

REARRANGEMENT INVARIANT SPACES WITH KATO PROPERTY

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Dedicated to Professor Lech Drewnowski on the occasion of his 70th birthday.

ABSTRACT. We study rearrangement invariant spaces on which the classes of strictly singular and compact operators coincide. The relation between this property and the fact that every normalized disjoint sequence in the space has a subsequence equivalent to the unit vector basis of ℓ_2 is analyzed.

1. INTRODUCTION

A classical result of J. Calkin [6] states that the only non-trivial closed ideal of operators in Hilbert space is the ideal of compact operators. In particular, as pointed out by T. Kato [16], on Hilbert spaces the ideals of strictly singular and compact operators coincide. This same property is also shared by the spaces ℓ_p (for $1 \leq p < \infty$) and c_0 (see also [12], [13]). More recently, this result has been extended to more instances of Banach lattices by means of the notion of disjointly homogeneous Banach lattice [9].

Our aim in this note is to study Banach spaces satisfying this property. Namely, we will say that a Banach space has Kato property when every strictly singular endomorphism is necessarily compact. Our interest here will focus on rearrangement invariant spaces, and in particular, on Lorentz, Orlicz and Marcinkiewicz spaces.

Let us begin by recalling the terminology employed in [9]. A Banach lattice E is called *disjointly homogeneous* if any two sequences $(x_n), (y_n)$ of pairwise disjoint normalized elements in E have an equivalent subsequence $(x_{n_k}) \sim (y_{n_k})$. Similarly, E will be called *2-disjointly homogeneous* when every sequence of pairwise disjoint normalized elements in E has a subsequence equivalent to the unit vector basis of ℓ_2 . Several properties of this class of spaces have been recently studied in [11], [9] and [10].

The importance of disjointly homogeneous spaces arises by the following facts that were given in [9]:

Theorem 1.1. *Let E be a discrete Banach lattice with a disjoint basis that is disjointly homogeneous. Then every strictly singular operator $T \in \mathcal{L}(E)$ is compact.*

Theorem 1.2. *Let E be a separable 2-disjointly homogeneous Banach lattice with finite cotype. Then every strictly singular operator $T \in \mathcal{L}(E)$ is compact.*

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Theorem 1.3. *Let E be a 2-disjointly homogeneous rearrangement invariant space on $[0, 1]$ with upper Boyd index $q_E < \infty$. Then every strictly singular operator $T \in \mathcal{L}(E)$ is compact.*

Our aim in this note is to study whether converse statements to these theorems hold.

Observe that since ℓ_p spaces are not 2-disjointly homogeneous for $p \neq 2$, but ℓ_p has Kato property, the converse of Theorem 1.2 is in general false. In a similar fashion, let X denote the space isomorphic to $L_{p,2}$ for $1 < p < 2$, with the lattice structure given by the unconditional Haar basis. Clearly, since $L_{p,2}$ is 2-disjointly homogeneous, by Theorem 1.2, X has Kato property, but for every $p < q \leq 2$, X contains a sequence of disjoint vectors spanning a subspace isomorphic to ℓ_q . Therefore, the converse to Theorem 1.1 is also false.

Our interest will hence focus on the converse to Theorem 1.3. Namely, we have the following:

Question 1.4. *Let X be an r.i. space on $[0, 1]$ such that every strictly singular operator $T \in \mathcal{L}(E)$ is compact. Must X be 2-disjointly homogeneous?*

The paper is organized as follows: after some preliminaries on rearrangement invariant spaces, in Section 3 we study several facts concerning Kato property and 2-disjointly homogeneous spaces. In Section 4, we show that non-reflexive spaces fail Kato property and provide a characterization of Lorentz spaces with Kato property. Finally, Section 5 is devoted to the study of Orlicz spaces with Kato property.

2. NOTATION AND PRELIMINARIES

Let us recall here some notions mainly related to rearrangement invariant spaces. The reader is referred to the monographs [5] and [19] for further considerations.

Let (Ω, Σ, μ) be $[0, 1]$ or \mathbb{R}_+ (with Lebesgue measure). A rearrangement invariant (r.i.) function space X over (Ω, Σ, μ) is a Banach lattice of measurable functions on Ω satisfying the following properties:

- (1) If $f \in X$, and g is a measurable function with the same distribution as f , then $g \in X$ and $\|f\| = \|g\|$.
- (2) The Köthe dual X' is a norming subset of X^* .

Observe that for an r.i. space X we have the following norm-one inclusions:

$$L_\infty(\Omega, \Sigma, \mu) \cap L_1(\Omega, \Sigma, \mu) \subset X \subset L_\infty(\Omega, \Sigma, \mu) + L_1(\Omega, \Sigma, \mu)$$

where $\|f\|_{L_\infty \cap L_1} = \max\{\|f\|_{L_1}, \|f\|_{L_\infty}\}$, and $\|f\|_{L_1 + L_\infty} = \int_0^1 f^*(t) dt$ (here, f^* denotes the decreasing rearrangement of f).

Given an r.i. space X , the function

$$\varphi_X(t) = \|\chi_{[0,t]}\|_X$$

for $t > 0$, is called the fundamental function of X . It is clear that φ_X is an increasing function in t , satisfying that

$$\varphi_X(t)\varphi_{X'}(t) = t.$$

In particular, $\varphi(t)/t$ is decreasing in t .

Among r.i. spaces, our attention will focus on Orlicz, Lorentz and Marcinkiewicz spaces. Let us recall their definitions.

Given an Orlicz function M (i.e. a continuous convex increasing function with $M(0) = 0$ and $\lim_{t \rightarrow \infty} M(t) = \infty$), the *Orlicz function space* $L^M(0, \infty)$ is the space of all measurable functions f on $(0, \infty)$ such that $\int_0^\infty M\left(\frac{|f(t)|}{r}\right) d\lambda < \infty$ for some $r > 0$. The norm is defined by

$$\|f\| = \inf \left\{ r > 0 : \int_0^\infty M\left(\frac{|f(t)|}{r}\right) d\lambda \leq 1 \right\}.$$

The space $L^M[0, 1]$ is defined similarly with functions on $[0, 1]$.

An Orlicz function M is said to satisfy the Δ_2 -condition at ∞ (respectively, at 0) when $\limsup_{t \rightarrow \infty} M(2t)/M(t) < \infty$ (resp. $\limsup_{t \rightarrow 0} M(2t)/M(t) < \infty$). Note that the space $L^M[0, 1]$ is separable if and only if M satisfies the Δ_2 condition at ∞ . If \bar{M} denotes the Young conjugate function, then

$$(L^M[0, 1])' = L^{\bar{M}}[0, 1].$$

Recall that given $1 \leq q < \infty$ and w a positive, non-increasing function in \mathbb{R}_+ , such that $\lim_{t \rightarrow 0} w(t) = \infty$, $\lim_{t \rightarrow \infty} w(t) = 0$, $\int_0^1 w(t) dt = 1$ and $\int_0^\infty w(t) dt = \infty$, the *Lorentz function space* $\Lambda_w^q(\mathbb{R}_+)$ [21] is the space of all measurable functions f on \mathbb{R}_+ such that

$$\|f\|_{\Lambda_w^q} = \left(\int_0^\infty f^*(t)^q w(t) dt \right)^{1/q} < \infty.$$

If the conditions on w are only imposed on the interval $[0, 1]$, and we define the norm by integrating over $[0, 1]$, then we obtain the Lorentz function space $\Lambda_w^q[0, 1]$.

As a particular case, for $1 < p < \infty$ and $1 \leq q \leq \infty$, we can consider the *Lorentz space* $L_{p,q}$ is the space of all measurable functions f in \mathbb{R}_+ such that

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty & \text{for } 1 \leq q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) < \infty, & \text{if } q = \infty. \end{cases}$$

Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an increasing function, with $\varphi(t)/t$ decreasing and such that $\varphi(t) = 0$ only when $t = 0$. Given such a function, we can consider the *Marcinkiewicz function space* $M_\varphi(\mathbb{R}_+)$ consisting of measurable functions such that

$$\|f\|_{M_\varphi} = \sup_{t>0} \frac{\varphi(t)}{t} \int_0^t f^*(s) ds < \infty.$$

Similarly, we can consider the Lorentz space Λ_φ which consists of those measurable functions with

$$\|f\|_{\Lambda_\varphi} = \int_0^\infty f^*(s) d\varphi(s) < \infty.$$

Note that if $\lim_{t \rightarrow 0} \varphi(t) = 0$ and we take w on \mathbb{R}_+ positive and decreasing such that $\varphi(t) = \int_0^t w(s)ds$, then $\Lambda_\varphi = \Lambda_w^1$. Observe also that the fundamental function of these spaces satisfy

$$\varphi_{\Lambda_\varphi} = \varphi_{M_\varphi} = \varphi.$$

Given an r.i. space X and its corresponding fundamental function φ_X , we can consider the Lorentz and Marcinkiewicz spaces associated with φ_X , and it holds that (see [5, Theorem 2.5.3] or [17])

$$\Lambda_{\varphi_X} \subset X \subset M_{\varphi_X}.$$

For an r.i. space X we also recall the definition of the *Boyd indices* (see [19, Section 2.b]) which are given by

$$p_X = \lim_{s \rightarrow \infty} \frac{\log s}{\log \|D_s\|} \quad q_X = \lim_{s \rightarrow 0^+} \frac{\log s}{\log \|D_s\|},$$

where $D_s : X \rightarrow X$ is the dilation operator which in case $\Omega = \mathbb{R}_+$ is given by $(D_s f)(t) = f(t/s)$, while in the case of $\Omega = [0, 1]$, it is given by

$$(D_s f)(t) = \begin{cases} f(t/s) & \text{for } t \leq \min(1, s) \\ 0 & \text{elsewhere.} \end{cases}$$

Recall that the Haar system $(h_n)_{n=1}^\infty$ is the sequence of functions on $[0, 1]$ given by $h_1(t) = 1$, and for $k = 0, 1, \dots, j = 1, 2, \dots, 2^k$

$$h_{2^k+j}(t) = \begin{cases} 1 & \text{for } t \in \left[\frac{j-1}{2^k}, \frac{2j-1}{2^{k+1}}\right], \\ -1 & \text{for } t \in \left[\frac{2j-1}{2^{k+1}}, \frac{j}{2^k}\right], \\ 0 & \text{otherwise.} \end{cases}$$

Written in this way, the vectors h_n are normalized in L_∞ , but we might consider their normalization in any other r.i. space. The Haar system is a monotone basis of every separable r.i. space on $[0, 1]$, which is unconditional if and only if $1 < p_X$ and $q_X < \infty$ (see [19, Proposition 2.c.1, Theorem 2.c.6]).

The Rademacher functions on $[0, 1]$ are given by

$$r_n(t) = \text{sign} \sin 2^n \pi t$$

for $n \in \mathbb{N}$. These form an orthonormal sequence in $L_2[0, 1]$, which are in fact equivalent to the unit vector basis of ℓ_2 in any r.i. space sufficiently far away from L_∞ ([19, Theorem 2.b.4]).

An r.i. space (or more generally, a Banach lattice) X is said to be *q-concave* for some $1 \leq q \leq \infty$, if there exists a constant $M < \infty$ so that

$$\left(\sum_{i=1}^n \|x_i\|^q \right)^{\frac{1}{q}} \leq M \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\|, \quad \text{if } 1 \leq q < \infty,$$

or

$$\max_{1 \leq i \leq n} \|x_i\| \leq M \left\| \bigvee_{i=1}^n |x_i| \right\|, \quad \text{if } q = \infty,$$

for every choice of $(x_i)_{i=1}^n$ in X (cf. [19, 1.d]). The smallest possible value of M is denoted by $M_{(q)}(X)$.

Similarly, X is p -convex for some $1 \leq p \leq \infty$, if there exists a constant $M < \infty$ such that

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right\| \leq M \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}}, \quad \text{if } 1 \leq p < \infty,$$

or

$$\left\| \bigvee_{i=1}^n |x_i| \right\| \leq M \max_{1 \leq i \leq n} \|x_i\|, \quad \text{if } p = \infty,$$

for every choice of $(x_i)_{i=1}^n$ in X . The smallest possible value of M is denoted by $M^{(p)}(X)$.

Recall that an operator between Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite dimensional subspace. This class forms a closed operator ideal that contains the compact operators and was introduced in connection with the perturbation theory of Fredholm operators [16]. In particular, the sum of a strictly singular operator and a Fredholm operator is again Fredholm with the same index (cf. [18]), and as a consequence, the spectra of strictly singular operators resembles that of compact operators. However, notice that, unlike compact operators, strictly singular operators are not stable under duality (cf. [25], [28]) and fail to have invariant subspaces ([26]).

Throughout, given a Banach space X we will denote by $\mathcal{S}(X)$ (respectively $\mathcal{K}(X)$) the ideal of strictly singular (resp. compact) endomorphisms on X .

3. KATO PROPERTY AND 2-DISJOINTLY HOMOGENEOUS SPACES

As we mentioned T. Kato showed that the classes of strictly singular and compact operators coincide on Hilbert spaces. Thus, we introduce the following notation.

Definition 3.1. *A Banach space X has Kato property whenever $\mathcal{S}(X) = \mathcal{K}(X)$.*

Taking into account the results presented in the introduction (see also [9]), examples of spaces with Kato property include Hilbert spaces, Lorentz spaces of the form $L_{p,2}[0,1]$ and $\Lambda(W,2)[0,1]$, Orlicz spaces $L^\varphi[0,1]$ where $\varphi(t) = t^2 \log^\alpha(1+t)$, sequence spaces like ℓ_p , c_0 , Tsirelson space (and some of its modifications)... But also, new “exotic” spaces such as the space \mathcal{X}_{AH} , constructed in [3] as a solution to the scalar-plus-compact problem, has Kato property.

Clearly, Kato property is an isomorphic property. Moreover, we have the following:

Proposition 3.2. *Let X be a Banach space with Kato property. Suppose that for some subspace $Y \subset X$, there is $Z \subset X$ such that $Y \simeq X/Z$, then Y also has Kato property.*

Proof. Suppose there is some operator $T : Y \rightarrow Y$ which is strictly singular but not compact. Let $Q : X \rightarrow Y$ be the onto mapping induced by the quotient X/Z , and $J : Y \rightarrow X$ be an

isomorphic embedding. Put $S = JTQ$:

$$\begin{array}{ccc} X & \xrightarrow{S} & X \\ Q \downarrow & & \uparrow J \\ Y & \xrightarrow{T} & Y \end{array}$$

Since T is strictly singular, so is S . Now, since T is not compact, there is some bounded sequence (y_n) in Y such that (Ty_n) has no convergent subsequence. Thus, using that Q is onto, by the open mapping theorem, we can find a bounded sequence (x_n) in X such that $Qx_n = y_n$. Clearly, since J is an isomorphic embedding, the sequence $S(x_n) = JTQ(x_n) = JT(y_n)$ has no convergent subsequence. Therefore, S is not compact in contradiction with the fact that X has Kato property. \square

Corollary 3.3. *Every complemented subspace of a space with Kato property also has Kato property.*

Let us recall the definition of 2-disjointly homogeneous Banach lattice and introduce a weaker version that will be useful for our purposes.

Definition 3.4.

- A Banach lattice E is 2-disjointly homogeneous (in short 2DH), if every normalized sequence of disjoint elements in E , has a subsequence equivalent to the unit vector basis of ℓ_2 .
- Let X be an r.i. space on $[0, 1]$. We say that X is restricted-2DH if for every sequence of disjoint sets $(A_n)_{n=1}^\infty$ in $[0, 1]$ there is a subsequence such that $(\frac{1}{\|\chi_{A_{n_k}}\|} \chi_{A_{n_k}})_{k=1}^\infty$ is equivalent to the unit vector basis of ℓ_2 .

It is clear if a space is 2DH, then it is restricted-2DH. We also have

Proposition 3.5. *Let X be an r.i. space on $[0, 1]$. The space X is restricted-2DH if and only if every subsequence of disjoint elements of the normalized Haar basis in X has a further subsequence equivalent to the unit vector basis of ℓ_2 .*

Proof. The direct implication is clear: if (h_{n_k}) is a subsequence of disjoint elements of the normalized Haar basis, then it is equivalent to $(|h_{n_k}|)$ which is a sequence of (normalized) characteristic functions over disjoint dyadic intervals. Thus, if the space X is restricted-2DH, then this sequence has a further subsequence equivalent to the unit vector basis of ℓ_2 .

For the converse implication, assume that every subsequence of disjoint elements of the normalized Haar basis in X has a further subsequence equivalent to the unit vector basis of ℓ_2 . As before, since we are dealing with a sequence of disjoint elements and using the fact that the space is rearrangement invariant, this is equivalent to the statement that for every sequence (D_k) of pairwise disjoint sets of dyadic measure (that is $\mu(D_k) = 2^{-n_k}$ for some $n_k \in \mathbb{N}$), there

is a subsequence such that the normalized characteristic functions

$$\frac{1}{\varphi(2^{-n_{k_j}})} \chi_{D_{k_j}}$$

are equivalent to the unit vector basis of ℓ_2 . Here, $\varphi(t) = \|\chi_{(0,t)}\|_X$ is the fundamental function of X .

Now, let (A_k) be a sequence of disjoint sets in $[0, 1]$. Without loss of generality we can assume that for some increasing sequence (n_k) of natural numbers, the measure of A_k satisfies

$$2^{-n_k} \leq \mu(A_k) \leq 2^{-n_k+1}.$$

Passing to a further subsequence if necessary, we can find sets

$$B_k \subset A_k \subset C_k$$

such that $\mu(B_k) = 2^{-n_k}$ and $\mu(C_k) = 2^{-n_k+1}$, with the additional assumption that (C_k) (as well as (B_k)) are pairwise disjoint. Now, by the previous paragraph, taking further subsequences (once for (C_k) and once more for (B_k)), we can suppose that both

$$\frac{1}{\varphi(2^{-n_k})} \chi_{B_k} \quad \text{and} \quad \frac{1}{\varphi(2^{-n_k+1})} \chi_{C_k}$$

are equivalent to the unit vector basis of ℓ_2 . Thus, for some constants $K > 0$ and every sequence of scalars $(a_k)_{k=1}^{\infty}$ we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(\mu(A_k))} \chi_{A_k} \right\| &\geq \frac{1}{2} \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(\mu(A_k)/2)} \chi_{A_k} \right\| \geq \frac{1}{2} \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(2^{-n_k})} \chi_{A_k} \right\| \\ &\geq \frac{1}{2} \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(2^{-n_k})} \chi_{B_k} \right\| \geq \frac{K}{2} \left(\sum_{k=1}^{\infty} a_k^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Where in the first inequality we have used that $\varphi(2t) \leq 2\varphi(t)$. Similarly, there is a constant $K' > 0$ satisfying that

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(\mu(A_k))} \chi_{A_k} \right\| &\leq 2 \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(2\mu(A_k))} \chi_{A_k} \right\| \leq 2 \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(2^{-n_k+1})} \chi_{A_k} \right\| \\ &\leq 2 \left\| \sum_{k=1}^{\infty} \frac{a_k}{\varphi(2^{-n_k+1})} \chi_{C_k} \right\| \leq 2K' \left(\sum_{k=1}^{\infty} a_k^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Thus, there is a subsequence of $(\chi_{A_k}/\varphi(\mu(A_k)))$ equivalent to the unit vector basis of ℓ_2 as we wanted to show. \square

This allows us to prove the following

Corollary 3.6. *Let X be an r.i. space on $[0, 1]$ which is isomorphic (as a Banach space) to a 2DH r.i. space Y . Then X is restricted-2DH.*

Proof. It follows from [15, Theorem 6.1], since either $X = Y$ up to equivalence of norms, or $X = L_2[0, 1]$, or the Haar basis in X is equivalent to a sequence of disjoint elements in Y and the result follows from Proposition 3.5. \square

Duality properties of DH Banach lattices have been recently studied in [10]. It turns out that restricted-2DH r.i. spaces are stable under duality.

Proposition 3.7. *An r.i. space X on $[0, 1]$ is restricted-2DH if and only if X' is restricted-2DH.*

Proof. Observe that Proposition 3.5 allows us to work with disjoint sequences of the Haar basis. Now, let (h_n) and (h_n^*) denote the normalized Haar basis of X and X^* respectively, and let $h_{n_k}^*$ be a disjoint sequence of the normalized Haar basis in X^* . Since X is restricted-2DH, there is a subsequence $(h_{n_{k_j}})$ which is equivalent to the unit vector basis of ℓ_2 . Therefore, we have

$$\begin{aligned} \left\| \sum_{j=1}^n a_j h_{n_{k_j}}^* \right\|_{X^*} &= \sup \left\{ \left\langle \sum_{j=1}^n a_j h_{n_{k_j}}^*, x \right\rangle : \|x\|_X = 1 \right\} \\ &= \sup \left\{ \left\langle \sum_{j=1}^n a_j h_{n_{k_j}}^*, \sum_{j=1}^n b_j h_{n_{k_j}} \right\rangle : \left\| \sum_{j=1}^n b_j h_{n_{k_j}} \right\|_X = 1 \right\} \\ &\approx \sup \left\{ \sum_{j=1}^n a_j b_j : \left(\sum_{j=1}^n b_j^2 \right)^{1/2} = 1 \right\} \\ &= \left(\sum_{j=1}^n a_j^2 \right)^{1/2}. \end{aligned}$$

This shows that the sequence $(h_{n_{k_j}}^*)$ is equivalent to the unit vector basis of ℓ_2 , and so X^* is also restricted-2DH. The converse is proved by duality. \square

We have several questions concerning restricted-2DH r.i. spaces.

We do not know if an r.i. space X on $[0, 1]$ which is restricted-2DH, must be 2-DH. If this were the case, then by Corollary 3.6, we would get that if an r.i. space X is isomorphic to another 2-DH r.i. space Y , then X is also 2-DH. We will see in Theorem 5.1 that this is the case in the setting of Orlicz spaces on $[0, 1]$.

On the other hand, in the Lorentz space $L_{p,\infty}$, every sequence of normalized characteristic functions of disjoint sets has a subsequence whose span is isomorphic to c_0 [7]. Thus, this is a restricted- ∞ DH space. However, $L_{p,\infty}$ is not ∞ -DH since it contains a normalized disjoint sequence whose span is isomorphic to ℓ_p (see [9]).

4. KATO PROPERTY IN LORENTZ AND MARCINKIEWICZ SPACES

Let us consider first the case of non-reflexive r.i. spaces.

Theorem 4.1. *If an r.i. space E contains a sublattice isomorphic to ℓ_1 , then there exists a non-compact strictly singular operator in E .*

Proof. By [22, Prop. 2.3.11], every sublattice isomorphic to ℓ_1 is complemented in E . Let us denote by X the complemented sublattice in E isomorphic to ℓ_1 , (x_n) a normalized sequence in X equivalent to the canonical basis of ℓ_1 and by P the projection onto X . Denote by $R(E)$ the subspace generated by the Rademacher system $r_k(t) = \text{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}$.

It is well-known that the canonical basis in $R(E)$ is symmetric and that $R(E) \neq \ell_1$ for any r.i. space $E \neq L_\infty$ ([19], 2.c.10). Now, since the strict inclusion from ℓ_1 into any symmetric sequence space is always strictly singular ([14], Cor. 3.2), we have that the inclusion operator

$$I : \ell_1 \rightarrow R(E)$$

is strictly singular. Denote by J the isometric embedding of $R(E)$ into E . Consider now the operator $T = JIP$ in E . Since I is strictly singular we have that also T is strictly singular. And since T sends (x_k) into (r_k) we have that T is not compact. □

In particular in every Lorentz space $\Lambda(\varphi)$ there exist non-compact strictly singular operators.

Theorem 4.2. *If an r.i. space E on $[0, 1]$ contains a subspace isomorphic to c_0 , then there exists a non-compact strictly singular operator in E .*

Proof. Given $f \in L_1[0, 1]$, let $(c_k(f))$ be its sequence of coefficients with respect to the Rademacher system, that is

$$c_k(f) = \int_0^1 f(t)r_k(t)dt,$$

for $k \in \mathbb{N}$. By Riemann-Lebesgue Lemma, the expression $R(f) = (c_k(f))_{k=1}^\infty$ defines a bounded operator

$$R : L_1 \rightarrow c_0.$$

Observe that the r.i. space E must be different from L_1 since it contains a subspace isomorphic to c_0 . Hence, by [4, Theorem 5], there exists a space F with symmetric basis such that $F \subset c_0$ with strict inclusion ($F \neq c_0$), satisfying that the operator $R : E \rightarrow F$ is bounded.

Let

$$J : c_0 \rightarrow E \quad \text{and} \quad I_{F,c_0} : F \hookrightarrow c_0$$

denote respectively an isomorphic embedding and the formal identity mapping. Now, consider the operator

$$T = JI_{F,c_0}R : E \rightarrow E.$$

By [14], since $F \neq c_0$, the operator I_{F,c_0} is strictly singular. Hence, so is T . Moreover, T is not compact since it maps the Rademacher sequence in E to a normalized sequence equivalent to the unit vector basis of c_0 . □

In particular, in every Marcinkiewicz space M_φ , there is a strictly singular non-compact operator, and the same holds for the order continuous part $(M_\varphi)_o$.

Corollary 4.3. *Let X be a non-reflexive r.i. space on $[0, 1]$. Then X fails Kato property.*

Proof. By Lozanovskii's theorem every non-reflexive Banach lattice contains a sublattice isomorphic to c_0 or ℓ_1 . Therefore, Theorems 4.1 and 4.2 together yield the conclusion. \square

For the class of Lorentz spaces $L_{p,q}[0, 1]$ the relation between Kato property and 2DH is clear:

Theorem 4.4. *Let $1 < p < \infty$, $1 \leq q \leq \infty$. The following conditions are equivalent*

- (1) $L_{p,q}[0, 1]$ has Kato property,
- (2) $L_{p,q}$ is 2DH,
- (3) $q = 2$.

Proof. The implication $2 \Rightarrow 1$ follows from [9]. By [7, Lemma 3.1] any normed disjointly supported sequence of $L_{p,q}$ contains a subsequence equivalent to the unit vector basis of ℓ_q . Therefore $3 \Rightarrow 2$.

So we must prove the implication $1 \Rightarrow 3$. Let $1 < q < 2$. Consider the operator $A = Ri_{q,2}S$ where S is the conditional expectation

$$Sx(t) = \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{2^{-k}}^{2^{-k+1}} x(s) ds \chi_{(2^{-k}, 2^{-k+1})}(t),$$

$i_{q,2}$ is the identical operator from l_q into l_2 and

$$R(x_1, x_2, \dots) = \sum_{k=1}^{\infty} x_k r_k(t),$$

$r_k(t)$ are the Rademacher functions, $k \in \mathbb{N}$. It is well known [19, Theorem 2.a.4], that S is a projection of norm 1 in any r.i. space, in particular in $L_{p,q}$. The subspace generated by sequence $2^{k/p} \chi_{(2^{-k}, 2^{-k+1})}(t)$ in $L_{p,q}$ is isomorphic to l_q , $1 < p < \infty$ [7]. Therefore S acts from $L_{p,q}$ into l_q . By Khintchine inequality R acts from l_2 into $L_{p,q}$. So, A is a bounded operator in $L_{p,q}$. Clearly $A \in SS(L_{p,q}) \setminus K(L_{p,q})$.

Let $2 < q < \infty$. Consider the operator $B = Qi_{2,q}P$ where P is the orthogonal projector on the Rademacher system, $i_{2,q}$ is the identical operator from l_2 into l_q and

$$Q(x_1, x_2, \dots) = \sum_{k=1}^{\infty} x_k 2^{\frac{k}{p}} \chi_{(2^{-k}, 2^{-k+1})}(t).$$

Since P is a bounded operator in L_r for any $r \in (1, \infty)$ and $L_{p,q}$ is an interpolation space with respect to L_r ($1 < r < \infty$) then P acts from $L_{p,q}$ into l_2 . The operator Q is an isomorphic embedding of l_q into $L_{p,q}$. Therefore B is a strictly singular and non-compact operator in $L_{p,q}$.

So, there exists a strictly singular and non compact operator in $L_{p,q}$ if $q \neq 2$. \square

Taking into account that every disjoint sequence in a Lorentz space of the form $\Lambda(W, p)[0, 1]$ has a subsequence equivalent to the unit vector basis of ℓ_p [8, Proposition 5.1], a similar argument as above shows that $\Lambda(W, p)[0, 1]$ has Kato property, if and only if it is 2DH, if and only if $p = 2$.

5. KATO PROPERTY IN ORLICZ SPACES

First, note that by Corollary 4.3, non-reflexive Orlicz spaces do not have Kato property. We need to recall the following classical notation [20]: For an Orlicz function M satisfying the Δ_2 -condition at ∞ , consider the set of (continuous) functions on $[0, 1]$

$$E_M^\infty = \bigcap_{s \geq 1} \overline{\left\{ \frac{M(r \cdot)}{M(r)} : r \geq s \right\}}.$$

Theorem 5.1. *For an Orlicz space $L^M[0, 1]$, the following are equivalent:*

- (1) $L^M[0, 1]$ is 2DH.
- (2) $L^M[0, 1]$ is restricted-2DH.
- (3) Every function in E_M^∞ is equivalent to the function $\varphi(t) = t^2$ at 0.

Proof. The equivalence between (1) and (3) has already been established in [9, Theorem 4.1]. Also, (1) implies (2) trivially. Let us prove that the implication (2) \Rightarrow (3) also holds.

First, note that for every $\psi \in E_M^\infty$, there exist a sequence of pairwise disjoint sets (A_n) such that the unit vector basis in the Orlicz space ℓ^ψ is equivalent to the sequence

$$\left(\frac{\chi_{A_n}}{\|\chi_{A_n}\|} \right)_{n=1}^\infty = (M^{-1}(1/\mu(A_n))\chi_{A_n})_{n=1}^\infty.$$

Indeed, given $\psi \in E_M^\infty$, take an increasing sequence (s_n) with $M(s_n) > 2^n$ such that for every $n \in \mathbb{N}$

$$\sup_{t \in [0, 1]} \left| \frac{M(s_n t)}{M(s_n)} - \psi(t) \right| < \frac{1}{2^n}.$$

Now, let (A_n) be a sequence of disjoint sets such that $\mu(A_n) = \frac{1}{M(s_n)} < \frac{1}{2^n}$. We have

$$\|\chi_{A_n}\| = \frac{1}{M^{-1}(1/\mu(A_n))},$$

so for arbitrary scalars $(\lambda_n)_{n=1}^\infty$ we have

$$\int_0^1 M\left(\sum_{n=1}^\infty \lambda_n \frac{\chi_{A_n}(t)}{\|\chi_{A_n}\|}\right) d\mu = \sum_{n=1}^\infty M\left(\lambda_n M^{-1}\left(\frac{1}{\mu(A_n)}\right)\right) \mu(A_n) = \sum_{n=1}^\infty \frac{M(\lambda_n s_n)}{M(s_n)} \approx \sum_{n=1}^\infty \psi(\lambda_n).$$

Therefore, $\left(\frac{\chi_{A_n}}{\|\chi_{A_n}\|}\right)_{n=1}^\infty$ is equivalent to the unit vector basis of ℓ^ψ .

Now, if (2) holds, then there is a subsequence of $\left(\frac{\chi_{A_n}}{\|\chi_{A_n}\|}\right)_{n=1}^\infty$ which is equivalent to the unit vector basis of ℓ_2 . Hence, the unit vector basis of ℓ^ψ has a subsequence equivalent to the basis of ℓ_2 . But since the basis of ℓ^ψ is symmetric, we get that $\ell^\psi = \ell_2$, or equivalently $\psi(t) \approx t^2$. Hence, (3) holds. \square

It follows from the results in [9] (see Theorem 1.3 above) that a 2DH Orlicz space on $[0, 1]$ has Kato property. A partial converse under proper convexity assumptions is given next.

Theorem 5.2. *Let M be an Orlicz function satisfying Δ_2 -condition at ∞ , and suppose that $L^M[0, 1]$ is 2-convex (or 2-concave). It holds that $L^M[0, 1]$ has Kato property if and only if it is 2DH.*

Proof. Suppose that $L^M[0, 1]$ is 2-convex and not 2DH. Then, by Theorem 5.1, there exists a function $\psi \in E_M^\infty$ that is not equivalent to t^2 at 0. Let us take an increasing sequence $s_n \rightarrow \infty$ such that

$$\psi(t) = \lim_{n \rightarrow \infty} \frac{M(s_n t)}{M(s_n)}$$

for $t \in [0, 1]$.

By the 2-convexity of $L^M[0, 1]$, the function M is equivalent at ∞ to a 2-convex function, and by [27, p. 28], we have that

$$\sup_{s \geq \frac{1}{t}} \frac{M(ts)}{M(s)} \leq Kt^2$$

for some constant $K > 0$ and every $0 < t < 1$. Therefore, $\psi(t) \leq Kt^2$ at 0.

Now, the function ψ is also 2-convex and not equivalent to t^2 at 0, so by [14, Proposition 5.10] it follows that the formal inclusion

$$i_{2,\psi} : \ell_2 \hookrightarrow \ell^\psi$$

is a strictly singular operator. Let us consider now, the operator

$$T : L^M[0, 1] \rightarrow L^M[0, 1]$$

given by $T = J i_{2,\psi} R$, where $R : L^M[0, 1] \rightarrow \ell_2$ denotes the projection onto the span of the Rademacher functions, and $J : \ell_\psi \rightarrow L^M[0, 1]$ denotes an isomorphic embedding. It is clear that T is strictly singular but not compact. Hence, $L^M[0, 1]$ fails Kato property.

Similarly, if the space $L^M[0, 1]$ is 2-concave, we follow the same steps by considering a 2-concave Orlicz sequence space ℓ^ψ different from ℓ_2 such that the formal inclusion

$$i_{\psi,2} : \ell^\psi \hookrightarrow \ell_2$$

defines a strictly singular operator (using also [14]). □

We do not know whether Theorem 5.2 can be extended to 2-convex or 2-concave r.i. spaces on $[0, 1]$.

It can be shown that condition (3) in Theorem 5.1 which characterizes 2DH Orlicz spaces can be rewritten as the formula

$$\sup_{0 < t < \infty} \limsup_{u \rightarrow \infty} \frac{M(tu)}{t^2 M(u)} < \infty.$$

Strengthening this condition slightly we get further necessary conditions for Kato property on an Orlicz space. We will separately prove two preliminary lemmas first.

Lemma 5.3. *Let M be an Orlicz function satisfying the Δ_2 -condition at ∞ and*

$$(1) \quad \lim_{t \rightarrow 0} \lim_{u \rightarrow \infty} \frac{M(tu)}{t^2 M(u)} = \infty.$$

Then there exist a sequence of disjoint measurable sets (A_k) in $[0, 1]$, and an Orlicz function F such that $(\chi_{A_k}/\|\chi_{A_k}\|)$ is equivalent to the unit vector basis of ℓ_F , the inclusion $\ell_F \subset \ell_2$ holds, and it is a strictly singular operator.

Proof. By assumption (1) there exist monotone sequences (b_k) and (d_k) increasing to ∞ such that

$$M\left(\frac{u}{\sqrt{k}}\right)k \geq b_k M(u)$$

for any $u \geq d_k$, and $k \in \mathbb{N}$. Let us denote $c_k = \frac{1}{M(d_k)}$. Then we have

$$M\left(\frac{1}{\sqrt{k}}M^{-1}\left(\frac{1}{c_k}\right)\right)c_k k \geq b_k$$

and moreover

$$(2) \quad M\left(\frac{1}{\sqrt{k}}M^{-1}\left(\frac{1}{c_{n_k}}\right)\right)c_{n_k} k \geq b_k$$

for any increasing subsequence (n_k) .

Without loss of generality we may assume that $\sum_{k=1}^{\infty} c_k \leq 1$. Let (F_k) be a sequence of disjoint measurable sets in $[0, 1]$, with $\mu(F_k) = c_k$, for every $k \in \mathbb{N}$. Then $\left\|M^{-1}\left(\frac{1}{c_k}\right)\chi_{F_k}\right\|_{L^M} = 1$. By [20, Prop. 3], there exist an Orlicz function F and a subsequence (n_k) such that $[x_{n_k}] = \ell_F$ where $x_k = M^{-1}\left(\frac{1}{c_k}\right)\chi_{F_k}$, $k \in \mathbb{N}$. Let $A_k = F_{n_k}$, for $k \in \mathbb{N}$.

Using (2) we get

$$\sum_{k=1}^m M\left(\frac{1}{\sqrt{m}}M^{-1}\left(\frac{1}{c_{n_k}}\right)\right)c_{n_k} \geq \frac{1}{a} \sum_{k=\frac{m}{2}}^m M\left(\frac{1}{\sqrt{k}}M^{-1}\left(\frac{1}{c_{n_k}}\right)\right)c_{n_k} \geq \frac{1}{a} b_{\frac{m}{2}} \sum_{k=\frac{m}{2}}^m \frac{1}{k} \geq \frac{\ln 2}{a} b_{\frac{m}{2}}$$

for any even m where a is the Δ_2 -constant of M . Hence

$$\lim_{m \rightarrow \infty} \frac{1}{\sqrt{m}} \left\| \sum_{k=1}^m x_{n_k} \right\|_{L^M} = \lim_{m \rightarrow \infty} \frac{1}{\sqrt{m}} \left\| \sum_{k=1}^m M^{-1}\left(\frac{1}{c_{n_k}}\right)\chi_{A_k} \right\|_{L^M} = \infty$$

and

$$\lim_{m \rightarrow \infty} \frac{1}{\sqrt{m}} \left\| \underbrace{(1, 1, \dots, 1)}_m, 0, 0, \dots \right\|_{\ell_F} = \infty.$$

This means that $\lim_{t \rightarrow 0} F(t)/t^2 = \infty$. Now, by Kalton's theorem [18, Thm. 4.a.10] the inclusion $\ell_F \subset \ell_2$ is strictly singular. \square

Lemma 5.4. *Let M be an Orlicz function, such that the complementary function \bar{M} satisfies Δ_2 -condition at ∞ , and*

$$\lim_{t \rightarrow 0} \lim_{u \rightarrow \infty} \frac{M(tu)}{t^2 M(u)} = 0.$$

Then, there exist a sequence of disjoint measurable sets (A_k) in $[0, 1]$, and an Orlicz function F such that $(\chi_{A_k}/\|\chi_{A_k}\|)$ is equivalent to the unit vector basis of ℓ_F , the inclusion $\ell_2 \subset \ell_F$ holds and it is a strictly singular operator.

Proof. We shall repeat the plan and notation of Lemma 5.3. There exist monotone sequences $b_k \rightarrow 0$, and $d_k \rightarrow \infty$ such that

$$M\left(\frac{u}{\sqrt{k}}\right)k \leq b_k M(u)$$

for any $u \geq d_k$, and $k \in \mathbb{N}$. If we denote $c_k = \frac{1}{M(d_k)}$, then

$$(3) \quad M\left(\frac{1}{\sqrt{k}}M^{-1}\left(\frac{1}{c_{n_k}}\right)\right)c_{n_k}k \leq b_k$$

for any increasing subsequence n_k , $k \in \mathbb{N}$.

Without loss of generality we may assume that $\sum_{k=1}^{\infty} c_k \leq 1$. Let (F_k) be a sequence of disjoint subsets of $[0, 1]$, with $\mu(F_k) = c_k$, for $k \in \mathbb{N}$. Then $\left\|M^{-1}\left(\frac{1}{c_k}\right)\chi_{F_k}\right\|_{L^M} = 1$. By [20, Prop. 3] there exist an Orlicz function F and subsequence (n_k) such that $[x_{n_k}] = \ell_F$ where $x_k = M^{-1}\left(\frac{1}{c_k}\right)\chi_{F_k}$. Let $A_k = F_{n_k}$, for $k \in \mathbb{N}$.

Using (3) we get

$$\sum_{k=m+1}^{2m} M\left(\frac{1}{\sqrt{k}}M^{-1}\left(\frac{1}{c_k}\right)\right)c_{n_k} \leq \sum_{k=m+1}^{2m} \frac{b_k}{k} \leq b_{m+1}$$

for any $m \in \mathbb{N}$. Since \bar{M} satisfies the Δ_2 -condition at ∞ , then

$$\lim_{m \rightarrow \infty} \left\| \frac{1}{\sqrt{m}} \sum_{k=m+1}^{2m} x_{n_k} \right\|_{L^M} = 0.$$

Hence

$$\lim_{m \rightarrow \infty} \frac{1}{\sqrt{m}} \left\| \underbrace{(0, 0, \dots, 0)}_m, \underbrace{(1, 1, \dots, 1)}_m, 0, 0, \dots \right\|_{l_F} = 0.$$

Therefore, we have that $\lim_{t \rightarrow 0} F(t)/t^2 = \infty$. And by Kalton's theorem [18, Thm. 4.a.10] we conclude that the inclusion $\ell_F \subset \ell_2$ is strictly singular. \square

These Lemmas allow us to give the following necessary conditions for Kato property on Orlicz spaces.

Corollary 5.5. *Let $L^M[0, 1]$ be a reflexive Orlicz space. If*

$$\lim_{t \rightarrow 0} \lim_{u \rightarrow \infty} \frac{M(tu)}{t^2 M(u)} \in \{0, \infty\},$$

then $L^M[0, 1]$ fails to have Kato property.

It remains open whether every Orlicz space $L^M[0, 1]$ with Kato property must be 2DH. However, in the infinite measure case we can provide examples of reflexive Orlicz function spaces $L^M(0, \infty)$ with Kato property which are not 2DH:

Theorem 5.6. *Consider the Orlicz function M defined by*

$$M(t) = \begin{cases} \frac{1}{\log 2} t^2 & t \in [0, 1] \\ \frac{t^2}{\log(1+t)} & t \in [1, \infty). \end{cases}$$

The reflexive Orlicz space $L^M(0, \infty)$ has Kato property but is not 2DH.

Proof. It follows from [15, Thm 8.6 and page 216] that the space $L^M = L^M(0, \infty)$ is isomorphic to the Orlicz space $L^M[0, 1]$ since M at 0 is t^2 . Now, the space $L^M[0, 1]$ is 2DH since $C_M^\infty = \{t^2\}$ (see Theorem 5.1), hence we get by Theorem 1.2, that $\mathcal{K}(L^M[0, 1]) = \mathcal{S}(L^M[0, 1])$. Thus, $\mathcal{K}(L^M) = \mathcal{S}(L^M)$, so L^M has Kato property.

Let us shown now that L^M is not a 2DH Banach lattice. According to [24, Theorem 1.1], we have to show that there is an Orlicz function F , non-equivalent to t^2 at 0, belonging to the set $C_M(0, \infty)$, where $C_M(0, \infty) := \overline{\text{conv}} E_M(0, \infty)$ in the space $C(0, 1)$. Recall that

$$E_M(0, \infty) = \overline{\{F \in C(0, 1) : F(t) = \frac{M(st)}{M(s)}, 0 < s < \infty, t \in (0, 1)\}}$$

Now, using [24, p.242], we have that every function $F \in C_M(0, \infty)$ can be expressed as a convex combination of three functions F_1, F_2 and F_3 , where $F_1 \in C_{M,1}$, $F_2 \in C_M^\infty$ and

$$F_3(t) = \int_1^\infty \frac{M(st)}{M(t)} d\mu(s)$$

for $t \in [0, 1]$, where μ is a probability measure in $[1, \infty)$ with $\mu(1) = 0$.

In our case we have clearly that $F_1(t) \sim t^2$ at 0, and $F_2(t) \sim t^2$ at 0.

Let us consider now the finite measure μ on $[1, \infty)$ defined by

$$\mu([1, \infty)) = \int_1^\infty \frac{ds}{s \log^2(1+s)},$$

Then the associated function $F_\mu \equiv F$ satisfies that

$$\begin{aligned} F_3(t) &= \int_1^{1/t} t^2 \log(1+s) d\mu(s) + \int_{1/t}^\infty \frac{t^2 \log(1+s)}{\log(1+ts)} d\mu(s) \\ &= t^2 \left(\int_1^{1/t} \frac{ds}{s \log(1+s)} + \int_{1/t}^\infty \frac{ds}{s \log(1+ts) \log(1+s)} \right) \end{aligned}$$

Now, it holds that

$$\int_1^{1/t} \frac{ds}{s \log(1+s)} \sim \log(|\log t|)$$

for t near 0. And the integral

$$\int_{1/t}^\infty \frac{ds}{s \log(1+ts) \log(1+s)} = \int_1^\infty \frac{du}{u \log(1+u) \log(1+\frac{u}{t})}$$

tends to 0 as t goes to 0.

Hence the function $F \in C_M(0, \infty)$ is equivalent to the function $t^2 \log(|\log t|)$ at 0. Therefore there is disjoint normalized function sequence in $L^M(0, \infty)$ which is not equivalent to the canonic basis of ℓ_2 and thus $L^M(0, \infty)$ is not 2DH. \square

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