathcal{L}^{1,s}_{\ell^1}(m_N) = \mathcal{L}^{1,s}_{\{e_i^* : i \in \mathbb{N}\}}(m_N).$$

This means that in order to check if a (class of) function(s) belongs to the space of weakly integrable functions it is enough to check its integrability with respect to all measures $\langle m_N, (\lambda_i) \rangle$ for (λ_i) in ℓ^1. However, it can be shown that for some particular vector measures m,

$$L^1(m) \subsetneq L^{1,s}_{\{e_i^* : i \in \mathbb{N}\}}(m) \subsetneq L^1_{\{e_i^* : i \in \mathbb{N}\}}(m),$$

again as a consequence of Example 2.13 in [11].

Therefore, integrability with respect to ℓ^∞-valued vector measures can be essentially checked by evaluating the scalar measures given by the elements of ℓ^1. In the same direction, let us finish the paper with a geometric description of the natural positively norming sets associated to this class of vector measures.

Theorem 5.3. *Let $X(\mu)$ be an order continuous Banach function space over the finite measure μ. Suppose that N is a positively norming set in $B^+_{X'}$, and let N_0 be the set of extreme points of the w^*-closed convex hull of N. Let us consider the vector measure $m_{N_0} : \Sigma \to \ell^\infty(N_0)$ defined as above. It holds that*

$$\overline{R(m_{N_0})}^{w^*} = \overline{\{\frac{d\langle m_{N_0}, y' \rangle}{d\mu} : y' \in B^+_{\ell^1(N_0)}\}}^{w^*} = \overline{co}^{w^*}(N).$$

Proof. Since $X(\mu)$ is order continuous, it follows by Lemma 2.2 that $B_{X(\mu)}$ is a relatively weakly compact set of $L^1(\mu)$. In particular, N_0 is also relatively weakly compact in $L^1(\mu)$. It is now easy to check that for $g \in N_0$ the Dirac measures $\delta_g \in B^+_{\ell^\infty(N_0)^*}$ correspond to the extreme points of the set

$$\{\frac{d\langle m, y'\rangle}{d\mu} : y' \in B^+_{\ell^1(N_0)}\} \subset B^+_{\ell^\infty(N_0)^*}.$$

Hence, by the Krein-Milman theorem we have

$$\overline{\{\frac{d\langle m, y'\rangle}{d\mu} : y' \in B^+_{\ell^1(N_0)}\}}^{w^*} = \overline{co}^{w^*}(N),$$

where $\overline{A}^{w^*}$ denotes the weak-$*$ closure of a set A in $(\ell^\infty(N_0))^*$.

Now, we claim that

$$\overline{\{\frac{d\langle m, x\rangle}{d\mu} : x \in B^+_{\ell^1(N_0)}\}}^{w^*} = \overline{\{\frac{d\langle m, x\rangle}{d\mu} : x \in B^+_{(\ell^\infty(N_0))^*}\}}^{w^*}.$$

Indeed, the second term equals $\overline{R(m)}^{w^*}$. Let $I_m : L^1(m) \to \ell^\infty(N_0)$ be the integration map associated to the vector measure m. Then

$$R(m) = I_m^*(B^+_{\ell^\infty(N_0)^*}) = I_m^*(\overline{B^+_{\ell^1(N_0)}}^{w^*}) \subset \overline{I_m^*(B^+_{\ell^1(N_0)})}^{w^*}$$

(we use Goldstein's theorem and the fact that I_m^* is (w^*, w^*)-continuous). This gives us the containment $\supseteq$. The other inclusion is clear. □

Notice that when N is norming we have that $\overline{co}^{w^*}(N)$ is the whole B_{X^*} and then the result above is uninformative. However, recall that a positively norming might consist of a single element, and so in this case its weak-$*$ closed convex hull is just a singleton.

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