THE CONVEX HULL OF A BANACH-SAKS SET

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Dedicated to the memory of Nigel J. Kalton

Abstract. A subset A of a Banach space is called Banach-Saks when every sequence in A has a Cesàro convergent subsequence. Our interest here focusses on the following problem: is the convex hull of a Banach-Saks set again Banach-Saks? By means of a combinatorial argument, we show that in general the answer is negative. However, sufficient conditions are given in order to obtain a positive result.

1. Introduction

A classical theorem of S. Mazur asserts that the convex hull of a compact set in a Banach space is again relatively compact. In a similar way, Krein-Šmulian's Theorem says that the same property holds for weakly compact sets, that is, these sets have relatively weakly compact convex hull. There is a third property, lying between these two main kinds of compactness, which is defined in terms of Cesàro convergence. Namely, a subset A of a Banach space X is called Banach-Saks if every sequence in A has a Cesàro convergent subsequence (i.e. every sequence $(x_n)_n$ in A has a subsequence $(y_n)_n$ such that the sequence of arithmetic means $((1/n)\sum_{i=1}^n y_i)_n$ is norm-convergent in X). In modern terminology, as it was pointed out by H. P. Rosenthal [27], this is equivalent to saying that no difference sequence in A generates an ℓ_1-spreading model.

The Banach-Saks property has its origins in the work of S. Banach and S. Saks [6], after whom the property is named. In that paper it was proved that the unit ball of L_p $(1 < p < \infty)$ is a Banach-Saks set. Recall that a Banach space is said to have the Banach-Saks property when its unit ball is a Banach-Saks set. This property has been widely studied in the literature (see for instance [5], [8], [15]) and more recently in [4] and [12]. Observe that since a Banach space with the Banach-Saks property must be reflexive [25], it is clear that neither L_1 nor L_∞ have this property. However, weakly compact sets in L_1 are Banach-Saks [30], and every sequence of disjoint elements in L_∞ is also a Banach-Saks set.

Since every compact set is Banach-Saks, and these sets are in turn weakly compact, taking into account both Mazur's and Krein-Šmulian's results, it may seem reasonable to expect that the convex hull of a Banach-Saks set is also Banach-Saks. We will show in Section 4 that this

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is not the case in general. We present a canonical example consisting of the weakly-null unit basis $(u_n)_n$ of a Schreier-like space $X_{\mathcal{F}}$ for a certain family of finite subsets $\mathcal{F}$ on $\mathbb{N}$ that we call a T-family (see Definitions 2.7 and 4.4). The role of the Schreier-like spaces and such families is not incidental. There are several equivalent conditions to the Banach-Saks property in terms of properties of certain families of finite subsets of $\mathbb{N}$ (see Theorem 2.4), and in fact we prove in Theorem 2.8 that a possible counterexample must be of the form $X_{\mathcal{F}}$ for a T-family $\mathcal{F}$. Therefore, an analysis of the families of finite subsets of integers is needed to understand the Banach-Saks property.

The example of a T-family we present is influenced on a classical construction P. Erdős and A. Hajnal [13] of a sequence of measurable subsets of the unit interval indexed by pairs of integers. These sequences of events behave in general in a completely different way than those indexed by integers, as it can be seen, for example, in the work of D. Fremlin and M. Talagrand [16]. Coming back to our space, every subsequence of the basis $(u_n)_n$ has a further subsequence which is equivalent to the unit basis of c_0, yet there is a block sequence of averages of $(u_n)_n$ generating an ℓ_1-spreading model. There is also the reflexive counterpart, either by considering a Baernstein space associated to $\mathcal{F}$, or from a more general approach considering a Davis-Figiel-Johnson-Pelczynski interpolation space of $X_{\mathcal{F}}$.

As far as we know, the main question considered in this paper appeared explicitly in [19], where the authors also proved that every Banach-Saks set in the Schreier space has Banach-Saks convex hull. We will see in Theorem 3.5 that this fact can be further extended to Banach-Saks sets contained in generalized Schreier spaces.

The paper is organized as follows: In Section 2 we introduce some notation, basic definitions and facts concerning the Banach-Saks property, with a special interest on its combinatorial nature. In Section 3 several sufficient conditions are given for the stability of the Banach-Saks property under taking convex hulls. This includes the study of Banach-Saks sets in Schreier-like spaces $X_{\mathcal{S}_\alpha}$ defined from any generalized Schreier family $\mathcal{S}_\alpha$. Finally, in Section 4 we present a canonical example of a Banach-Saks set whose convex hull is not, as well as the corresponding reflexive version.

2. Notation, basic definitions and facts

We use standard terminology in Banach space theory from the monographs [1] and [21]. Let us introduce now some basic concepts in infinite Ramsey theory, that will be used throughout this paper. Unless specified otherwise, by a family $\mathcal{F}$ on a set I we mean a collection of finite subsets of I. We denote infinite subsets by capital letters $M, N, P, \dots$, and finite ones with $s, t, u, \dots$. Given a family $\mathcal{F}$ on $\mathbb{N}$, and $M \subseteq \mathbb{N}$, we define the *trace* $\mathcal{F}[M]$ of $\mathcal{F}$ in M and the *restriction* $\mathcal{F}\restriction M$ of $\mathcal{F}$ in M as

$$\begin{aligned}\mathcal{F}[M] :=&\{s\cap M \,:\, s\in\mathcal{F}\},\\ \mathcal{F}\restriction M :=&\{s\in\mathcal{F} \,:\, s\subseteq M\},\end{aligned}$$

respectively. A family $\mathcal{F}$ on I is called compact, when it is compact with respect to the topology induced by the product topology on 2^I. The family $\mathcal{F}$ is pre-compact, or relatively compact, when the topological closure of $\mathcal{F}$ consists only of finite subsets of I. The family $\mathcal{F}$ is *hereditary* when for every $s\subseteq t\in\mathcal{F}$ one has that $s\in\mathcal{F}$. The $\subseteq$-closure of $\mathcal{F}$ is the minimal hereditary family $\widehat{\mathcal{F}}$ containing $\mathcal{F}$, i.e. $\widehat{\mathcal{F}}:=\{t\subseteq s \,:\, s\in\mathcal{F}\}$. It is easy to see that $\mathcal{F}$ is pre-compact if and only if $\widehat{\mathcal{F}}$ is compact. Typical examples of pre-compact families are

$$\begin{aligned}[][I]^n \;:=&\{s\subseteq I: \#s=n\},\\ [I]^{\le n} :=&\{s\subseteq I: \#s\le n\},\\ [I]^{<\omega} :=&\{s\subseteq I: \#s<\infty\}.\end{aligned}$$

A natural procedure to obtain pre-compact families is to consider, given a relatively weakly-compact subset $\mathcal{K}$ of c_0 and $\varepsilon,\delta>0$, the sets

$$\begin{aligned}\mathrm{supp}_{\,\varepsilon}(\mathcal{K}) :=&\{\mathrm{supp}_{\,\varepsilon}x \,:\, x\in\mathcal{K}\},\\ \mathrm{supp}_{\,\varepsilon,+}(\mathcal{K}) :=&\{\mathrm{supp}_{\,\varepsilon,+}x \,:\, x\in\mathcal{K}\},\\ \mathrm{supp}_{\,\varepsilon}^{\,\delta}(\mathcal{K}) :=&\{\mathrm{supp}_{\,\varepsilon,+}x \,:\, x\in(\mathcal{K})_\varepsilon^\delta\},\end{aligned}$$

where $\mathrm{supp}_{\,\varepsilon}x := \{n\in\mathbb{N} \,:\, |(x)_n|\ge\varepsilon\}$, $\mathrm{supp}_{\,\varepsilon,+}x := \{n\in\mathbb{N} \,:\, (x)_n\ge\varepsilon\}$, $(\mathcal{K})_\varepsilon^\delta := \{x\in\mathcal{K} \,:\, \sum_{n\notin\mathrm{supp}_{\,\varepsilon}x}|(x)_n|\le\delta\}$, and $(x)_n$ denotes the n^{th} coordinate of x in the canonical unit basis of c_{00}.

In particular, when $(x_n)_n$ is a weakly-convergent sequence to x in some Banach space X, and $\mathcal{M}$ is an arbitrary subset of B_{X^*} the family $\mathcal{K}:=\{(x^*(x_n-x))_n \,:\, x^*\in\mathcal{M}\}\subseteq c_0$ is relatively weakly-compact. Given $\varepsilon,\delta>0$ and $\mathcal{M}\subseteq B_{X^*}$, we define

$$\begin{aligned}\mathcal{F}_\varepsilon((x_n)_n,\mathcal{M}) :=&\mathrm{supp}_{\,\varepsilon}(\mathcal{K}),\\ \mathcal{F}_\varepsilon^\delta((x_n)_n,\mathcal{M}) :=&\mathrm{supp}_{\,\varepsilon}^{\,\delta}(\mathcal{K}).\end{aligned}$$

When $\mathcal{M}=B_{X^*}$ we will simply omit $\mathcal{M}$ in the terminology above.

Given $n\in\mathbb{N}$, a family $\mathcal{F}$ on I is called *n-large* in some $J\subseteq I$ when for every infinite $K\subseteq J$ there is $s\in\mathcal{F}$ such that $\#(s\cap K)\ge n$. Or equivalently, when $\mathcal{F}[K]\not\subseteq[K]^{\le n-1}$ for any $K\subseteq J$. The family $\mathcal{F}$ is *large* on J when it is n-large on J for every $n\in\mathbb{N}$. Perhaps the first known example of a compact, hereditary and large family is the Schreier family

$$\mathcal{S}:=\{s\subseteq\mathbb{N} \,:\, \#s\le\min s\}.$$

Generalizing ideas used for families of sets, given $\mathcal{K} \subseteq c_0$ and $M \subseteq \mathbb{N}$, we define $\mathcal{K}[M] := \{\mathbb{1}_M \cdot x : x \in \mathcal{K}\}$ as the image of $\mathcal{K}$ under the natural restriction to the coordinates in M. The following is a list of well-known results on compact families, commonly used by the specialist, which are necessary to understand most of the properties of Banach-Saks sets.

Theorem 2.1. *Let $\mathcal{K}$ be a relatively weakly-compact subset of c_0, $\varepsilon, \delta > 0$. Then there is an infinite subset $M \subseteq \mathbb{N}$ such that*

(a) $\mathrm{supp}_\varepsilon(\mathcal{K}[M]) = \mathrm{supp}_\varepsilon^\delta(\mathcal{K}[M])$ *and* $\mathrm{supp}_\varepsilon(\mathcal{K}[M])$ *is hereditary, and*

(b.1) *either there is some $k \in \mathbb{N}$ such that* $\mathrm{supp}_\varepsilon(\mathcal{K}[M]) = [M]^{\leq k}$,

(b.2) *or else* $_*(\mathcal{S} \upharpoonright M) := \{s \setminus \{\min s\} : s \in \mathcal{S} \upharpoonright M\} \subseteq \mathrm{supp}_\varepsilon(\mathcal{K}[M])$, *and consequently* $\mathrm{supp}_\varepsilon(\mathcal{K}[M])$ *is large in M.*

The proofs of these facts are mostly based on the Ramsey property of a particularly relevant type of pre-compact families called *barriers* on some set M, that were introduced by C. ST. J. A. Nash-Williams [24]. These are families $\mathcal{B}$ on M such that every further subset $N \subseteq M$ has an initial segment in $\mathcal{B}$, and such that there do not exist two different elements of $\mathcal{B}$ which are subsets one of the other. Examples of barriers are $[\mathbb{N}]^n$, $n \in \mathbb{N}$, and the *Schreier barrier* $\mathfrak{S} := \{s \in \mathcal{S} : \#s = \min s\}$. As it was proved by Nash-Williams, barriers have the Ramsey property, and in fact provide a characterization of it. The final ingredient is the fact that if $\mathcal{F}$ is pre-compact, then there is a trace $\mathcal{F}[M]$ of $\mathcal{F}$ which is the closure of a barrier on M (we refer the reader to [2],[22]).

Definition 2.2. A subset A of a Banach space X is a Banach-Saks set (or has the Banach-Saks property) if every sequence $(x_n)_n$ in A has a *Cesàro*-convergent subsequence $(y_n)_n$, i.e. the sequence of averages $((1/n)\sum_{k=1}^n y_k)_n$ is norm-convergent in X.

It is easy to see that compact sets are Banach-Saks, that the Banach-Saks property is hereditary (every subset of a Banach-Saks set is again Banach-Saks), it is closed under sums, and that it is preserved under the action of a bounded operator. It is natural to ask the following.

Question 1. *Is the convex hull of a Banach-Saks set again a Banach-Saks set?*

Using the localized notion of the Banach-Saks property, a space has the Banach-Saks property precisely when its unit ball is a Banach-Saks set. A classical work by T. Nishiura and D. Waterman [25] states that a Banach space with the Banach-Saks property is reflexive. Here is the local version of this fact.

Proposition 2.3. *Every Banach-Saks set is relatively weakly-compact.*

PROOF. Let A be a Banach-Saks subset of a Banach space X, and fix a sequence $(x_n)_n$ in A. By Rosenthal's ℓ_1 Theorem, there is a subsequence $(y_n)_n$ of $(x_n)_n$ which is either equivalent to the unit basis of ℓ_1 or weakly-Cauchy. The first alternative cannot occur, since the unit basis of ℓ_1 is not a Banach-Saks set. Let now $x^{**} \in X^{**}$ be the weak*-limit of $(y_n)_n$. Since A is a Banach-Saks subset of X, there is a further subsequence $(z_n)_n$ of $(y_n)_n$ which is Cesàro-convergent to some $x \in X$. It follows that $x^{**} = x$, and consequently $(z_n)_n$ converges weakly to $x \in X$. □

As the previous proof suggests, the unit basis of ℓ_1 plays a very special role for the Banach-Saks property. This is fully explained by the following characterization, due to H. P. Rosenthal [27] and S. Mercourakis [23] in terms of the asymptotic notions of *Spreading models* and *uniform weakly-convergence.*

Definition 1. *Let X be a Banach space and let $(x_n)_n$ be a sequence in X converging weakly to $x \in X$. Recall that $(x_n)_n$* generates an ℓ_1-spreading model *when there is $\delta > 0$ such that*

$$\Big\| \sum_{n\in s} a_n(x_n - x)\Big\| \geq \delta \sum_{n\in s} |a_n| \tag{2.1}$$

for every $s \subseteq \mathbb{N}$ with $\#s \leq \min s$ and every sequence $(a_n)_{n\in s}$ of scalars.

The sequence $(x_n)_n$ uniformly weakly-converges *to x when for every $\varepsilon > 0$ there is an integer $n(\varepsilon) > 0$ such that for every functional $x^* \in B_{X^*}$*

$$\#(\{n \in \mathbb{N} : |x^*(x_n - x)| \geq \varepsilon\}) \leq n(\varepsilon). \tag{2.2}$$

The notion of ℓ_1 spreading model is orthogonal to the Banach-Saks property: Suppose that $(x_n)_n$ weakly-converges to x and generates an ℓ_1-spreading model. Let $\delta > 0$ be witnessing that. Set $y_n = x_n - x$ for each n. Since $\|y_n\| \geq \delta$ for all n, it follows by Mazur's Lemma that there is a subsequence $(z_n)_n$ of $(y_n)_n$ which is a 2-basic sequence. We claim that no further subsequence of $(z_n)_n$ is Cesàro-convergent: Fix an arbitrary subset $s \subseteq \mathbb{N}$ with even cardinality. Then the upper half part t of s satisfies that $\#t \leq \min t$. So, using also that $(z_n)_n$ is 2-basic,

$$\Big\| \frac{1}{\#s}\sum_{n\in s} z_n\Big\| \geq \frac{1}{2}\Big\| \frac{1}{\#s}\sum_{n\in t} z_n\Big\| \geq \frac{\delta}{2}\frac{\#t}{\#s} = \frac{\delta}{4}. \tag{2.3}$$

This immediately gives that no subsequence of $(z_n)_n$ is Cesàro-convergent to 0.

On the other hand if $(x_n)_n$ is uniformly weakly-convergent to some x, then every subsequence of $(x_n)_n$ is Cesàro-convergent (indeed these conditions are equivalent [23]): Suppose that $(y_n)_n$ is a subsequence of $(x_n)_n$. Now for each $\varepsilon > 0$ let $n(\varepsilon)$ be witnessing that (2.2) holds. Set $z_n = y_n - x$ for each n. Now suppose that s is an arbitrary finite subset of $\mathbb{N}$ with cardinality $\geq n(\varepsilon)$. Then, given $x^* \in B_{X^*}$, and setting $t := \{n \in s : |x^*(z_n)| \geq \varepsilon\}$, we have that

$$\Big| x^*(\frac{1}{\#s}\sum_{n\in s} z_n)\Big| \leq \frac{1}{\#s}\sum_{n\in t}|x^*(z_n)| + \frac{1}{\#s}\sum_{n\in s\setminus t}|x^*(z_n)| \leq \frac{n(\varepsilon)}{\#s}C + \varepsilon. \tag{2.4}$$

Hence,

$$\Big\| \frac{1}{\#s}\sum_{n\in s} z_n\Big\| \leq \frac{n(\varepsilon)}{\#s}C + \varepsilon. \tag{2.5}$$

This readily implies that $(z_n)_n$ is Cesàro-convergent to 0, or, in other words, $(y_n)_n$ is Cesàro-convergent to x. Next result summarizes the relationship between these three notions.

Theorem 2.4. *Let A be an arbitrary subset of a Banach space X. The following are equivalent:*

(a) *A is a Banach-Saks subset of X.*
(b) *A is relatively weakly-compact and for every weakly-convergent sequence in A it never generates an ℓ_1-spreading model.*
(c) *A is relatively weakly-compact and for every weakly-convergent sequence $(x_n)_n$ in A and every $\varepsilon > 0$ the family $\mathcal{F}_\varepsilon((x_n)_n)$ is not large in $\mathbb{N}$.*
(d) *A is relatively weakly-compact and for every weakly convergent sequence $(x_n)_n$ in A there is some norming set $\mathcal{N}$ such that for every $\varepsilon > 0$ the family $\mathcal{F}_\varepsilon((x_n)_n, \mathcal{N})$ is not large.*
(e) *For every sequence $(a_n)_n$ in A there is a subsequence $(b_n)_n$ and some norming set $\mathcal{N}$ such that for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that $\mathcal{F}_\varepsilon((b_n)_n, \mathcal{N}) \subseteq [\mathbb{N}]^{\leq m}$.*
(f) *Every sequence in A has a uniformly weakly-convergent subsequence.*

Recall that a λ-norming set, $0 < \lambda \leq 1$ is a subset $\mathcal{N} \subseteq B_{X^*}$ such that

$$\lambda \|x\| \leq \sup_{f \in \mathcal{N}} |f(x)| \text{ for every } x \in X.$$

The subset $\mathcal{N} \subseteq B_{X^*}$ is norming when it is λ-norming for some $0 < \lambda \leq 1$. Note we could rephrase (e) as saying that the sequence $(b_n)_n$ is uniformly weakly-convergent with respect to $\mathcal{N}$.

The equivalences between (a) and (b), and between (a) and (f) are due to Rosenthal [27] and Mercourakis [23], respectively. For the sake of completeness, we give now hints of the proof of Theorem 2.4 using, mainly, Theorem 2.1:

(a) implies (b) because we have already seen that if a sequence $(x_n)_n$ converges weakly to x, generates an ℓ_1-spreading model and is such that $(x_n - x)_n$ is basic, then it does not have Cesàro-convergent subsequences. We prove that (b) implies (c) by using Theorem 2.1. Let $(x_n)_n$ be a weakly convergent sequence in A with limit x, and let us see that $\mathcal{F}_\varepsilon((x_n))$ is not large for any $\varepsilon > 0$. Otherwise, by Theorem 2.1, there is some M such that

$$_*(\mathcal{S} \upharpoonright M) \subseteq \mathcal{F}_\varepsilon^\delta((x_n)_n)[M] = \mathcal{F}_\varepsilon((x_n)_n)[M].$$

Set $y_n := x_n - x$ for each $n \in M$. It follows that $(y_n)_{n \in M}$ is a non-trivial weakly-null sequence, hence by Mazur's Lemma, there is $N \subseteq M$ such that $(y_n)_{n \in N}$ is a 2-basic sequence. We claim that then $(y_n)_{n \in N}$ generates an ℓ_1-spreading model, which is impossible: Let $s \in \mathcal{S} \upharpoonright N$, and let $(\lambda_k)_{k \in s}$ be a sequence of scalars. Let $t \subseteq s$ be such that $\lambda_k \cdot \lambda_l \geq 0$ for all $k, l \in t$, $|\sum_{k \in t} \lambda_k| \geq 1/4 \sum_{k \in s} |\lambda_k|$ and $t \in {}_*(S \upharpoonright N)$. Then let $x^* \in B_{X^*}$ be such that

$$x^*(y_n) \geq \varepsilon \text{ for } n \in t, \text{ and } \sum_{n \in M \setminus t} |x^*(y_n)| \leq \frac{\varepsilon}{4}.$$

It follows that

$$\Big\|\sum_{k\in s}\lambda_k y_k\Big\| \geq \Big|x^*(\sum_{k\in s}\lambda_k y_k)\Big| \geq \Big|\sum_{k\in t}\lambda_k x^*(y_k)\Big| - \frac{\varepsilon}{4}\max_{k\in s}|\lambda_k| \geq \varepsilon\Big|\sum_{k\in t}\lambda_k\Big| - \frac{\varepsilon}{4}\max_{k\in s}|\lambda_k|$$
$$\geq \frac{\varepsilon}{4}\sum_{k\in s}|\lambda_k| - \varepsilon\Big\|\sum_{k\in s}\lambda_k y_k\Big\|,$$

and consequently,

$$\Big\|\sum_{k\in s}\lambda_k y_k\Big\| \geq \frac{\varepsilon}{4(1+\varepsilon)}\sum_{k\in s}|\lambda_k|. \tag{2.6}$$

Now, we have that (c) implies (d) and (d) implies (e) trivially. For the implication (e) implies (f) we use the following classical result by J. Gillis [18].

Lemma 2.5. *For any $\varepsilon,\delta>0$ and $m\in\mathbb{N}$ there is $n:=\mathbf{n}(\varepsilon,\delta,m)$ such that whenever (Ω,Σ,μ) is a probability space and $(A_i)_{i=1}^n$ is a sequence of μ-measurable sets with $\mu(A_i)\geq\varepsilon$ for every $1\leq i\leq n$, there is $s\subseteq\{1,\dots,n\}$ of cardinality m such that*

$$\mu(\bigcap_{i\in s}A_i)\geq(1-\delta)\varepsilon^m.$$

Incidentally, the counterexample by P. Erdős and A. Hajnal of the natural generalization of Gillis' result concerning double-indexed sequences will be crucial for our solution to Question 1 (see Section 4).

We pass now to see that (e) implies (f): Fix a sequence $(x_n)_n$ in A converging weakly to x and $\varepsilon>0$. By (e), we can find a subsequence $(y_n)_n$ of $(x_n)_n$ and a λ-norming set $\mathcal{N}$, $0<\lambda\leq 1$, such that $(y_n)_n$ uniformly-weakly-converges with respect to $\mathcal{N}$. Going towards a contradiction, suppose $(y_n)_n$ does not uniformly weakly-converge to x. Fix then $\varepsilon>0$ such that there are arbitrary large sets in $\mathcal{F}_\varepsilon((y_n)_n)$. In this case we see that then $\mathcal{F}_{\lambda\varepsilon(1-\delta)}((y_n)_n,\mathcal{N})$ has also arbitrary large sets, contradicting our hypothesis. Set $z_n:=y_n-x$ for every $n\in\mathbb{N}$. Now given $m\in\mathbb{N}$, let $x^*\in B_{X^*}$ be such that

$$s:=\{n\in M\,:\,|x^*(z_n)|\geq\varepsilon\}\text{ has cardinality }\geq\mathbf{n}(\frac{\varepsilon\delta\lambda}{2K},\frac{1}{2},m),$$

where $K:=\sup_n\|z_n\|$. By a standard separation result, there are $f_1,\dots,f_l\in\mathcal{N}$ and $\nu_1,\dots,\nu_l$ such that $\sum_{i=1}^l|\nu_i|\leq\lambda^{-1}$ and

$$\Big|\sum_{i=1}^l\nu_i f_i(z_n)\Big|\geq\varepsilon(1-\frac{\delta}{2})\text{ for every }n\in s. \tag{2.7}$$

Now on $\{1,2,\dots,l\}$ define the probability measure induced by the convex combination

$$\Big(\frac{1}{\sum_{j=1}^l|\nu_j|}|\nu_i|\Big)_{i=1}^l.$$

For each $n\in s$, let

$$A_n:=\{j\in\{1,\dots,l\}\,:\,|f_j(z_n)|\geq\varepsilon(1-\delta)\}.$$

Then, for every $n \in s$ one has that

$$\varepsilon(1-\frac{\delta}{2}) \leq \Big|\sum_{j=1}^{l} \nu_j f_j(z_n)\Big| \leq \sum_{j\in A_n} |\nu_j| K + \varepsilon(1-\delta).$$

Hence,

$$\mu(A_n) \geq \frac{\delta\varepsilon\lambda}{2K}.$$

By Gillis' Lemma, it follows in particular that there is some $t \subseteq \{1,\dots,l\}$ of cardinality m such that $\bigcap_{n\in t} A_n \neq \emptyset$, so let j be in that intersection. It follows then that $|f_j(z_n)| \geq \lambda\varepsilon(1-\delta)$ for every $n \in t$, hence $t \in \mathcal{F}_{\lambda\varepsilon(1-\delta)}((y_n)_n, \mathcal{N})$.

(f) implies (a) because uniformly weakly-convergent sequences are Cesàro-convergent. This finishes the proof.

Hence, Question 1 for weakly-null sequences can be reformulated as follows:

Question 2. *Suppose that $(x_n)_n$ is a weakly-null sequence such that some sequence in* $\mathrm{co}(\{x_n\}_n)$ *generates an ℓ_1-spreading model. Does there exist a subsequence of $(x_n)_n$ generating an ℓ_1-spreading model?*

As a consequence of Theorem 2.4 we obtain the following well-known 0-1-law by P. Erdös and M. Magidor [14].

Corollary 2.6. *Every bounded sequence in a Banach space has a subsequence such that either all its further subsequences are Cesàro-convergent, or none of them.*

To see this, let $(x_n)_n$ be a sequence in a Banach space. If $A := \{x_n\}_n$ is Banach-Saks, then, by (e) above, there is a uniformly weakly-convergent subsequence $(y_n)_n$ of $(x_n)_n$, and as we have mentioned above, every further subsequence of $(y_n)_n$ is Cesàro-convergent. Now, if A is not Banach-Saks, then by (b) there is a weakly-convergent sequence $(y_n)_n$ in A with limit y generating an ℓ_1-spreading model. We have already seen that if $(z_n)_n$ is a basic subsequence of $(y_n - y)_n$, then no further subsequence of it is Cesàro-convergent.

We introduce now the Schreier-like spaces, which play an important role for the Banach-Saks property.

Definition 2.7. Given a family $\mathcal{F}$ on $\mathbb{N}$, we define the Schreier-like norm $\|\cdot\|_{\mathcal{F}}$ on $c_{00}(\mathbb{N})$ as follows. For each $x \in c_{00}$ let

$$\|x\|_{\mathcal{F}} = \max\{\|x\|_\infty, \sup_{s\in\mathcal{F}} \sum_{n\in s} |(x)_n|\}, \tag{2.8}$$

where $(x)_n$ denotes the n^{th}-coordinate of x in the usual Hamel basis of $c_{00}(\mathbb{N})$. We define the Schreier-like space $X_{\mathcal{F}}$ as the completion of c_{00} under the $\mathcal{F}$-norm.

Note that $X_{\mathcal{F}} = X_{\widehat{\mathcal{F}}}$ for every family $\mathcal{F}$, so the hereditary property of $\mathcal{F}$ plays no role for the corresponding space. It is clear that the unit vector basis $(u_n)_n$ is a 1-unconditional Schauder basis of $X_{\mathcal{F}}$, and it is weakly-null if and only if $\mathcal{F}$ is pre-compact. In fact, otherwise there will be a subsequence of $(u_n)_n$ 1-equivalent to the unit basis of ℓ_1. So, Schreier-like spaces will be assumed to be constructed from pre-compact families. It follows then that for pre-compact families $\mathcal{F}$, the space $X_{\mathcal{F}}$ is c_0-saturated. This can be seen, for example, by using Pták's Lemma, or by the fact that $X_{\mathcal{F}} = X_{\widehat{F}} \hookrightarrow C(\widehat{\mathcal{F}})$ isometrically, and the fact that the function spaces $C(K)$ for K countable are c_0-saturated, by a classical result of A. Pelczynski and Z. Semadeni [26].

Observe that the unit basis of the *Schreier space* $X_{\mathcal{S}}$ generates an ℓ_1-spreading model, so no subsequence of it can be Cesàro-convergent. In fact, the same holds for the Schreier-like space $X_{\mathcal{F}}$ of an arbitrary large family $\mathcal{F}$. However, it was proved by M. González and J. Gutiérrez in [19] that the convex hull of a Banach-Saks subset of the Schreier space $X_{\mathcal{S}}$ is again Banach-Saks. In fact, we will see in Subsection 3.1 that the same holds for the spaces $X_{\mathcal{F}}$ where $\mathcal{F}$ is a generalized Schreier family. Still, a possible counterexample for Question 1 has to be a Schreier like space, as we see from the following characterization.

Theorem 2.8. *The following are equivalent:*

(a) *There is a normalized weakly-null sequence having the Banach-Saks property and whose convex hull is not a Banach-Saks set.*
(b) *There is a Shreier-like space $X_{\mathcal{F}}$ such that its unit basis $(u_n)_n$ is Banach-Saks and its convex hull is not.*
(c) *There is a compact and hereditary family $\mathcal{F}$ on $\mathbb{N}$ such that:*
   (c.1) *$\mathcal{F}$ is not large in any $M \subseteq \mathbb{N}$.*
   (c.2) *There is a partition $\bigcup_n I_n = \mathbb{N}$ in finite sets I_n a probability measure μ_n on I_n and $\delta > 0$ such that the set*
$$\mathcal{G}_\delta^{\bar{\mu}}(\mathcal{F}) := \{t \subseteq \mathbb{N} : \text{there is } s \in \mathcal{F} \text{ such that } \min_{n\in t} \mu_n(s \cap I_n) \geq \delta\} \tag{2.9}$$
   is large.

For the proof we need the following useful result.

Lemma 2.9. *Let $(x_n)_n$ and $(y_n)_n$ be two bounded sequences in a Banach space X.*

(a) *If $\sum_n \|x_n - y_n\| < \infty$, then $\{x_n\}_n$ is Banach-Saks if and only if $\{y_n\}_n$ is Banach-Saks.*
(b) $\mathrm{co}(\{x_n\}_n)$ *is a Banach-Saks set if and only if every block sequence in* $\mathrm{co}(\{x_n\}_n)$ *has the Banach-Saks property.*

Proof. The proof of (a) is straightforward. Let us concentrate in (b): Suppose that $\mathrm{co}(\{x_n\}_n)$ is not Banach-Saks, and let $(y_n)_n$ be a sequence in $\mathrm{co}(\{x_n\}_n)$ without Cesàro-convergent subsequences. Write $y_n := \sum_{k\in F_n} \lambda_k^{(n)} x_k$, $(\lambda_k^{(n)})_{k\in F_n}$ a convex combination, for each n. By a Cantor diagonalization process we find M such that $((\lambda_k^{(n))_{k\in\mathbb{N}}})_{n\in M}$ converges pointwise to a (possibly infinite) convex sequence $(\lambda_k)_k \in B_{\ell_1}$. Set $\mu_k^{(n)} := \lambda_k^{(n)} - \lambda_k$ for each $n \in M$. Then there is an infinite subset $N \subseteq M$ and a block sequence $((\eta_k^{(n)})_{k\in s_n})_{n\in N}$, $\sum_{k\in s_n} |\eta_k^{(n)}| \le 2$, such that

$$\sum_{n\in N}\sum_{k\in\mathbb{N}} |\mu_k^{(n)} - \eta_k^{(n)}| < \infty. \tag{2.10}$$

Setting $z_n := \sum_{k\in s_n} \eta_k^{(n)} x_k$ for each n, it follows from (2.10) that

$$\sum_{n\in N} \|y_n - z_n\| < \infty. \tag{2.11}$$

By (a), no subsequence of $(z_n)_{n\in N}$ is Cesàro-convergent. Now set $t_n := \{k \in s_n \,:\, \eta_k^{(n)} \ge 0\}$, $u_n = s_n \setminus t_n$, $z_n^{(0)} := \sum_{k\in t_n} \eta_k^{(n)}$ and $z_n^{(1)} := z_n - z_n^{(0)}$. Then, either $\{z_n^{(0)}\}_{n\in N}$ or $\{z_n^{(1)}\}_{n\in N}$ is not Banach-Saks. So, without loss of generality, let us assume that $\{z_n^{(0)}\}_{n\in N}$ is not Banach-Saks. Then, using again (a), and by going to a subsequence if needed, we may assume that $\sum_{k\in t_n} \eta_k^{(n)} = \eta$ for every $n \in N$. It follows that the block sequence $((1/\eta)\sum_{k\in t_n} \eta_k^{(n)} x_n)_{n\in N}$ in $\mathrm{co}(\{x_n\}_n)$ does not have that Banach-Saks property. □

Proof of Theorem 2.8. It is clear that (b) implies (a). Let us prove that (c) implies (b). We fix a family $\mathcal{F}$ as in (c). We claim that $X_{\mathcal{F}}$ is the desired Schreier space: Let $(u_n)_n$ be the unit basis of $X_{\mathcal{F}}$, and let

$$\mathcal{N} := \{\pm u_n^*\}_n \cup \{\sum_{n\in s} \pm u_n^* \,:\, s \in \mathcal{F}\},$$

where (u_n^*) is the biorthogonal sequence to $(u_n)_n$. Then

$$\mathcal{F}_\varepsilon((u_n), \mathcal{N}) = \mathcal{F} \cup [\mathbb{N}]^1$$

for every $\varepsilon > 0$, so it follows from our hypothesis (c.1) and Theorem 2.4 (d) that $\{u_n\}_n$ is Banach-Saks. Define now for each $n \in \mathbb{N}$, $x_n := \sum_{k\in I_n} (\mu_n)_k u_k$. Then

$$\mathcal{F}_\delta((x_n)_n, \mathcal{N}) = \mathcal{G}_\delta(\mathcal{F})$$

so $\mathcal{F}_\delta((x_n)_n)$ is large, hence $\{x_n\}_n \subseteq \mathrm{co}(\{u_n\}_n)$ is not Banach-Saks.

Finally, suppose that (a) holds and we work to see that (c) also holds. Let $(x_n)_n$ be a weakly-null sequence in some space X with the Banach-Saks property but such that $\mathrm{co}(\{x_n\}_n)$ is not Banach-Saks. By the previous Lemma 2.9 (b), we may assume that there is a block sequence $(y_n)_n$ with respect to $(x_n)_n$ in $\mathrm{co}(\{x_n\}_n)$ without the Banach-Saks property. By Theorem 2.4 there is some subsequence $(z_n)_n$ of $(y_n)_n$ and $\varepsilon > 0$ such that

$$\mathcal{F}_\varepsilon((z_n)_n) \text{ is large.} \tag{2.12}$$

By re-enumeration if needed, we may assume that $\bigcup_n \mathrm{supp}\, z_n = \mathbb{N}$, where the support is taken with respect to (x_n). Let

$$\mathcal{F} := \mathcal{F}_{\frac{\varepsilon}{2}}((x_n)_n).$$

On the other hand, since $(x_n)_n$ is weakly-null, it follows that $\mathcal{F}$ is pre-compact, and, since it is hereditary by definition, it is compact. Again by invoking Theorem 2.4 we know that $\mathcal{F}$ is not large in any $M \subseteq \mathbb{N}$. Now let $I_n := \operatorname{supp} z_n$ and let μ_n be the convex combination with support I_n such that $z_n = \sum_{k\in I_n} (\mu_n)_k x_k$ for each $n \in \mathbb{N}$. Then $(I_n)_n$ is a partition of $\mathbb{N}$ and μ_n is a probability measure on I_n. We see now that (2.9) holds for $\delta := \varepsilon/2$: Fix an infinite subset $M \subseteq \mathbb{N}$, and fix $m \in \mathbb{N}$. By (2.12), we can find $x^* \in B_{X^*}$ such that

$$s := \{n \in M \,:\, |x^*(z_n)| \geq \varepsilon\} \text{ has cardinality } \geq m. \tag{2.13}$$

We claim that $s \in \mathcal{G}^{\bar{\mu}}_{\varepsilon/2}(\mathcal{F})$: Fix $n \in s$, and let $s_n := \{k \in I_n \,:\, |x^*(x_k)| \geq \varepsilon/2\}$ and $t_n := I_n \setminus s_n$. Then

$$\varepsilon \leq x^*(z_n) \leq \sum_{k\in s_n} (\mu_n)_k + \sum_{k\in t_n} (\mu_n)_k \frac{\varepsilon}{2} \leq \sum_{k\in s_n} (\mu_n)_k + \frac{\varepsilon}{2}$$

hence $\mu_n(s_n) \geq \varepsilon/2$, and so $s \in \mathcal{G}^{\bar{\mu}}_{\varepsilon/2}(\mathcal{F})$. □

3. Stability under convex hull: positive results

Recall that a Banach space X is said to have the weak Banach-Saks property if every weakly convergent sequence in X has a Cesàro convergent subsequence. Equivalently, every weakly compact set in X has the Banach-Saks property. Examples of Banach spaces with the weak Banach-Saks property but without the Banach-Saks property are L^1 and c_0 (see [30]).

The following simple observation provides our first positive result concerning the stability of Banach-Saks sets under convex hulls.

Proposition 3.1. *Let X be Banach space with the weak Banach-Saks property. Then the convex hull of a Banach-Saks subset of X is also Banach-Saks.*

Proof. If $A \subseteq X$ has the Banach-Saks property, then A is relatively weakly compact. Therefore, by Krein-Šmulian's Theorem, $\operatorname{co}(A)$ is also relatively weakly compact. Since X has the weak Banach-Saks property, it follows that $\operatorname{co}(A)$ has the Banach-Saks property. □

However, the weak Banach-Saks property is far from being a necessary condition. For instance, the Schreier space $X_{\mathcal{S}}$ does not have the weak Banach-Saks property [30], but the convex hull of any Banach-Saks set is again a Banach-Saks set (see [19, Corollary 2.1]). In Section 3.1, we will see that this result can be extended to generalized Schreier spaces.

Another partial result is the following.

Proposition 3.2. *Let $(x_n)_n$ be a sequence in a Banach space X such that every subsequence is Cesàro convergent. Then $\operatorname{co}(\{x_n\})$ is a Banach-Saks set.*

Proof. As we mentioned in Section 2, the hypothesis is equivalent to saying that $(x_n)_n$ is uniformly weakly-convergent to some $x \in X$ [23, Theorem 1.8]. Now, by Lemma 2.9 (b), it suffices to prove that every block sequence $(y_n)_n$ with respect to $(x_n)_n$ in $\mathrm{co}(\{x_n\}_n)$ is Banach-Saks. Indeed we are going to see that such sequence $(y_n)_n$ is uniformly weakly-convergent to x. Fix $\varepsilon > 0$, and let m be such that

$$\mathcal{F}_\varepsilon((x_n)_n) \subseteq [\mathbb{N}]^{\leq m}. \tag{3.1}$$

We claim that $\mathcal{F}((y_n)_n, \varepsilon) \subseteq [\mathbb{N}]^{\leq m}$ as well: So, let $x^* \in B_{X^*}$ and define $s := \{n \in \mathbb{N} : |x^*(y_n - x)| \geq \varepsilon\}$. Using that $\{y_n\}_n \subseteq \mathrm{co}(\{x_n\}_n)$ we can find for each $n \in s$,an integer $l(n) \in \mathbb{N}$ such that $|x^*(x_{l(n)} - x)| \geq \varepsilon$. Since $(y_n)_n$ is a block sequence with respect to $(x_n)_n$, it follows that $(l(n))_{n\in s}$ is a 1-1 sequence. Finally, since $\{l(n)\}_{n\in s} \in \mathcal{F}_\varepsilon((x_n)_n)$, it follows from (3.1) that $\#s \leq m$. □

It is worth to point out that the hypothesis and conclusion in the previous proposition are not equivalent: The unit basis of the space $(\bigoplus_n \ell_1^n)_{c_0}$ is not uniformly weakly-convergent (to 0) but its convex hull is a Banach-Saks set.

Recall that for a σ-field Σ over a set Ω and a Banach space X, a function $\mu : \Sigma \to X$ is called a (countably additive) vector measure if it satisfies

1. $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, whenever $E_1, E_2 \in \Sigma$ are disjoint, and
2. for every pairwise disjoint sequence $(E_n)_n$ in Σ we have that $\mu(\cup_{n=1}^\infty E_n) = \sum_{n=1}^\infty \mu(E_n)$ in the norm of X.

Proposition 3.3. *If a Banach-Saks set A is contained in the range of some vector measure, then* $\mathrm{co}(A)$ *is also Banach-Saks.*

Proof. J. Diestel and C. Seifert proved in [11] that every set contained in the range of a vector measure is Banach-Saks. Although the range of a vector measure $\mu(\Sigma)$ need no be a convex set, by a classical result of I. Kluvanek and G. Knowles [20, Theorems IV.3.1 and V.5.1], there is always a (possibly different) vector measure μ' whose range contains the convex hull of $\mu(\Sigma)$. Thus if a set A is contained in the range of a vector measure, then $\mathrm{co}(A)$ is also a Banach-Saks set. □

However, there are Banach-Saks sets which are not the range of a vector measure: consider for instance the unit ball of ℓ_p for $1 < p < 2$ [11].

3.1. **A result for generalized Schreier spaces.** We present here a positive answer to Question 1 for a large class of Schreier-like spaces, the spaces $X_\alpha := X_{\mathcal{S}_\alpha}$ constructed from the generalized Schreier families $\mathcal{S}_\alpha$ for a countable ordinal number α.

Recall that given two families $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{N}$, we define

$$\mathcal{F} \oplus \mathcal{G} := \{s \cup t \, : \, s \in \mathcal{G}, \, t \in \mathcal{F} \text{ and } s < t\}$$

$$\mathcal{F} \otimes \mathcal{G} := \{s_0 \cup \cdots \cup s_n \, : \, (s_i) \text{ is a block sequence in } \mathcal{F} \text{ and } \{\min s_i\}_{i\leq n} \in \mathcal{G}\},$$

where $s < t$ means that $\max s < \min t$.

Definition 3.4. For each countable limit ordinal number α we fix a strictly increasing sequence $(\beta_n^{(\alpha)})_n$ such that $\sup_n \beta_n^{(\alpha)} = \alpha$. We define now

(a) $\mathcal{S}_0 := [\mathbb{N}]^{\leq 1}$.
(b) $\mathcal{S}_{\alpha+1} = \mathcal{S}_\alpha \otimes \mathcal{S}$.
(c) $\mathcal{S}_\alpha := \bigcup_{n\in\mathbb{N}} \mathcal{S}_{\beta_n^{(\alpha)}} \upharpoonright [n+1, \infty[$.

Then each S_α is a compact, hereditary and spreading family with Cantor-Bendixson rank equal to ω^α. These families have been widely used in Banach space theory. As an example of their important role we just mention that given a pre-compact family $\mathcal{F}$ there exist an infinite set M, a countable ordinal number α and $n \in \mathbb{N}$ such that $\mathcal{S}_\alpha \otimes [M]^{\leq n} \subseteq \mathcal{F}[M] \subseteq \mathcal{S}_\alpha \otimes [M]^{\leq n+1}$. It readily follows that every subsequence of the unit basis of $X_{\mathcal{F}}$ has a subsequence equivalent to a subsequence of the unit basis of X_α. The main result of this part is the following.

Theorem 3.5. *Let α be a countable ordinal number. $A \subseteq X_\alpha$ has the Banach-Saks property if and only if $\mathrm{co}(A)$ has the Banach-Saks property.*

The particular case $\alpha = 0$ is a consequence of the weak-Banach-Saks property of c_0 and Proposition 3.1. For $\alpha \geq 1$ the spaces X_α are not weak-Banach-Saks. Still, González and Gutiérrez proved the case $\alpha = 1$ in [19]. Implicitly, the case $\alpha < \omega$ was proved by I. Gasparis and D. Leung [17] since it follows from their result stating that every seminormalized weakly-null sequence in X_α, $\alpha < \omega$, has a subsequence equivalent to a subsequence of the unit basis of X_β, $\beta \leq \alpha$. We conjecture that the same should be true for an arbitrary countable ordinal number α.

The next can be proved by transfinite induction.

Proposition 3.6. *Let $\beta < \omega_1$.*

(1) *For every $\alpha < \beta$ there is some $n \in \mathbb{N}$ such that $(\mathcal{S}_\alpha \otimes \mathcal{S}) \upharpoonright (\mathbb{N}/n) \subseteq \mathcal{S}_\beta$.*
(2) *For every $n \in \mathbb{N}$ there are $\alpha_0, \dots, \alpha_n < \beta$ such that*

$$(\mathcal{S}_\alpha)_{\leq n} := \{s \in \mathcal{S}_\beta \,:\, \min s \leq n\} \subseteq \mathcal{S}_{\alpha_0} \oplus \cdots \oplus \mathcal{S}_{\alpha_n}.$$

□

Fix a countable ordinal number α. We introduce now a property in X_α that will be used to characterize the Banach-Saks property for subsets of X_α.

Definition 3.7. We say that a weakly null sequence $(x_n)_n$ in X_α is $< \alpha$*-null* when

for every $\beta < \alpha$ and every $\varepsilon > 0$ the set $\{n \in \mathbb{N} \,:\, \|x_n\|_\beta \geq \varepsilon\}$ is finite.

Proposition 3.8. *Suppose that $(x_n)_n$ is a bounded sequence in X_α such that there are $\varepsilon > 0$, $\beta < \alpha$ and a block sequence $(s_n)_n$ in $\mathcal{S}_\beta$ such that $\sum_{k\in s_n} |(x_n)_k| \geq \varepsilon$. Then $\{x_n\}_n$ is not Banach-Saks.*

Proof. Let $K = \sup_n \|x_n\|$. Let $\bar{n} \in \mathbb{N}$ be such that $(\mathcal{S}_\beta \otimes \mathcal{S}) \upharpoonright [\bar{n}, \infty[\subseteq \mathcal{S}_\alpha$. Fix a subsequence $(x_n)_{n\in M}$.

Claim 1. For every $\delta > 0$ there is a subsequence $(x_n)_{n\in N}$ such that for every $n \in N$ one has that

$$\sum_{m\in N,\, m<n} \max\left\{\sum_{k\in s_n} |(x_m)_k|, \sum_{k\in s_m} |(x_n)_k|\right\} \leq \delta. \tag{3.2}$$

The proof of this claim is the following. Using that $(u_n)_n$ is a Schauder basis of X_α and that $(s_n)_n$ is a block, we can find a subsequence $(x_n)_{n\in N}$ such that for every $n \in N$ one has that

$$\sum_{m\in N,\, m<n} \sum_{k\in s_n} |(x_m)_k| \leq \delta. \tag{3.3}$$

We color each pair $\{m_0 < m_1\} \in [\mathbb{N}]^2$ by

$$c(\{m_0, m_1\}) = \begin{cases} 0 & \text{if } \sum_{k\in s_{m_0}} |(x_{m_1})_k| \geq \delta \\ 1 & \text{otherwise.} \end{cases}$$

By the Ramsey Theorem, there is some infinite subset $P \subseteq N$ such that c is constant on $[P]^2$ with value $i = 0, 1$. We claim that $i = 1$. Otherwise, suppose that $i = 0$. Let $m_0 \in P$, $m_0 > \bar{n}$ be such that $m_0 \cdot \delta > K$, and let $m_1 \in P$ be such that $t = [m_0, m_1[\cap P$ has cardinality m_0. Then $n_0 < m_0 \leq \min s_{m_0}$, and hence $s = \bigcup_{m\in t} s_m \in \mathcal{S}_\alpha$. But then,

$$K \geq \|x_{m_1}\| \geq \sum_{k\in s} |(x_{m_1})_k| = \sum_{m\in t} \sum_{k\in s_m} |(x_{m_1})_k| \geq \#t \cdot \delta > K,$$

a contradiction. Now it is easy to find $P \subseteq N$ such that for every $n \in P$,

$$\sum_{m\in P,\, m<n} \sum_{k\in s_m} |(x_n)_k| \leq \delta. \tag{3.4}$$

Using the Claim 1 repeatedly, we can find $N \subseteq M$ such that

$$\sum_{n\in N} \sum_{m\neq n\in N} \sum_{k\in s_m} |(x_n)_k| \leq \frac{\varepsilon}{2}.$$

In other words, $(x_n, \sum_{k\in s_n} \theta_k^{(n)} u_k^*)_{n\in N}$ behaves almost like a biorthogonal sequence for every sequence of signs $((\theta_k^{(n)})_{k\in s_n})_{n\in N}$. We see now that $(x_n)_{n\in N}$ generates an ℓ_1-spreading model with constant $\geq \varepsilon/2$. We assume without loos of generality that $\bar{n} < N$. Let $t \in \mathcal{S} \upharpoonright N$, and let

$(a_n)_{n\in t}$ be a sequence of scalars such that $\sum_{n\in t}|a_n|=1$. Then $s=\bigcup_{n\in t}s_n\in\mathcal{S}_\alpha$, and hence,

$$\|\sum_{n\in t}a_nx_n\|\ge\sum_{k\in s}|(\sum_{n\in t}a_nx_n)_k|=\sum_{n\in t}\sum_{k\in s_n}|(\sum_{m\in t}a_mx_m)_k|\ge$$
$$\ge\sum_{n\in t}|a_n|\sum_{k\in s_n}|(x_n)_k|-\sum_{n\in t}\sum_{k\in s_n}\sum_{m\in t\setminus\{n\}}|(x_m)_k|\ge\varepsilon\sum_{n\in t}|a_n|-\frac{\varepsilon}{2}\ge\frac{\varepsilon}{2}\sum_{n\in t}|a_n|.$$

□

The following characterizes the Banach-Saks property of subsets of X_α.

Proposition 3.9. *Let $(x_n)_n$ be a weakly null sequence in X_α. The following are equivalent:*

(1) *Every subsequence of $(x_n)_n$ has a further subsequence dominated by the unit basis of c_0.*
(2) *Every subsequence of $(x_n)_n$ has a further norm-null subsequence or a subsequence equivalent to the unit basis of c_0.*
(3) *$\{x_n\}_n$ is a Banach-Saks set.*
(4) *$(x_n)_n$ is $<\alpha$-null.*

Proof. (1) ⇒ (2) ⇒ (3) trivially. (3) implies (4): Suppose otherwise that $(x_n)_n$ is not $<\alpha$-null. Fix $\varepsilon>0$ and $\beta<\alpha$ such that

$$M:=\{n\in\mathbb{N}\,:\,\|x_n\|_\beta\ge\varepsilon\}\text{ is infinite.}$$

For each $n\in M$, let $s_n\in\mathcal{S}_\beta$ such that $\sum_{k\in s_n}|(x_n)_k|\ge\varepsilon$. Since $(x_n)_{n\in\mathbb{N}}$ is weakly-null, we can find $N\subseteq\mathbb{N}$ and $t_n\subseteq s_n$ for each $n\in N$ such that $(t_n)_{n\in N}$ is a block sequence and $\sum_{k\in t_n}|(x_n)_k|\ge\varepsilon/2$. Then by Proposition 3.8, $\{x_n\}_{n\in N}$ is not Banach-Saks, and we are done.

(4) implies (1). Let $K:=\sup_{n\in\mathbb{N}}\|x_n\|$. Let $(x_n)_{n\in M}$ be a subsequence of $(x_n)_{n\in\mathbb{N}}$. If $\alpha=0$, Then X_α is isometric to c_0, and so we are done. Let us suppose that $\alpha>0$. Fix $\varepsilon>0$.

Claim 2. There is $N=\{n_k\}_k\subseteq M$, $n_k<n_{k+1}$, such that for every $i<j$ and every $s\in\mathcal{S}_\alpha$

$$\text{if }\sum_{k\in s}|(x_{n_i})_k|>\varepsilon/2^{i+1},\text{ then }\sum_{k\in s}|(x_{n_j})_k|\le\frac{\varepsilon}{2^j}.$$

□

Its proof is the following: Let $n_0=\min M$. Let $m_0\in\mathbb{N}$ be such that

$$\sum_{k>m_0}|(x_{n_0})_k|\le\frac{\varepsilon}{2}. \tag{3.5}$$

In other words,

$$\{s\in\mathcal{S}_\alpha\,:\,\sum_{k\in s}|(x_{n_0})_k|>\frac{\varepsilon}{2}\}\subseteq(\mathcal{S}_\alpha)_{\le m_0}. \tag{3.6}$$

By Proposition 3.6 (2) there are $\alpha_0^{(0)},\dots,\alpha_{l_0}^{(0)}<\alpha$ such that

$$(\mathcal{S}_\alpha)_{\le m_0}\subseteq\mathcal{S}_{\alpha_0^{(0)}}\oplus\cdots\oplus\mathcal{S}_{\alpha_{l_0}^{(0)}}. \tag{3.7}$$

We use that $(x_n)_n$ is $< \alpha$-null to find $n_1 \in M$, $n_1 > n_0$, be such that for every $n \geq n_1$ one has that

$$\|x_n\|_{(\mathcal{S}_\alpha)_{\leq m_0}} \leq \frac{\varepsilon}{2}. \tag{3.8}$$

Let now $m_1 > \max\{n_1, m_0\}$ be such that

$$\sum_{k > m_1} |(x_{n_1})_k| \leq \frac{\varepsilon}{4}. \tag{3.9}$$

Then there are $\alpha_0^{(1)}, \dots, \alpha_{l_1}^{(1)} < \alpha$ such that

$$\{s \in \mathcal{S}_\alpha \,:\, \sum_{k \in s} |(x_{n_1})_k| > \frac{\varepsilon}{4}\} \subseteq (\mathcal{S}_\alpha)_{\leq m_1} \subseteq \mathcal{S}_{\alpha_0^{(1)}} \oplus \dots \oplus \mathcal{S}_{\alpha_{l_1}^{(1)}}. \tag{3.10}$$

Let now $n_2 \in M$, $n_2 > n_1$ be such that for every $n \geq n_2$ one has that

$$\|x_n\|_{(\mathcal{S}_\alpha)_{\leq m_1}} \leq \frac{\varepsilon}{4}. \tag{3.11}$$

In general, suppose defined n_i, let $m_i > \max\{n_i, m_{i-1}\}$ be such that

$$\sum_{k > m_i} |(x_{n_i})_k| \leq \frac{\varepsilon}{2^{i+1}}. \tag{3.12}$$

Then,

$$\{s \in \mathcal{S}_\alpha \,:\, \sum_{k \in s} |(x_{n_i})_k| > \frac{\varepsilon}{2^{i+1}}\} \subseteq (\mathcal{S}_\alpha)_{\leq m_i} \subseteq \mathcal{S}_{\alpha_0^{(i)}} \oplus \dots \oplus \mathcal{S}_{\alpha_{l_i}^{(i)}}, \tag{3.13}$$

for some $\alpha_0^{(i)}, \dots, \alpha_{l_i}^{(i)} < \alpha$. Let $n_{i+1} \in M$, $n_{i+1} > n_i$ be such that for all $n \geq n_{i+1}$ one has that

$$\|x_n\|_{(\mathcal{S}_\alpha)_{\leq m_i}} \leq \frac{\varepsilon}{2^{i+1}}. \tag{3.14}$$

We have therefore accomplish the properties we wanted for N.

Now fix N as in Claim 2. Then $(x_n)_{n \in N}$ is dominated by the unit basis of c_0. To see this, fix a finite sequence of scalars $(a_i)_{i \in t}$, and $s \in \mathcal{S}_\alpha$. If $\sum_{k \in s} |(x_{n_i})_k| \leq \varepsilon/2^{i+1}$ for every $i \in t$, then,

$$\sum_{k \in s} |(\sum_{i \in t} a_i x_{n_i})_k| \leq \max_{i \in t} |a_i| \cdot \sum_{k \in s} \sum_{i \in t} |(x_{n_i})_k| \leq \max_{i \in t} |a_i| \sum_{i \in t} \frac{\varepsilon}{2^{i+1}} \leq \varepsilon \max_{i \in t} |a_i|.$$

Otherwise, let i_0 be the first $i \in t$ such that $\sum_{k \in s} |(x_{n_i})_k| > \varepsilon/2^{i+1}$. It follows from the claim that

$$\sum_{k \in s} |(x_{n_j})_k| \leq \frac{\varepsilon}{2^j} \text{ for every } i_0 < j \tag{3.15}$$

Hence,

$$\sum_{k \in s} |(\sum_{i \in t} a_i x_{n_i})_k| \leq \sum_{k \in s} |(\sum_{i \in t,\, i < i_0} a_i x_{n_i})_k| + |a_{i_0}| \cdot \sum_{k \in s} |(x_{n_{i_0}})|_k + \sum_{k \in s} |(\sum_{i > i_0} a_i x_{n_i})_k| \leq$$

$$\leq \max_{i \in t} |a_i| \sum_{i < i_0} \frac{\varepsilon}{2^{i+1}} + |a_{i_0}| \|x_{n_{i_0}}\| + \max_{i \in t} |a_i| \sum_{i > i_0} \frac{\varepsilon}{2^i} \leq (\varepsilon + K) \max_i |a_i|.$$

□

Proof of Theorem 3.5. Suppose that A is Banach-Saks, and suppose that $(x_n)_n$ is a sequence in $\mathrm{co}(A)$ without Cesàro-convergent subsequences. Since $\mathrm{co}(A)$, is relatively weakly-compact, we may assume that $x_n \to_n x \in X_\alpha$ weakly. Let $y_n := x_n - x$ for each $n \in \mathbb{N}$. Then $(y_n)_n$ is a weakly-null sequence without Cesàro-convergent subsequences. Hence, by Proposition 3.9, there is some $\varepsilon > 0$ and some $\beta > 0$ such that

$$M = \{n \in \mathbb{N} : \|y_n\|_\beta \geq \varepsilon\} \text{ is infinite.}$$

For each $n \in M$, let $s_n \in \mathcal{S}_\beta$ such that

$$\sum_{k \in s_n} |(y_n)_k| \geq \varepsilon.$$

For each $n \in M$, write as convex combination, $x_n = \sum_{a \in F_n} \lambda_a \cdot a$, where $F_n \subseteq A$ is finite. Since $(y_n)_n$ is weakly-null, it follows that by going to a subsequence if needed that we may assume that $(s_n)_n$ is a block sequence. Let n_0 be such that for all $n \geq n_0$ one has that $\sum_{k \in s_n} |(x)_k| \leq \varepsilon/2$. Hence for every $n \geq n_0$ one has that

$$\begin{aligned} \varepsilon \leq \sum_{k \in s_n} |(y_n)_k| \leq \sum_{k \in s_n} |(x)_k| + \sum_{k \in s_n} \sum_{a \in F_n} \lambda_a |(a)_k| = \sum_{k \in s_n} |(x)_k| + \sum_{a \in F_n} \lambda_a \sum_{k \in s_n} |(a)_k| \leq \\ \leq \sum_{k \in s_n} |(x)_k| + \max_{a \in F_n} \sum_{k \in s_n} |(a)_k| \leq \frac{\varepsilon}{2} + \max_{a \in F_n} \sum_{k \in s_n} |(a)_k|. \end{aligned}$$

So for each $n \geq n_0$ we can find $a_n \in F_n$ such that $\sum_{k \in s_n} |(a_n)_k| \geq \varepsilon/2$. Then, by Proposition 3.8, $(a_n)_n$ is not Banach-Saks. $\square$

Conjecture 1. Let $\mathcal{F}$ be a compact, hereditary and spreading family on $\mathbb{N}$. Then the convex hull of any Banach-Saks subset $A \subseteq X_{\mathcal{F}}$ is again Banach-Saks.

4. A Banach-Saks set whose convex hull is not Banach-Saks

The purpose of this section is to present an example of a Banach-Saks set whose convex hull is not. To do this, using our characterization in Theorem 2.8, it suffices to find a special pre-compact family $\mathcal{F}$ as in (c) of that proposition. The requirement of $\mathcal{F}$ being hereditary is not essential here because $X_{\mathcal{F}} = X_{\widehat{\mathcal{F}}}$.

We introduce now some notions of special interest. In what follows, $I = \bigcup_{n \in \mathbb{N}} I_n$ is a partition of I into finite pieces I_n. A *transversal* (relative to $(I_n)_n$) is an infinite subset T of I such that $\#(T \cap I_n) \leq 1$ for all n. By reformulating naturally Theorem 2.1 we obtain the following.

Lemma 4.1. *Let $T \subseteq I$ be a transversal and $n \in \mathbb{N}$.*

(a) *If $\mathcal{F}$ is not n-large in T, then there exist a transversal $T_0 \subseteq T$ and $m \leq n$ such that $\mathcal{F}[T_0] = [T_0]^{\leq m}$.*
(b) *If $\mathcal{F}$ is not large in T then there is some transversal $T_0 \subseteq T$ and $n \in \mathbb{N}$ such that $\mathcal{F}[T_0] = [T_0]^{\leq n}$.*

(c) *If $\mathcal{F}$ is n-large in T, then there exists a transversal $T_0 \subseteq T$ such that $[T_0]^{\le n} \subseteq \mathcal{F}[T_0]$.*

Definition 4.2. For every $0 < \lambda < 1$ and $s \in \mathcal{F}$ let us define

(a) $s[\lambda] := \{n \in \mathbb{N} : \#(s \cap I_n) \ge \lambda \# I_n\}$,
(b) $s[+] := \{n \in \mathbb{N} : s \cap I_n \ne \emptyset\}$,

and the families of finite sets of $\mathbb{N}$

(c) $\mathcal{G}_\lambda(\mathcal{F}) := \{s[\lambda] : s \in \mathcal{F}\}$,
(d) $\mathcal{G}_+(\mathcal{F}) := \{s[+] : s \in \mathcal{F}\}$.

Proposition 4.3. *Suppose that $\mathcal{F}$ is a T-family on I. For every $0 < \lambda < 1$ and every sequence of scalars $(a_n)_n$, we have that*

$$\lambda \Big\| \sum_n a_n u_n \Big\|_{\mathcal{G}_\lambda(\mathcal{F})} \le \max \left\{ \Big\| \sum_n a_n \Big(\frac{1}{\# I_n} \sum_{j \in I_n} u_j \Big) \Big\|_{\mathcal{F}}, \sup_n |a_n| \right\} \le \Big\| \sum_n a_n u_n \Big\|_{\mathcal{G}_+(\mathcal{F})}. \tag{4.1}$$

Proof. For each n, set

$$x_n := \frac{1}{\# I_n} \sum_{j \in I_n} u_j.$$

Given $(a_n)_n$, by Definition 2.7, for every $s \in \mathcal{F}$, we have that

$$\begin{aligned} \sum_{k \in s} \Big| \Big(\sum_n a_n x_n \Big)_k \Big| &= \sum_{n \in s[+]} \sum_{k \in s \cap I_n} \frac{|a_n|}{\# I_n} = \sum_{n \in s[+]} |a_n| \frac{\#(s \cap I_n)}{\# I_n} \\ &\le \sum_{n \in s[+]} |a_n| \le \Big\| \sum_n a_n u_n \Big\|_{\mathcal{G}_+(\mathcal{F})}, \end{aligned}$$

and

$$\sup_k \left| \left(\sum_n a_n x_n \right)_k \right| \le \sup_n \frac{|a_n|}{\# I_n} \le \sup_n |a_n \le \| \sum_n a_n u_n \|_{\mathcal{G}_+(\mathcal{F})}.$$

This proves the second inequality in (4.1). Now, given $t \in \mathcal{G}_\lambda(\mathcal{F})$, let $s \in \mathcal{F}$ be such that $s[\lambda] = t \subseteq s[+]$. Then

$$\sum_{k \in s} \Big| \Big(\sum_n a_n x_n \Big)_k \Big| \ge \sum_{n \in s[\lambda]} \sum_{k \in s \cap I_n} |a_n| \frac{1}{\# I_n} = \sum_{n \in s[\lambda]} |a_n| \frac{\#(s \cap I_n)}{\# I_n} \ge \lambda \sum_{n \in t} |a_n|.$$

This proves the first inequality in (4.1). □

Observe that the use of the sup-norm of $(a_n)_n$ in the middle term of (4.1) can be explained by the fact that the sequence of averages $(x_n)_n$ is not always seminormalized, independently of the family $\mathcal{F}$. However, for the families we will consider $(x_n)_n$ will be normalized and 1-dominating the unit basis of c_0, so the term $\sup_n |a_n|$ will disappear in (4.1).

Definition 4.4. A pre-compact family $\mathcal{F}$ on I is called a *T-family* when there is a partition $(I_n)_n$ of I into finite pieces I_n such that

(a) $\mathcal{F}$ is not large in any $J \subseteq I$.
(b) There is $0 < \lambda \le 1$ such that $\mathcal{G}_\lambda(\mathcal{F})$ is large in $\mathbb{N}$.

Observe that the pre-compactness of $\mathcal{F}$ follows from (a) above.

Proposition 4.5. *Let $\mathcal{F}$ be a T-family on $I = \bigcup_n I_n$. Then*

(a) *the block sequence of averages $\left(1/\#I_n \sum_{i\in I_n} u_i\right)_n$ is not Banach-Saks in $X_\mathcal{F}$.*
(b) *Every subsequence $(u_i)_{i\in T}$ of $(u_i)_{i\in I}$ has a further subsequence $(u_i)_{i\in T_0}$ equivalent to the unit basis of c_0. Moreover its equivalence constant is at most the integer n such that $\mathcal{F}[T_0] = [T_0]^{\le n}$.*

Proof. Set $x_n := 1/\#I_n \sum_{i\in I_n} u_i$ for each $n \in \mathbb{N}$. (a): From Theorem 2.1 there is $M \subseteq \mathbb{N}$ such that $[M]^1 \subseteq \mathcal{G}_\lambda(\mathcal{F})[M]$. This readily implies that $\|x_n\|_\mathcal{F} \ge \lambda$ for every $m \in M$. Therefore, $(x_n)_{n\in M}$ is a seminormalized block subsequence of the unit basis $(u_n)_n$, and it follows that $(x_n)_{n\in M}$ dominates the unit basis of c_0. From the left inequality in (4.1) in Proposition 4.3 we have that $(x_n)_{n\in M}$ also dominates the subsequence $(u_n)_{n\in M}$ of the unit basis of $X_{\mathcal{G}_\lambda(\mathcal{F})}$. Since $\mathcal{G}_\lambda(\mathcal{F})$ is large, no subsequence of its unit basis is Banach-Sack and therefore $(x_n)_n$ is not Banach-Saks.

(b) Let $(u_i)_{i\in T}$ be a subsequence of the unit basis of $X_\mathcal{F}$. Without loss of generality, we assume that T is a transversal of I. Using our hypothesis (a), the Lemma 4.1 (b) gives us another transversal $T_0 \subset T$ and $n \in \mathbb{N}$ such that $\mathcal{F}[T_0] = [T_0]^{\le n}$. Then the subsequence $(u_i)_{i\in T_0}$ is equivalent to the unit basis of c_0 and therefore Cesàro convergent to 0. In fact, for every $s \in \mathcal{F}$ and for every scalar sequence $(a_j)_{j\in T_0}$

$$\sum_{i\in s} |a_i| = \sum_{i\in s\cap T_0} |a_i| \le \max\{|a_i| : i \in s \cap T_0\}\#(s\cap T_0) \le n\|(a_i)\|_\infty.$$

On the other hand it is clear that $\|(a_i)\|_\infty \le \|\sum_{i\in T_0} a_i e_i\|_\mathcal{F}$. □

This is the main result.

Theorem 4.6. *There is a T-family on $\mathbb{N}$. More precisely, for every $0 < \varepsilon < 1$ there is a partition $\bigcup_n I_n$ of $\mathbb{N}$ in finite pieces I_n and a pre-compact family $\mathcal{F}$ on $\mathbb{N}$ such that*

(a) *$\mathcal{F}$ is not 4-large in any $M \subseteq \mathbb{N}$.*
(b) *$\mathcal{G}_{1-\varepsilon}(\mathcal{F}) = \mathcal{G}_+(\mathcal{F}) = \mathfrak{S}$, the Schreier barrier.*
(c) *For every $s \in \mathcal{G}_+(\mathcal{F})$ one has that $s \cap I_n = I_n$, where n is the minimal m such that $s \cap I_m \ne \emptyset$.*

Corollary 4.7. *For every $\varepsilon > 0$ there is a Schreier-like space $X_\mathcal{F}$ such that every subsequence of the unit basis of it has a further subsequence 4-equivalent to the unit basis of c_0, yet there is a block sequence of averages $((1/\#I_n)\sum_{i\in I_n} u_i)_n$ which is $1+\varepsilon$-equivalent to the unit basis of the Schreier space $X_\mathcal{S}$.*

Proof. From Proposition 4.3, it only rests to see that $\|\sum_n a_n x_n\|_{\mathcal{F}} \geq \sup_n |a_n|$, where $x_n = 1/\#I_n \sum_{i\in I_n} u_i$ for every $n \in \mathbb{N}$. To see this, fix a finite sequence of scalars $(a_n)_{n\in t}$, and fix $m \in t$. Let $u \in \mathfrak{S}$ be such that $\min u = m$ and $u \cap t = \{m\}$, and let $s \in \mathcal{F}$ such that $s[+] = u$. Then, by the properties of $\mathcal{F}$, it follows that $s \cap I_m = I_m$, while $s \cap I_n = \emptyset$ for $n \in t \setminus \{m\}$. Consequently,

$$\|\sum_{n\in t} a_n x_n\|_{\mathcal{F}} \geq \sum_{k\in s} \left| \left(\sum_{n\in t} a_n \frac{1}{\#I_n} \sum_{i\in I_n} u_i \right)_k \right| = |a_m|.$$

□

The construction of our family as in Theorem 4.6 is strongly influenced by the following counterexample of Erdős and Hajnal [13] to the natural generalization of Gillis' Lemma 2.5 to double-indexed sequences of large measurable sets.

Lemma 4.8. *For every $m \in \mathbb{N}$ and $\varepsilon > 0$ there is probability space (Ω, Σ, μ) and a sequence $(A_{i,j})_{1\leq i<j\leq n}$ with $\mu(A_{i,j}) \geq \varepsilon$ for every $1 \leq i < j \leq n$ such that for every $s \subseteq \{1, \dots, n\}$ of cardinality m one has that*

$$\bigcap_{\{i,j\}\in[s]^2} A_{i,j} = \emptyset.$$

Proof. Given $n, r \in \mathbb{N}$ let $\Omega := \{1, \dots, r\}^n$, and let μ be the probability counting measure on r^n. Given $1 \leq i < j \leq n$ we define the subset of n-tuples

$$A_{i,j}^{(n,r)} := \{(a_l)_{l=1}^n \in \{1, \dots, r\}^n \ : \ a_i \neq a_j\}. \tag{4.2}$$

This is the desired counterexample. In fact,

(a) $\#A_{i,j}^{(n,r)} = r^n(1 - 1/r)$ for every $1 \leq i < j \leq n$, and
(b) $\bigcap_{\{i,j\}\in[s]^2} A_{i,j}^{(n,r)} = \emptyset$ for every $s \in [\{1, \dots, n\}]^{r+1}$.

To see (a), given $1 \leq i < j \leq n$

$$\{1, 2, \dots, r\}^n \setminus A_{i,j}^{(n,r)} = \bigcup_{\theta=1}^{r} \{(a_l)_{l=1}^n \in \{1, 2, \dots, r\}^k \ : \ a_i = a_j = \theta\}$$

being the last union disjoint. Since

$$\#\{(a_l)_{l=1}^n \in \{1, 2, \dots, r\}^n : a_i = a_j = \theta\} = r^{n-2},$$

it follows that $\#A_{i,j}^{(n,r)} = r^n(1 - 1/r)$. It is easy to see (b) holds since otherwise we would have found a subset of $\{1, \dots, r\}$ of cardinality $r + 1$. □

Proof of Theorem 4.6. For practical reasons we will define such family not in $\mathbb{N}$ but in a more appropriate countable set I. Fix $0 < \lambda < 1$. We define first the disjoint sequence $(I_n)_n$. For each $m \in \mathbb{N}, m \geq 4$, let r_m be such that

$$\left(1 - \frac{1}{r_m}\right)^{\binom{m-2}{2}} \geq \lambda. \tag{4.3}$$

Let $4 \leq m \leq n$ be fixed. Let

$$I_{m,n} := \{1, \ldots, r_m\}^{n \times [\{2,\ldots,m-1\}]^2}.$$

Let $I_n = \{n\}$ for $n = 1, 2, 3$. For $n \geq 4$ let

$$I_n := \prod_{4 \leq m \leq n} I_{m,n} = \prod_{4 \leq m \leq n} \{1, \ldots, r_m\}^{n \times [\{2,\ldots,m-1\}]^2}.$$

Observe that for $n \neq n'$ one has that $I_n \cap I_{n'} = \emptyset$. Let $I := \bigcup_n I_n$. Now, given $4 \leq m_0 \leq n$ and $2 \leq i_0 < j_0 \leq m_0 - 1$, let

$$\pi_{i_0,j_0}^{(n,m_0)} : I_n \to \{1, 2, \ldots, r_{m_0}\}^n$$

be the natural projection,

$$\pi_{i_0,j_0}^{(n,m_0)}(\,((b_{i,j}^{(l,m)}))_{4 \leq m \leq n,\, 1 \leq l \leq n,\, 2 \leq i < j \leq m-1}\,) := (b_{i_0,j_0}^{(l,m_0)})_{l=1}^n \in \{1, 2, \ldots, r_{m_0}\}^n.$$

We start with the definition of the family $\mathcal{F}$ on I. Recall that $\mathfrak{S} := \{s \subseteq \mathbb{N} : \#s = \min s\}$ is the Schreier barrier. We define $F : \mathfrak{S} \to [I]^{<\infty}$ such that $F(u) \subseteq \bigcup_{n \in u} I_n$ and then we will define $\mathcal{F}$ as the image of F. Fix $u = \{n_1 < \cdots < n_{n_1}\} \in \mathfrak{S}$:

(i) For $u = \{1\}$, let $F(u) := I_1$.

(ii) For $u := \{2, n\}$, $2 < n$, let $F(u) := I_2 \cup I_n$.

(iii) For $u := \{3, n_1, n_2\}$, $3 < n_1 < n_2$, let $F(u) := I_3 \cup I_{n_1} \cup I_{n_2}$.

(iv) For $u = \{n_1, \ldots, n_{n_1}\}$ with $3 < n_1 < n_2 < \cdots < n_{n_1}$, then let

$$F(u) \cap I_{n_k} := I_{n_k} \text{ for } k = 1, 2, 3,$$

and for $3 < k \leq n_1$, let

$$F(u) \cap I_{n_k} := \bigcap_{1 < i < j < k} \big(\pi_{i,j}^{(n_k,n_1)}\big)^{-1}\big(A_{n_i,n_j}^{(n_k,r_{n_1})}\big) \tag{4.4}$$

Where the A's are as in (4.2). Explicitly,

$$F(u) \cap I_{n_k} = \{\big((b_{i,j}^{(l,m)})\big)_{4 \leq m \leq n_k,\, 1 \leq l \leq n_k,\, 2 \leq i < j \leq m-1} \in I_{n_k} \,:\, b_{i,j}^{(n_i,n_1)} \neq b_{i,j}^{(n_j,n_1)},\ 1 < i < j < k\}.$$

Observe that it follows from (4.4) that

$$\pi_{i,j}^{(n_k,n_1)}\big(F(u) \cap I_{n_k}\big) = A_{n_i,n_j}^{(n_k,r_{n_1})} \subset \{1, 2, \ldots, r_{n_1}\}^{n_k} \tag{4.5}$$

for every $1 < i < j < k$.

From the definition of $\mathcal{F}$ it follows that $u = F(u)[+]$ for every $u \in \mathfrak{S}$. Now, we claim that given $u \in \mathfrak{S}$, we have that $u = F(u)[\lambda]$, or, in other words, $\#(F(u) \cap I_n) \geq \lambda \# I_n$ for every $n \in u$. The only non-trivial case is when $u = \{n_1 < \cdots < n_{n_1}\}$ with $n_1 > 3$, and $n = n_k$ is such that $3 < k \leq n_1$. It follows from the equality in (4.4), (a) in the proof of Lemma 4.8, and the choice of r_{n_1} in (4.3) that

$$\frac{\#(F(u)\cap I_{n_k})}{\#(I_{n_k})} = \prod_{1<i<j<k} \frac{\#(A_{n_i,n_j}^{(n_k,r_{n_1})})}{(r_{n_1})^{n_k}} = \prod_{1<i<j<k}\left(1-\frac{1}{r_{n_1}}\right) \geq \left(1-\frac{1}{r_{n_1}}\right)^{\binom{n_1-2}{2}} \geq \lambda$$

Summarizing, $\mathcal{G}(\mathcal{F},\lambda) = \mathcal{G}(\mathcal{F},+) = \mathfrak{S}$. Thus, $\mathcal{F}$ satisfies the property (b) in Theorem 4.6. For the property (a) we use the following fact.

Lemma 4.9. *Suppose that $\mathcal{A} \subseteq \mathfrak{S}$ is a subset such that*

(a) $\min u = \min v = n_1 > 3$ *for all* $u, v \in \mathcal{A}$.

(b) *there are* $1 < i < j < n_1$ *and a set* $w \subset \mathbb{N}$ *such that*

  (b.1) $\#w \geq r_{n_1} + 2$ *and* $n_1 < \min w$.

  (b.2) *For every* $l_1 < l_2 < \max w$ *in* w *there is* $u \in \mathcal{A}$ *such that* $\{n_1, l_1, l_2, \max w\} \subset u$, $\#(u \cap \{1,2,\dots,l_1\}) = i$ *and* $\#(u \cap \{1,2,\dots,l_2\}) = j$.

Then

$$I_{\max w} \cap \bigcap_{u\in\mathcal{A}} F(u) = \emptyset.$$

Proof of Lemma 4.9. Observe that for $l \in u$, $\#(u \cap \{1,2,\dots,l\}) = i$ just means that l is the i^{th}−element of u. For every couple $\{l_1 < l_2\} \in [w \setminus \{\max w\}]^2$, take $u_{l_1,l_2} \in \mathcal{A}$ satisfying the condition of (b.2). Since, $u_{l_1,l_2} = \{n_1 < \cdots < n_i = l_1 < \cdots < n_j = l_2 < \cdots < \max w < \cdots \leq n_{n_1}\}$, it follows from the equality in (4.4) that

$$\pi_{i,j}^{(\max w, n_1)}\big(F(u_{l_1,l_2}) \cap I_{\max w}\big) = A_{l_1,l_2}^{(\max w, r_{n_1})}.$$

Hence

$$\begin{aligned}\pi_{i,j}^{(\max w,n_1)}(I_{\max w} \cap \bigcap_{u\in\mathcal{A}} F(u)) \subseteq & \bigcap_{\{l_1,l_2\}\in[w\setminus\{\max w\}]^2} \pi_{i,j}^{(\max w,n_1)}(I_{\max w}\cap F(u_{l_1,l_2})) = \\ = & \bigcap_{\{l_1,l_2\}\in[w\setminus\{\max w\}]^2} A_{l_1,l_2}^{(\max w,r_{n_1})} = \emptyset\end{aligned}$$

where the last equality follows from (b) in the proof of Lemma 4.8, since $\#w \geq r_{n_1} + 2$.

□

We continue with the proof property (a) of $\mathcal{F}$ in Theorem 4.6. Suppose otherwise that there exists a transversal T of I such that $\mathcal{F}$ is 4-large in T. By Lemma 4.1 (c), there exists $T_0 \subseteq T$ such that $[T_0]^4 \subseteq \mathcal{F}[T_0]$. For every $k \in T_0$, $n(k)$ denotes the unique integer m for which $k \in I_m$. It is easy to see that if $k_1, k_2 \in T_0$ with $k_1 < k_2$, then $n(k_1) < n(k_2)$. Now, for each $t = \{k_0 < k_1 < k_2 < k_3\}$ in $[T_0]^4$, let us choose $U(t) \in \mathfrak{S}$ such that

$$t \subset F(U(t)).$$

Observe that $\{n(k_0), n(k_1), n(k_2), n(k_3)\} \subset U(t)$, and hence $\#U(t) \leq n(k_0)$. Now, let

$$\bar{k} := \min T_0 \text{ and } \bar{n} := n(\bar{k}).$$

Define the coloring $\Theta : [T_0 \setminus \{\bar{k}\}]^3 \to [\{1, 2, \ldots, n(\bar{k})\}]^3$ for each $t = \{k_1 < k_2 < k_3\}$ in $T_0 \setminus \{\bar{k}\}$ as

$$\Theta(t) = \big(\#(U(\{\bar{k}\} \cup t) \cap \{1, \ldots, n(k_1)\}), \#(U(\{\bar{k}\} \cup t) \cap \{1, \ldots, n(k_2)\}), \min U(\{\bar{k}\} \cup t)\big).$$

By the Ramsey theorem, there exist $1 < i < j < n_1 \leq n(\theta)$ and $T_1 \subseteq T_0 \setminus \{\theta\}$ such that Θ is constant on T_1 with value $\{i, j, n_1\}$. Choose $k_1 < \cdots < k_{r_{\bar{n}}+2}$ in T_1, and set

$$\mathcal{A} := \{U(\{\bar{k}, k_{l_1}, k_{l_2}, k_{r_{\bar{n}}+2}\}) : 1 \leq l_1 < l_2 < r_{\bar{n}} + 2\}.$$

Notice that $\mathcal{A}$ fulfills the hypothesis of Lemma 4.9 with respect to the set $w = \{n(t_{l_1}) : 1 \leq l_1 \leq r_{n(\theta)} + 2\}$, and therefore

$$I_{n(t_{r_{n(\theta)}+2})} \cap \bigcap_{u \in \mathcal{A}} F(u) = \emptyset, \tag{4.6}$$

which contradicts the fact that

$$k_{r_{\bar{n}}+2} \in I_{n(k_{r_{\bar{n}}+2})} \cap \bigcap_{u \in \mathcal{A}} F(u).$$

The family $\mathcal{F}$ clearly has property (c) from the statement of Theorem 4.6 by construction. This finishes the proof of the desired properties of $\mathcal{F}$. $\square$

A similar analysis will be used now to prove that the closed linear span of the sequence

$$x_n = \frac{1}{\#I_n} \sum_{j \in I_n} u_j$$

is not a complemented subspace of $X_{\mathcal{F}}$. Let $(x_n^*)_n$ denote the sequence of biorthogonal functionals to $(x_n)_n$ on $[x_n]^*$.

Proposition 4.10. *If $T : [u_k]_k \to [x_n]_n$ is a linear mapping such that*

$$\lim_{k \to \infty} \langle x_{n(k)}^*, T u_k \rangle \neq 0,$$

then T cannot be bounded. In particular, there does not exist a projection $P : X_{\mathcal{F}} \to [x_n]_n$.

Proof. Let us suppose that T is bounded. Since $\lim_{k \to \infty} \langle x_{n(k)}^*, T u_k \rangle \neq 0$, let $\alpha > 0$ be such that $|\langle x_{n(k_j)}^*, T u_{k_j} \rangle| \geq \alpha$ for every $j \in \mathbb{N}$. Moreover, since $(u_k)_k$ is weakly null, up to equivalence we can assume that $(T u_{k_j})_j$ are disjoint blocks with respect to $(x_n)_n$.

By Proposition 4.5(b), passing to a further subsequence it holds that $(u_{k_j})_j$ is 3-equivalent to the unit basis of c_0. Now, let $0 < \lambda \leq 1$ such that $\mathcal{G}_\lambda(\mathcal{F}) = \mathfrak{S}$, and take $n_0 > \frac{3\|T\|}{\alpha\lambda}$. Let $u \in \mathfrak{S}$ with $\min u = n_0$. We have

$$3 \geq \Big\| \sum_j u_{k_j} \Big\| \geq \frac{1}{\|T\|} \Big\| \sum_j T u_{k_j} \Big\|_{X_{\mathcal{F}}} \geq \sum_{i \in F(u)} |\langle u_i^*, \sum_j T u_{k_j} \rangle| \geq \frac{n_0 \alpha \lambda}{\|T\|}.$$

This is a contradiction with the choice of n_0. □

Remark 4.11. The Cantor-Bendixson rank of a T-family must be infinite. To see this, observe that if $f : I \to J$ is finite-to-one [1] then f preserves the rank $\varrho(\mathcal{F})$ of pre-compact families $\mathcal{F}$ in I. Since $n(\cdot) : I \to \mathbb{N}$, $n(i) = n$ if and only if $i \in I_n$ is finite-to-one and since $n(\mathcal{F}) = \{\{n(i)\}_{i\in s} : s \in \mathcal{F}\} = \mathcal{G}_+(\mathcal{F}) \supseteq \mathcal{G}_\lambda(\mathcal{F})$ is large, it follows that $\varrho(n(\mathcal{F})) = \varrho(\mathcal{F})$ is infinite. In this way our T-family $\mathcal{F}$ in Theorem 4.6 is minimal because $\varrho(\mathcal{F}) = \varrho(n(\mathcal{F})) = \varrho(\mathfrak{S}) = \omega$.

4.1. **A reflexive counterexample.** There is a reflexive counterpart of our example $X_\mathcal{F}$. Indeed we are going to see that the Baernstein space $X_{\mathcal{F},2}$ for our family $\mathcal{F}$ is such space. It is interesting to note that the corresponding construction $X_{\mathcal{S},2}$ for the Schreier family $\mathcal{S}$ was used by A. Baernstein II in [5] to provide the first example of a reflexive space without the Banach-Saks property. This construction was later generalized by C. J. Seifert in [29] to obtain $X_{\mathcal{S},p}$.

Definition 4.12. Given a pre-compact family $\mathcal{F}$, and given $1 \le p \le \infty$, one defines on $c_{00}(\mathbb{N})$ the norm $\|x\|_{\mathcal{F},p}$ for a vector $x \in c_{00}(\mathbb{N})$ as follows:

$$\|x\|_{\mathcal{F},p} := \sup\{\|(\|E_i x\|_\mathcal{F})_{i=1}^n\|_p \, : \, E_1 < \cdots < E_n, n \in \mathbb{N}\} \tag{4.7}$$

where $E_1 < \cdots < E_n$ are finite sets and Ex is the natural projection on E defined by $Ex := \mathbb{1}_E \cdot x$. Let $X_{\mathcal{F},p}$ be the corresponding completion of $(c_{00}, \|\cdot\|_{\mathcal{F},p})$.

Again, the unit Hamel basis of c_{00} is a 1-unconditional Schauder basis of $X_{\mathcal{F},p}$. Notice also that this construction generalizes the Schreier-like spaces, since $X_{\mathcal{F},\infty} = X_\mathcal{F}$.

Proposition 4.13. *The space $X_{\mathcal{F},p}$ is ℓ_p-saturated. Consequently, if $1 < p < \infty$, the space $X_{\mathcal{F},p}$ is reflexive.*

Proof. The case $p = \infty$ was already treated when we introduced the Schreier-like spaces after Definition 2.7. So, suppose that $1 \le p < \infty$.

Claim 3. Suppose that $(x_n)_n$ is a normalized block sequence of $(u_n)_n$. Then

$$\|\sum_n a_n x_n\|_{\mathcal{F},p} \ge \|(a_n)_n\|_p. \tag{4.8}$$

To see this, for each n, let $(E_i^{(n)})_{i=1}^{k_n}$ be a block sequence of finite sets such that

$$1 = \sum_{i=1}^{k_n} \|E_i^{(n)} x_n\|_\mathcal{F}^p. \tag{4.9}$$

[1] $f : I \to J$ is finite-to-one when $f^{-1}\{j\}$ is finite for every $j \in J$.

Without loss of generality we may assume that $\bigcup_{i=1}^{k_n} E_i^{(n)} \subseteq \operatorname{supp} x_n$, hence $E_{k_n}^{(n)} < E_1^{(n+1)}$ for every n. Set $x = \sum_n a_n x_n$. It follows that

$$(\|\sum_n a_n x_n\|_{\mathcal{F},p})^p \geq \sum_n \sum_{i=1}^{k_n} \|E_i^{(n)} x\|_{\mathcal{F}}^p = \sum_n |a_n|^p.$$

This finishes the proof of Claim 3. It follows from this claim that $c_0 \not\hookrightarrow X_{\mathcal{F},p}$. Fix now a normalized block sequence $(x_n)_n$ of $(u_n)_n$ and $\varepsilon > 0$. Let $(\varepsilon_n)_n$ be such that $\sum_n \varepsilon_n^p \leq \varepsilon/2$, $\varepsilon_n > 0$ for each n. Since $c_0 \not\hookrightarrow X_{\mathcal{F},p}$ and since $X_{\mathcal{F}}$ is c_0-saturated, we can find a $\|\cdot\|_{\mathcal{F},p}$-normalized block sequence $(y_n)_n$ of $(x_n)_n$ such that

$$\|y_n\|_{\mathcal{F}} \leq \varepsilon_n. \tag{4.10}$$

Claim 4. For every sequence of scalars $(a_n)_n$ we have that

$$\|(a_n)_n\|_p \leq \|\sum_n a_n y_n\|_{\mathcal{F},p} \leq (1+\varepsilon)\|(a_n)_n\|_p. \tag{4.11}$$

Once this is established, we have finished the proof of this proposition. The first inequality in (4.11) is consequence of Claim 3. To see the second one, fix a block sequence $(E_i)_{i=1}^l$ of finite subsets of $\mathbb{N}$. For each n, let $B_n := \{j \in \{1,\dots,l\} \,:\, E_j x_n \neq \emptyset\}$, and for n such that $B_n \neq \emptyset$, let $i_n := \min B_n$, $j_n := \max B_n$. Observe that $i_n, j_n \in B_m$ for at most one $m \neq n$. Then, setting $y = \sum_n a_n y_n$,

$$\begin{aligned}\sum_{i=1}^l \|E_i y\|_{\mathcal{F}}^p = \sum_{i\in\bigcup_n B_n} \|E_i y\|_{\mathcal{F}}^p \leq \sum_n \sum_{i\in B_n} \|E_i y\|_{\mathcal{F}}^p \leq |a_1|^p \sum_{i\in B_1} \|E_i y_1\|_{\mathcal{F}}^p + \|E_{j_1} y\|_{\mathcal{F}}^p + \\ + \sum_{n\geq 2} \left(|a_n|^p \sum_{i\in B_n} \|E_i y_n\|_{\mathcal{F}}^p + \|E_{i_n} y\|_{\mathcal{F}}^p + \|E_{j_n} y\|_{\mathcal{F}}^p \right) \leq \\ \leq \sum_n |a_n|^p \|y_n\|_{\mathcal{F},p}^p + 2\max_n |a_n|^p \sum_n \varepsilon_n^p \leq (1+\varepsilon)\sum_n |a_n|^p.\end{aligned}$$

□

Proposition 4.14. *Given $0 < \lambda < 1$, let $\mathcal{F}$ be a T-family for λ as in Theorem 4.6 with respect to some $\bigcup_n I_n$. Then*

(a) *Every subsequence of the unit basis of $X_{\mathcal{F},p}$ has a further subsequence 6-equivalent to the unit basis of ℓ_p.*
(b) *The sequence of averages*

$$\left(\frac{1}{\#I_n} \sum_{i\in I_n} u_i \right)_n$$

is λ-equivalent to the unit basis of the Seifert space $X_{\mathcal{S},p}$.

PROOF. (a): Fix a subsequence $(u_n)_{n\in M}$ of $(u_n)_n$ and let $(u_n)_{n\in N}$ be a further sequence of it such that $\mathcal{F}[N]\subseteq [N]^{\le 3}$. Fix also a sequence of scalars $(a_n)_{n\in N}$ such that $x=\sum_{n\in N} a_n u_n\in X_{\mathcal{F},p}$. Given a finite subset $E\subseteq \mathbb{N}$ we obtain that

$$\|Ex\|_{\mathcal{F}}\le 3 \max_{n \text{ is such that } Ex_n\neq 0} |a_n|. \tag{4.12}$$

Now given a block sequence $(E_i)_{i=1}^l$ of finite subsets of $\mathbb{N}$, and given $i=1,\dots,l$, let $A_i=\{n\in N\,:\, E_i x_n\neq 0\}$ and let $B:=\{i\in\{1,\dots l\}\,:\, A_i\neq\emptyset\}$. Then we obtain that

$$\sum_{i=1}^l \|E_i x\|_{\mathcal{F}}^p \le 3\sum_{i\in B}(\max_{n\in A_i}|a_n|)^p\le 6\sum_n |a_n|^p$$

the last inequality because $A_i\cap A_j=\emptyset$ if $i<j$ are not consecutive in B, and if $i<j$ are consecutive, then $\#(A_i\cap A_j)\le 1$. The other inequality is proved in the Claim 3 of Proposition 4.13.

Let us prove (b): First of all, observe that by definition we have that $X_{\mathcal{S},p}=X_{\mathfrak{S},p}$. Set $x_n:=(1/\#I_n)\sum_{i\in I_n} u_i$ for each $n\in\mathbb{N}$, and fix a sequence of scalars $(a_n)_n$. Set also

$$x=\sum_n a_n x_n \text{ and } u=\sum_n a_n u_n.$$

Let $(E_i)_{i=1}^l$ be a block sequence of finite subsets of $\mathbb{N}$ such that

$$\|\sum_n a_n u_n\|_{\mathfrak{S},p}^p=\sum_{i=1}^l \|E_i u\|_{\mathfrak{S}}^p. \tag{4.13}$$

For each $i=1,\dots,l$, let $t_i\in\mathfrak{S}$ be such that $\|E_i u\|_{\mathfrak{S}}=\sum_{n\in t_i\cap E_i}|a_n|$. For each $i=1,\dots,l$ let $s_i\in\mathcal{F}$ be such that $s_i[\lambda]=t_i$, and set $F_i:=\bigcup_{n\in E_i} I_n$. Notice that $(F_i)_{i=1}^l$ is a block sequence of finite subsets of $\bigcup_n I_n=\mathbb{N}$. Then

$$\begin{aligned}\|\sum_n a_n x_n\|_{\mathcal{F},p}^p &\ge \sum_{i=1}^l \|F_i(\sum_n a_n x_n)\|_{\mathcal{F}}^p=\sum_{i=1}^l \|\sum_{n\in E_i} a_n x_n\|_{\mathcal{F}}^p\ge\\ &\ge \sum_{i=1}^l\left(\sum_{k\in s_i}|(\sum_{n\in E_i} a_n x_n)_k|\right)^p\ge \sum_{i=1}^l(\lambda\sum_{n\in E_i\cap t_i}|a_n|)^p=\lambda^p\|\sum_n a_n u_n\|_{\mathfrak{S},p}^p.\end{aligned}$$

For the other inequality, let $(F_i)_{i=1}^l$ be a block sequence such that

$$\|\sum_n a_n x_n\|_{\mathcal{F},p}^p=\sum_{i=1}^l\|F_i x\|_{\mathcal{F}}^p. \tag{4.14}$$

For each $i=1,\dots,l$, let $s_i\in\mathcal{F}$ be such that $\|F_i x\|_{\mathcal{F}}=\sum_{k\in s_i}|(F_i x)_k|$, and $E_i:=\{n\in\mathbb{N}\,:\, F_i\cap I_n\neq\emptyset\}$. Then, setting $t_i:=s_i[+]\in\mathfrak{S}$, we have that

$$\|F_i x\|_{\mathcal{F},p}=\sum_{n\in s_i[+]\cap E_i}|a_n|\frac{\#((s_i\cap F_i)\cap I_n)}{\#I_n}\le\sum_{n\in s_i[+]}|(E_i u)_n|\le\|E_i u\|_{\mathfrak{S}}. \tag{4.15}$$

Since $(E_i)_{i=1}^l$ is a block sequence it follows that

$$\|\sum_n a_n u_n\|_{\mathfrak{S},p}^p \geq \sum_{i=1}^l \|E_i u\|_{\mathfrak{S}}^p \geq \sum_{i=1}^l \|F_i x\|_{\mathcal{F}}^p = \|\sum_n a_n x_n\|_{\mathcal{F},p}^p.$$

□

There is another, more general, approach to find a reflexive counterexample to Question 1. This can be done by considering the interpolation space $\Delta_p(W, X)$, $1 < p < \infty$, where W is the closed absolute convex hull of a Banach-Saks subset of X which it is not Banach-Saks itself.

Recall that given a convex, symmetric and bounded subset W of a Banach space X, and $1 < p < \infty$, one defines the Davis-Figiel-Johnson-Pelczynski [10] interpolation space $Y := \Delta_p(W, X)$ as the space

$$\{x \in X : \|x\|_Y < \infty\},$$

where

$$\|x\|_Y := \|(|x|_n)_n\|_p$$

and where for each n,

$$|x|_n := \inf\{\lambda > 0 : \frac{x}{\lambda} \in 2^n W + \frac{1}{2^n} B_X\}.$$

The key is the following.

Lemma 4.15. *A subset A of W is a Banach-Saks subset of X if and only if A is a Banach-Saks subset of $Y := \Delta_p(W, X)$.*

Proof. Fix $A \subseteq W$, and set $Y := \Delta_p(W, X)$ Since the identity $j : Y \to X$ is a bounded operator, it follows that if A is a Banach-Saks subset of Y then $A = j(A)$ is also a Banach-Saks subset of X.

Now suppose that A is a Banach-Saks subset of X. Going towards a contradiction, we fix a weakly convergent sequence $(x_n)_n$ in A with limit x generating an ℓ_1-spreading model. Let δ witnessing that, and set $y_n := x_n - x \in 2W$ for each n. Observe that it follows from the definition that

(a) For every $\lambda > 0$ and every $\varepsilon > 0$ there is n_0 such that for every $x \in \lambda W$ we have that $\sum_{n > n_0} |x|_n^p \leq \varepsilon$.

Since A is Banach-Saks in X, we assume without loss of generality that the sequence $(y_n)_n$ is uniformly weakly-convergent (to 0). Observe that then

(b) For every $\varepsilon > 0$ there is n such that if $\#s = n$, then $\|\sum_{n \in s} y_n\|_X \leq \varepsilon \# s$.

Consequently,

(c) For every $\varepsilon > 0$ and r there is m such that if $\#s = m$, then $\sum_{n \leq r} |\sum_{k \in s} y_k|^p \leq \varepsilon$.

Now let $k \in \mathbb{N}$ be such that $k^{1/p} < \delta k$, and $\varepsilon > 0$ such that $k^{1/p} + \varepsilon < \delta k$. Using (a) and (c) above we can find finite sets $s_1 < \dots s_n$ such that

(d) $s = \bigcup_{i=1}^n s_i \in \mathcal{S}$.

(e) Setting $z_i := (1/\#s_i) \sum_{k \in s_i} y_k$ for each $i = 1, \dots, k$, then there is a block sequence $(v_i)_{i=1}^k$ in ℓ_p such that $\|v_i\|_p \leq 1$, $i = 1, \dots, k$, and such that

$$\|(|z_1 + \cdots + z_k|_n)_n - (v_1 + \cdots + v_k)\|_p \leq \varepsilon. \tag{4.16}$$

It follows then from (d), (e) and the fact that $(y_n)_n$ generates an ℓ_1-spreading model with constant δ that

$$\delta k \leq \|\frac{1}{k} \sum_{i=1}^k z_i\|_Y \leq \|v_1 + \cdots + v_k\|_p + \varepsilon \leq k^{\frac{1}{p}} + \varepsilon < \delta k,$$

a contradiction. □

Let now $X := X_{\mathcal{F}}$ where $\mathcal{F}$ is a T-family, let W be the closed absolute convex hull of the unit basis $\{u_n\}_n$ of $X_{\mathcal{F}}$

Proposition 4.16. *The interpolation space $Y := \Delta_p(W, X_{\mathcal{F}})$, $1 < p < \infty$, is a reflexive space with a weakly-null sequence which is a Banach-Saks subset of Y, but its convex hull is not.* □

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