EXTRAPOLATION ON $L^{p,\infty}(\mu)$

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Abstract. We solve the extrapolation problem concerning bounded operators on $L^{p,\infty}(\mu)$; that is, we give end-point estimates for sublinear operators T such that $T : L^{p,\infty}(\mu) \to L^{p,\infty}(\nu)$ is bounded with constant less than or equal to $1/(p-1)^m$. Applications to the Hardy-Littlewood maximal operator, the Hilbert transform and composition of operators are also given.

1. Introduction and motivation

In 1951, Yano ([30]) proved that if T is a sublinear operator such that for every $1 < p < p_0$ (with p_0 fixed)

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$, $m > 0$, with μ and ν finite measures, then

$$T : L(\log L)^m(\mu) \longrightarrow L^1(\nu)$$

is bounded, where

$$L(\log L)^m(\mu) = \left\{ f \in L^0(\mu);\ ||f||_{L(\log L)^m(\mu)} = \int_0^\infty f_\mu^*(s)\Big(1 + \log^+ \frac{1}{s}\Big)^m ds < \infty \right\}.$$

As usual, $L^0(\mu)$ denotes the space of μ-measurable functions and f_μ^* is the decreasing rearrangement of f with respect to the measure μ. Moreover, this result is sharp, in the sense that $L(\log L)^m(\mu)$ is the biggest domain space satisfying that $Tf \in L^1(\nu)$, as one can see taking $d\nu = d\mu = \chi_{(0,1)}dx$ and $T = M$, the Hardy-Littlewood maximal operator.

This result is known as Yano's extrapolation theorem. Moreover, the condition on the measures μ and ν can be weakened and one can consider μ and ν σ-finite measures. In this case, the conclusion is that (see [13])

$$T : L(\log L)^m(\mu) \longrightarrow E_m(\nu) \tag{1.1}$$

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is bounded, where

$$E_m(\nu) = \left\{ f \in L^0(\mu); \sup_{t>0} \frac{t f_\nu^{**}(t)}{(1+\log^+ t)^m} < \infty \right\}$$

and $f_\mu^{**}(t) = \frac{1}{t}\int_0^t f_\mu^*(s)ds$. Observe that if ν is finite, $E_m(\nu) = L^1(\nu)$ for every m.

Throughout this paper we shall work with μ and ν σ-finite non-atomic measures and we shall use the following notation: For every $m \geq 2$, let

$$\log_m x = \log_{m-1} \log_1 x,$$

with

$$\log_1 x = 1 + \log^+ x,$$

and

$$L \prod_{j \in J} \big(\log_j L \big)^{m_j} (\mu) = \left\{ f;\ \int_0^\infty f_\mu^*(t) \prod_{j\in J} \Big(\log_j \frac{1}{t} \Big)^{m_j} dt < \infty \right\}.$$

In connection with the so-called weak type extrapolation, it is known (see [1], [14]) that if T is a sublinear operator such that for every $1 < p < p_0$ (p_0 fixed)

$$T : L^p(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$, then

$$T : L(\log_1 L)^m \log_3 L(\mu) \longrightarrow R_m(\nu) \tag{1.2}$$

is bounded where

$$R_m(\nu) = \left\{ f \in L^0(\mu); ||f||_{R_m} = \sup_{t>0} \frac{t f_\nu^*(t)}{(\log_1 t)^m} < \infty \right\}.$$

If $m = 1$, it is known that the space $L \log_1 L \log_3 L(\mu)$ is not optimal, in the sense that there exists a space D such that $L \log_1 L \log_3 L(\mu)$ is strictly embedded in D (see [2] and [14]) with

$$T : D \longrightarrow R_1(\nu)$$

bounded. However, it has recently been proved (see [15]) that $L \log_1 L \log_3 L(\mu)$ is essentially the largest Lorentz space embedded in D. In particular, the space $L \log_1 L \log_4 L(\mu)$ does not embed in D.

Remark 1.1. We should mention here that in order to get the boundedness (1.1), it is enough to have a weaker hypothesis on T, namely that

$$T : L^{p,1}(\mu) \longrightarrow L^p(\nu)$$

is bounded with constant less than or equal to $1/(p-1)$ (see [13]). Similarly, in order to have (1.2), it is enough to assume that

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$.

Let us now explain the original motivation of this work. In [5], the following operator was introduced

$$Af(x) = \left\| \frac{f\chi_{(0,x)}}{x - \cdot} \right\|_{L^{1,\infty}(0,1)}.$$

This operator plays an important role in connection with the following problem in ergodic theory. Recall that a dynamical system is a probability space (Ω, Σ, μ), together with a measure preserving transformation τ. The following return times theorem was proved in [11] (see also [9], [10] and [29]):

Theorem 1.2. *Let $(\Omega, \Sigma, \mu, \tau)$ be a dynamical system. For $1 \leq p \leq \infty$, given $f \in L^p(\mu)$, there is a set $\Omega_f \subset \Omega$ with $\mu(\Omega_f) = 1$, such that for any other dynamical system $(\Omega', \Sigma', \mu', \varrho)$, $g \in L^q(\mu')$ (with $\frac{1}{p} + \frac{1}{q} = 1$), and $x \in \Omega_f$, the sequence of means*

$$\frac{1}{n}\sum_{k=1}^{n} f(\tau^k x) g(\varrho^k y)$$

converges μ'-almost everywhere.

The question is to understand whether the fact that f and g lie in dual spaces is a necessary assumption for this theorem to hold (see [16]). In an attempt to break this duality, in [3] and [4], it was proved that given a dynamical system $(\Omega, \Sigma, \mu, \tau)$, if $f \in L^p(\mu)$ for any $p > 1$ (or even if $f \in L \log L(\mu)$), then there is a set $\Omega_f \subset \Omega$ with $\mu(\Omega_f) = 1$ satisfying that for every sequence (X_k) of i.i.d. random variables on a probability space (Ω', Σ', ν) with $X_k \in L_1(\nu)$ and any $x \in \Omega_f$

$$\frac{1}{n}\sum_{k=1}^{n} f(\tau^k x) X_k$$

converges ν-almost everywhere. However, for a general function in $L_1(\mu)$ this is no longer true [5]. One way to prove this is to show that the operator A described above is not of weak type $(1, 1)$.

Therefore, the question is to find which is the largest space X such that for every $f \in X$, the above property holds. In this direction, it was proved in [17] that

$$A : L \log_2 L(0, 1) \longrightarrow L^{1,\infty}(0, 1)$$

is bounded and in that paper it is also stated that A cannot be bounded in any Orlicz space strictly bigger that $L \log_3 L(0, 1)$, leaving as an open question whether

$$A : L \log_3 L(0, 1) \longrightarrow L^{1,\infty}(0, 1)$$

is bounded or not.

In our attempt to solve this question we made the following observation: given a locally integrable function f, if we consider M the Hardy-Littlewood maximal operator, it is well-known that

$$(Mf)^*(t) \approx \frac{1}{t}\int_0^t f^*(s)ds.$$

Therefore, it holds that

$$||Mf||_{L\log_3 L} \approx ||f||_{L\log_1 L\log_3 L}$$

and hence, if we were able to prove that

$$A \circ M : L^p \longrightarrow L^{p,\infty} \tag{1.3}$$

is bounded with constant $C/(p-1)$, then by the weak extrapolation result mentioned above

$$A \circ M : L\log_1 L\log_3 L \longrightarrow L^{1,\infty}$$

would be bounded. Thus,

$$||Ag||_{L^{1,\infty}} \lesssim ||g||_{L\log_3 L}$$

at least when g belongs to the collection of functions of the form Mf (this would partially answer in the positive the open question on the operator A).

Hence, since $M : L^p \longrightarrow L^{p,\infty}$ is bounded with a uniform constant independent of p, it is clear that (1.3) would hold provided that

$$A : L^{p,\infty} \longrightarrow L^{p,\infty}$$

is bounded with constant $C/(p-1)$. At this point, we should mention that this estimate remains open and although we have succeeded in proving the above bound with constant smaller than $C\frac{\log^2 \frac{1}{p-1}}{p-1}$, this is not enough to conclude the desired result.

However, we still would have to solve the following extrapolation problem: what kind of endpoint estimate can we get from the boundedness of $T : L^{p,\infty} \longrightarrow L^{p,\infty}$ with constant less than or equal to $1/(p-1)$? Can we get a better estimate than from the hypothesis $T : L^p \longrightarrow L^{p,\infty}$ with the same behavior of the constant? Considering Remark 1.1, it might seem that the answer is negative, but as we shall later see this is not the case.

In order to make things clearer and also to introduce the context where some of the applications will take place, let us also consider the following simple and classical situation. Let T be a sublinear operator such that T is of weak type $(1,1)$ and bounded in L^∞, and assume that we are interested in studying the boundedness properties at the extreme point $p = 1$ of the iterated operators $T^{(2)}f = T(Tf)$ or, more generally, $T^{(n)}f = T^{(n-1)}(Tf)$.

To this end, using the classical interpolation and extrapolation theory we can obtain the following result.

Proposition 1.3. *Under the above hypothesis in the operator T, it holds that, for every $n \in \mathbb{N}$,*

$$T^{(n)} : L(\log_1 L)^n \longrightarrow E_n \tag{1.4}$$

is bounded.

Proof. By interpolation, it is known that $T^{(n)} : L^p \longrightarrow L^p$ with constant less than or equal to $C(p-1)^{-n}$ and hence, by Yano's extrapolation theorem the result follows. □

On the other hand, if an operator T satisfies the above hypothesis, then

$$(Tf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)ds \tag{1.5}$$

and hence, by iterating we get

$$(T^{(n)}f)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\Big(\log\frac{t}{s}\Big)^{n-1} ds,$$

from which the boundedness

$$T^{(n)} : L(\log_1 L)^{n-1} \longrightarrow R_{n-1} \tag{1.6}$$

follows. Observe that

$$L(\log_1 L)^n \subset L(\log_1 L)^{n-1}, \qquad \text{and} \qquad R_{n-1} \subset R_n$$

and hence (1.6) improves essentially (1.4) both in the domain and in the range spaces. Under this situation, trying to understand why the domain of $T^{(n)}$ could be improved and if this improvement was consequence of some extrapolation argument, we found out that the hidden estimates that we have to use in our extrapolation argument are that

$$T : L^{p,\infty} \longrightarrow L^{p,\infty}$$

is bounded with constant less than or equal to $C(p-1)^{-1}$

Solving this new extrapolation result is the main contribution of this paper. To be more precise, our first contribution is Theorem 3.5 where we prove that if T is a sublinear operator such that

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$. Then,

$$T : [L(\log_1 L)^{m-1}\log_3 L(\mu)]_1 \longrightarrow R_1(\nu)$$

is bounded where

$$L\,(\log_1 L)^m \log_3 L \subsetneq [L(\log_1 L)^{m-1}\log_3 L]_1 \subsetneq L(\log_1 L)^{m-1}\log_3 L.$$

Thus, we get a better result than if we only use the information that T is weak type (p,p) with the same behavior of the constant. We believe this fact provides renewed motivation in the study of bounds for $T : L^{p,\infty} \to L^{p,\infty}$, with the hope of improving hitherto known endpoint estimates.

Afterwards, we will apply this result to the case of composition of operators and we prove in Theorem 3.8 (this is our second main contribution) that if $T = T_1 \circ T_2$

with $T_1 : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu)$ bounded with constant less than or equal to $1/(p-1)^m$ and T_2 satisfies certain mild conditions, then

$$T : L(\log_1 L)^{m-1} \log_3 L(\mu) \longrightarrow R_1(\nu)$$

is bounded; that is we can get rid of the brackets $[\cdot]$ and improve the domain even further. Applications of our main two results are given in the setting of the Hardy-Littlewood maximal operator and the Hilbert transform.

We also study the same extrapolation problem for operators bounded on $L^{p,\infty}$ for every $p > p_0$ with constant less than or equal to $1/(p-p_0)^m$ and $p_0 > 1$ fixed.

The final section is devoted to study the functional properties of the brackets spaces that have appeared in our results and in the appendix we present some further extrapolation results which may be of interest for the applications.

Throughout the text C will denote a constant independent of the parameters involved (including p). As usual, the symbol $\lesssim$ denotes that an inequality $\leq$ holds up to some constant C, and similarly, $\approx$ means that both $\lesssim$ and $\gtrsim$ hold.

2. The class $\mathcal{A}^*$

First, let us define the following classes of operators.

Definition 2.1. Let $\mathcal{A}$ be a set of measurable functions on $(0,\infty)$. We say that a sublinear operator T is in the class $\mathcal{A}^*(\mu,\nu)$ if, there exists a function $a \in \mathcal{A}$ such that, for every $f \in L^0(\mu)$ for which Tf is well defined,

$$(Tf)^*_\nu(t) \lesssim \int_0^\infty a(s) f^*_\mu(st)\,ds. \tag{2.1}$$

If the measures μ and ν coincide with the Lebesgue measure, we simply write $\mathcal{A}^*$.

2.1. Examples.
Let us first see examples of operators in several classes $\mathcal{A}^*$ or $\mathcal{A}^*(\mu,\nu)$.

I) Let $\mathcal{A} = L^\infty$, then:

I.1) The Hardy-Littlewood operator $M \in \mathcal{A}^*$ since $a(s) = \chi_{(0,1)}(s)$.

I.2) Let $T = H$ be the Hilbert transform. Then, it is known (see [7]) that

$$(Tf)^*(t) \lesssim \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty f^*(s)\frac{ds}{s} = \int_0^\infty \min\left(1, \frac{1}{u}\right) f^*(tu)\,du$$

and hence, since $a(u) = \min\left(1, \frac{1}{u}\right)$, $H \in \mathcal{A}^*$.

I.3) With the same proof we can include in $\mathcal{A}^*(\mu,\nu)$ all the operators which are of joint weak type $(1,1;\infty,\infty)$ with respect the measures μ and ν (see [8, Chapter 3]): If μ and ν coincide with the Lebesgue measure these include the Riesz transform and some singular integral operators.

I.4) The Laplace transform defined by

$$Lf(x) = \int_0^\infty e^{-\frac{t}{x}} f(t) \frac{dt}{x}$$

clearly satisfies that $L \in \mathcal{A}^*$ since

$$Lf(x) \leq Lf^*(x) = \int_0^\infty e^{-t} f^*(tx) dt.$$

I.5) The Riemann-Liouville operator with parameter $\alpha \geq 1$ defined by

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^x (x-t)^{\alpha-1} f(t) dt$$

is in $\mathcal{A}^*$, since

$$R_\alpha f(x) = \int_0^1 (1-s)^{\alpha-1} f(sx) ds$$

and hence

$$(R_\alpha f)^*(t) \leq \int_0^1 (1-s)^{\alpha-1} f^*(st) ds$$

and the function $a(s) = (1-s)^{\alpha-1}\chi_{(0,1)}(s)$ satisfies the required conditions.
II) Let now $\mathcal{A}$ be the set of decreasing functions with compact support in $(0,1)$.
II.1) Clearly again, $M \in \mathcal{A}^*$ and $R_\alpha \in \mathcal{A}^*$.
II.2) Let

$$T_\varphi f(x) = \sup_{h>0} \frac{1}{h} \int_0^h \varphi\Big(\frac{t}{h}\Big) |f(x-t)| dt$$

where φ is a function with compact support in $(0,1)$. It was proved in [19], that

$$(T_\varphi f)^*(\xi) \lesssim \int_0^1 \varphi^*(t) f^*(t\xi) dt,$$

and hence $T_\varphi \in \mathcal{A}^*$. For instance, for every $0 < \alpha \leq 1$, the operators

$$M_\alpha^+ f(x) = \sup_{r>x} \frac{1}{(r-x)^\alpha} \int_x^r \frac{|f(s)|}{(r-s)^{1-\alpha}} ds$$

and

$$M_\alpha^- f(x) = \sup_{r<x} \frac{1}{(x-r)^\alpha} \int_r^x \frac{|f(s)|}{(s-r)^{1-\alpha}} ds$$

are particular operators of this kind. These operators were studied in [19], [23] and [24] in connection with C_α sumability criterium for the Lebesgue Differentiation theorem.
II.3) In general, if we have a sublinear operator T which is bounded in L^∞ and satisfies a restricted weak type inequality

$$T : L^{p,1}(\mu) \longrightarrow L^{p,\infty}(\nu),$$

then standard techniques in interpolation theory show that

$$(Tf)^*_\nu(t) \lesssim \int_0^1 s^{\frac{1}{p}-1} f^*_\mu(st)dt$$

and therefore $T \in \mathcal{A}^*(\mu,\nu)$. This is the case, for example, for the Hardy-Littlewood maximal operator and any measure $d\mu = d\nu = u(x)dx$ with u a weight in the Muckenhoupt class A_1; that is $M \in \mathcal{A}^*(u,u)$ for every $u \in A_1$ (see [25]).

III) Let now $1 \leq p_0 < p_1 \leq \infty$ and let us define

$$\mathcal{A}_{p_0,p_1} = \Big\{ a \downarrow; a(t) \leq \min(t^{\frac{1}{p_0}-1}, t^{\frac{1}{p_1}-1}), \forall t > 0 \Big\}.$$

Then every operator T of joint weak type $(p_0,\infty;p_1,\infty)$ with respect to μ and ν (see [8, Chapter 3]) satisfies that $T \in \mathcal{A}^*_{p_0,p_1}(\mu,\nu)$.

2.2. **Boundedness properties for operators in the classes $\mathcal{A}^*$.**

Proposition 2.2. *If $\mathcal{A}$ is the set of decreasing and bounded functions satisfying*

$$\int_1^\infty \frac{a(t)}{t^\alpha} dt < \infty,$$

for some $0 < \alpha < 1$, then for every $T \in \mathcal{A}^*(\mu,\nu)$
i)

$$T : L^1(\mu) \longrightarrow L^{1,\infty}(\nu)$$

is bounded.
ii) There exists p_0 such that for every $1 < p < p_0$

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $C/(p-1)$.

Remark 2.3. As we shall see below, in general, estimates of the form ii) need not imply the weak type $(1,1)$ boundedness of T. However, the main advantage of estimates of the form ii) is that, if $\mu = \nu$, then these can be iterated to conclude that

$$T^{(n)} : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant less than or equal to $C/(p-1)^n$ and this will give us some information at the end-point of the operator $T^{(n)}$ that cannot be obtained directly from i).

Proof. For simplicity, we omit the measures μ and ν. The first part follows directly by a simple change of variable since

$$\begin{aligned} ||Tf||_{L^{1,\infty}} &= \sup_{t>0} t(Tf)^*(t) \leq \sup_{t>0} t \int_0^\infty a(s) f^*(st) ds \\ &= \sup_{t>0} \int_0^\infty a\Big(\frac{s}{t}\Big) f^*(s) ds \lesssim ||f||_1. \end{aligned}$$

On the other hand, by (2.1), it is clear that

$$(Tf)^*(t) \lesssim ||f||_{L^{p,\infty}} \int_0^\infty a(s)\frac{1}{(ts)^{1/p}} ds$$

and hence,

$$\begin{aligned} ||Tf||_{L^{p,\infty}} &\lesssim ||f||_{L^{p,\infty}} \int_0^\infty a(s)\frac{1}{s^{1/p}} ds \\ &\approx ||f||_{L^{p,\infty}} \left(\int_0^1 a(s)\frac{1}{s^{1/p}} ds + \int_1^\infty a(s)\frac{1}{s^{1/p}} ds \right) = ||f||_{L^{p,\infty}}(I + II). \end{aligned}$$

Now, to estimate I, we simply use that a is bounded and we get $I \lesssim \frac{1}{p-1}$. In order to estimate II, we observe that for every $p < 1/\alpha$,

$$\int_1^\infty \frac{a(t)}{t^{1/p}} dt \le \int_1^\infty \frac{a(t)}{t^\alpha} dt = C < \infty,$$

and hence the result follows. □

Similarly:

Proposition 2.4. *If $\mathcal{A}_{p_0,p_1}$ is the set given in example III), then, for every $T \in \mathcal{A}^*_{p_0,p_1}(\mu,\nu)$,*
i)

$$T : L^{p_0,1}(\mu) \longrightarrow L^{p_0,\infty}(\nu)$$

is bounded.
ii) For every $p_0 < p < p_1$

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $\frac{C}{(p-p_0)(p_1-p)}$.

Proof. The first part follows again by a simple change of variable since

$$\begin{aligned} ||Tf||_{L^{p_0,\infty}} &= \sup_{t>0} t^{1/p_0}(Tf)^*(t) \le \sup_{t>0} t^{1/p_0} \int_0^\infty a(s) f^*(st) ds \\ &= \sup_{t>0} t^{1/p_0 - 1} \int_0^\infty a\left(\frac{s}{t}\right) f^*(s) ds \le \int_0^\infty s^{1/p_0 - 1} f^*(s) ds = ||f||_{L^{p_0,1}}. \end{aligned}$$

Also,

$$(Tf)^*(t) \lesssim ||f||_{L^{p,\infty}} \int_0^\infty a(s)\frac{1}{(ts)^{1/p}} ds$$

and hence

$$\begin{aligned} ||Tf||_{L^{p,\infty}} &\lesssim ||f||_{L^{p,\infty}} \int_0^\infty a(s)\frac{1}{s^{1/p}} ds \lesssim ||f||_{L^{p,\infty}} \int_0^\infty \min(s^{\frac{1}{p_0}-1}, s^{\frac{1}{p_1}-1})\frac{1}{s^{1/p}} ds \\ &\approx ||f||_{L^{p,\infty}} \left(\int_0^1 s^{\frac{1}{p_0}-1}\frac{1}{s^{1/p}} ds + \int_1^\infty s^{\frac{1}{p_1}-1}\frac{1}{s^{1/p}} ds \right), \end{aligned}$$

from which the result follows. □

3. Extrapolation on $L^{p,\infty}$ with $p>1$

Let T be a sublinear operator such that for every $1<p<p_0$, with p_0 fixed,

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$, for some $m>0$. For simplicity in our presentation, we shall start with the case $m=1$ and $\mu=\nu$ the Lebesgue measure. Since $M : L^p \longrightarrow L^{p,\infty}$ is bounded with constant uniformly bounded in p, we have that

$$T\circ M : L^p \longrightarrow L^{p,\infty}$$

is bounded with the same behavior of the constant and, by Antonov's result

$$T\circ M : L\log_1 L\log_3 L \longrightarrow R_1$$

is bounded; that is,

$$\begin{aligned}||T(Mf)||_{R_1} &\lesssim \int_0^\infty f^*(t)\log_1\frac{1}{t}\log_3\frac{1}{t}dt \approx ||f||_1 + \int_0^1\left(\frac{1}{t}\int_0^t f^*(s)ds\right)\log_3\frac{1}{t}dt\\ &\approx ||Mf||_{L^{1,\infty}} + \int_0^1 (Mf)^*(t)\log_3\frac{1}{t}dt.\end{aligned}$$

Therefore, if

$$E=\left\{g\in L^{1,\infty};\ g=Mf,\ \text{for some } f\in L^1_{loc}\right\}$$

then

$$T : E\cap \underline{L\ \log_3 L} \longrightarrow R_1$$

is bounded, where the domain set is embedded with the quasi-norm

$$||f||_{L^{1,\infty}\cap\underline{L\ \log_3 L}} = ||f||_{L^{1,\infty}} + \int_0^1 f^*(t)\log_3\frac{1}{t}dt.$$

We denote with the underline notation $\underline{L\ \log_3 L}$ the fact that we only integrate in $(0,1)$.

Question: Which is the best space contained in $L^{1,\infty}\cap\underline{L\ \log_3 L}$ where T is bounded?

Observe that if we replace the operator M by any operator S such that, for some positive constant $B>0$,

$$\frac{1}{B}\frac{1}{t}\int_0^t f^*(s)ds \le (Sf)^*(t) \le B\frac{1}{t}\int_0^t f^*(s)ds, \tag{3.1}$$

then, with the same argument as before, we get that

$$T : E_S\cap\underline{L\ \log_3 L} \longrightarrow R_1$$

is bounded where

$$E_S=\left\{g\in L^{1,\infty}; \exists f\in L^1_{loc}\ \text{ with }\ g=Sf\right\}.$$

Moreover, for every $g \in E_S$,

$$||Tg||_R \lesssim C_B ||g||_{L^{1,\infty} \cap \underline{L \log_3 L}}$$

being C_B a constant depending only on B. Therefore, if for a fixed constant B, we define E^B the set of functions $g \in L^{1,\infty}$ such that, there exists an operator S satisfying (3.1) and a function f with $g = Sf$, then

$$T : E^B \cap \underline{L \log_3 L} \longrightarrow R_1$$

is bounded.

Remark 3.1. Now, if $g \in E^B$, then $g^*(t) \approx \frac{1}{t} \int_0^t f^*(s)ds$ and hence $\sup_{t \leq y} tg^*(t) \approx yg^*(y)$ and $\lim_{t \to 0} tg^*(t) = 0$ and, conversely, if $\sup_{t \leq y} tg^*(t) \approx yg^*(y)$ and $\lim_{t \to 0} tg^*(t) = 0$ then the function $G(y) = \sup_{t \leq y} tg^*(t)$ is quasi-increasing, $G(y)/y$ is decreasing and $G(0+) = 0$. Therefore, G is quasi-concave and hence equivalent to a concave function (cf. [8, Chapter 2]); that is

$$G(y) = \int_0^y \tilde{g}(s)ds$$

with $\tilde{g}$ decreasing. Consequently $g^*(t) \approx \frac{1}{t} \int_0^t \tilde{g}(s)ds$. From here, it follows that

$$g^*(t) = h(t) \frac{1}{t} \int_0^t \tilde{g}(s)ds$$

with $h, h^{-1} \in L^\infty$ and hence there exists a measure preserving transformation satisfying that

$$g(x) = h(\sigma(x)) \frac{1}{\sigma(x)} \int_0^{\sigma(x)} \tilde{g}(s)ds$$

and defining $Sf(x) = h(\sigma(x)) \frac{1}{\sigma(x)} \int_0^{\sigma(x)} f^*(s)ds$ we obtain that $g = S\tilde{g}$ and S satisfies (3.1) for some B.

That is, E^B can be essentially described as the set

$$\left\{ g \in L^{1,\infty}; \frac{1}{B} g^*(t) \leq \frac{1}{t} \int_0^t \tilde{g}(s)ds \leq Bg^*(t),\ \tilde{g} \ \text{ decreasing} \right\},$$

and we obtain the following result. In fact, we can state it for arbitrary non-atomic σ-finite measures μ and ν, since in this case every decreasing function is the decreasing rearrangement, with respect to the corresponding measure, of some function.

Proposition 3.2. *Let T be a sublinear operator such that for $1 < p < p_0$, with p_0 fixed,*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)$. Then, for every $B > 0$,

$$T : E^B \cap \underline{L \log_3 L}(\mu) \longrightarrow R_1(\nu)$$

is bounded, where

$$E^B = \left\{ g \in L^{1,\infty}(\mu); \exists h, \frac{1}{B} g_\mu^*(t) \le \frac{1}{t} \int_0^t h_\mu^*(s)ds \le B g_\mu^*(t) \right\}.$$

Before stating the next result, which is a consequence of this proposition, let us introduce some notation. Given a quasi-Banach r.i. space X over a measure space (Ω, Σ, μ), for each $p \ge 1$ let us denote

$$[X]_p = \left\{ g \in L^{p,\infty}(\mu); \frac{\sup_{t \le y} t^{1/p} g_\mu^*(t)}{y} \chi_{[0,1]}(y) \in \widetilde{X} \right\}$$

endowed with the quasi-norm

$$\|g\|_{[X]_p} = \|g\|_{L^{p,\infty}} + \left\| \frac{\sup_{t \le y} t^{1/p} g_\mu^*(t)}{y} \chi_{[0,1]}(y) \right\|_{\widetilde{X}}.$$

Here $\widetilde{X}$ denotes the canonical representation of the space X on the line $(0,\infty)$, that is $\|f\|_X = \|f_\mu^*\|_{\widetilde{X}}$. The basic properties of the spaces $[X]_p$ will be collected later (see Section 5).

Theorem 3.3. *Let T be a sublinear operator such that for $1 < p < p_0$ (with p_0 fixed)*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)$. Then,

$$T : [L \log_3 L(\mu)]_1 \longrightarrow R_1(\nu)$$

is bounded.

Proof. Let $g \in [L \log_3 L(\mu)]_1$ and let $H(y) = \sup_{t \le y} t g_\mu^*(t)$. Then $g_\mu^*(y) \le H(y)/y$ and hence there exists $k \in L^\infty$ with $||k||_\infty \le 1$ such that $g_\mu^*(y) = k(y)H(y)/y$. Let σ be the measure preserving transformation such that $g_\mu^*(\sigma(x)) = g(x)$. Then

$$g(x) = k(\sigma(x)) \frac{H(\sigma(x))}{\sigma(x)}.$$

Let us define $T_k(f) = T((k \circ \sigma) f)$. Then clearly T_k satisfies the hypothesis of the previous proposition. Now

$$Tg(x) = T_k \left(\frac{H(\sigma(x))}{\sigma(x)} \right)$$

and $H_\sigma(x) = \frac{H(\sigma(x))}{\sigma(x)}$ satisfies that $(H_\sigma)_\mu^* = H(y)/y$ with H quasi-concave. Hence, there exists a concave function G such that

$$\frac{1}{2} H(y) \le G(y) \le 2H(y).$$

Now, $G(y) = \int_0^y g(s)ds$ with g a decreasing function and since $g(s) = h_\mu^*(s)$ for some function h, we obtain that $H_\sigma \in E^B$ with $B = 2$. Therefore, by the previous proposition we have that

$$||Tg||_{R_1(\nu)} = ||T_k(H_\sigma)||_{R_1(\nu)} \lesssim ||H_\sigma||_{L^{1,\infty}(\mu)\cap \underline{L\ \log_3 L}(\mu)} = ||g||_{[L \log_3 L(\mu)]_1},$$

and we are done. □

Remark 3.4. In this remark we shall omit (by simplicity) the measure μ. We have that

$$L\ \log_1 L \log_3 L \subsetneq [L \log_3 L]_1. \tag{3.2}$$

Indeed, if $g \in L\ \log_1 L \log_3 L$, then $g^*(t) \leq \frac{1}{t}\int_0^t g^*(s)ds \in L\ \log_3 L(0,1)$, and hence $\frac{\sup_{t\leq y} tg^*(t)}{y} \in L\ \log_3 L$. Also $g \in L^1$ and hence $g \in L^{1,\infty}$. Moreover, the embedding is strict: if we take g such that

$$g^*(t) = \frac{1}{t \log_1 \frac{1}{t} \log_2 \frac{1}{t} \big(\log_3 \frac{1}{t}\big)^3} \tag{3.3}$$

then clearly $g \notin L\ \log_1 L \log_3 L$ but $\frac{\sup_{t\leq y} tg^*(t)}{y} = g^*(y) \in L^{1,\infty} \cap L\ \log_3 L(0,1)$.

Therefore, this shows that the end-point estimate that we obtain for an operator T bounded on $L^{p,\infty}$ with constant less than or equal to $\frac{1}{p-1}$ is better than the one obtained if we only use the information that such operator is of weak type (p,p) with the same behavior of the constant, as was mentioned in the introduction.

With the obvious changes, we also obtain the following result:

Theorem 3.5. *Let T be a sublinear operator such that*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$. Then,

$$T : [L(\log_1 L)^{m-1} \log_3 L(\mu)]_1 \longrightarrow R_m(\nu)$$

is bounded.

Moreover

$$L\,(\log_1 L)^m \log_3 L \subsetneq [L(\log_1 L)^{m-1} \log_3 L]_1.$$

In the case of finite measures, the above result reads as follows:

Corollary 3.6. *Let μ and ν be two finite measures and let T be a sublinear operator such that*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-1)^m$. Then,

$$T : [L\,(\log_1 L)^{m-1} \log_3 L(\mu)]_1 \longrightarrow L^{1,\infty}(\nu)$$

is bounded, where now we have

$$||g||_{[L\,(\log_1 L)^{m-1}\log_3 L(\mu)]_1} = \left\| \frac{\sup_{t\le y} t g_\mu^*(t)}{y} \right\|_{L\,(\log_1 L)^{m-1}\log_3 L}.$$

Corollary 3.7. *If $\{T_j\}_{j=1}^n$ satisfy the hypothesis of Theorem 3.3, then*

$$T_1 \circ T_2 \circ \cdots \circ T_n : [L(\log_1 L)^{n-1}\log_3 L(\mu)]_1 \longrightarrow R_n(\mu)$$

is bounded. In particular, this boundedness is satisfied by the iterated operator $T^{(n)}$ with T satisfying the hypothesis of Theorem 3.3.

It is clear that if T_1 satisfies the hypothesis of Theorem 3.3 and

$$T_2 : L^p(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant uniform in $p > 1$, then

$$T_1 \circ T_2 : L\,(\log_1 L)\log_3 L(\mu) \longrightarrow R_1(\nu)$$

is bounded. However we shall prove in our next theorem that we can obtain the same result for a wide class of operators T_2 which do not satisfy necessarily the uniform bound assumed above. Moreover, T_2 may not be bounded on L^p.

Theorem 3.8. *Let $n \ge 1$, and let us consider $\mathcal{A}_n$ the set of decreasing and bounded functions such that*

$$\int_1^\infty a(s)(\log_1 s)^{n-1}\log_3 s\,\frac{ds}{s} < \infty.$$

Let us suppose

$$T_1 : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant $1/(p-1)^n$ and $T_2 \in \mathcal{A}_n^*$. Then

$$T_1 \circ T_2 : L\,(\log_1 L)^n \log_3 L(\mu) \longrightarrow R_n(\mu)$$

is bounded.

Observe that all the examples given in Section 2 except those in III) are in the class $\mathcal{A}_n$. The case III) will be considered in the next section.

Proof. The idea is to apply Theorem 3.5 to the operator T_1 and work with $(T_1 \circ T_2)f = T_1(T_2 f)$. Using (2.1) and the fact that a is decreasing, we have

$$\begin{aligned} sup_{t\le y} t(T_2 f)_\mu^*(t) &\lesssim sup_{t\le y} t\int_0^\infty a(s) f_\mu^*(st)ds = sup_{t\le y}\int_0^\infty a\Big(\frac{u}{t}\Big) f_\mu^*(u)du \\ &= \int_0^\infty a\Big(\frac{u}{y}\Big) f_\mu^*(u)du = y\int_0^\infty a(s) f_\mu^*(sy)ds. \end{aligned}$$

From here, it follows first since $a \in L^\infty$ that

$$T_2 : L^1(\mu) \longrightarrow L^{1,\infty}(\mu)$$

is bounded and hence,

$$\begin{aligned} ||(T_1\circ T_2)f||_{R_n(\mu)} &= ||T_1(T_2f)||_{R_n(\mu)} \lesssim ||T_2f||_{[L(\log_1 L)^{n-1}\log_3 L(\mu)]_1} \\ &= ||T_2f||_{L^{1,\infty}(\mu)} + \int_0^1 \frac{\sup_{t\le y} t(T_2f)^*_\mu(t)}{y}\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}dy \\ &\lesssim ||f||_1 + \int_0^1\Big(\int_0^\infty a(s)f^*_\mu(sy)ds\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}dy \\ &= ||f||_1 + \int_0^\infty f^*_\mu(s)\bigg(\int_0^1 a\Big(\frac{s}{y}\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\frac{dy}{y}\bigg)ds. \end{aligned}$$

Now, if $s\le 1$,

$$\begin{aligned} \int_0^1 a\Big(\frac{s}{y}\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\frac{dy}{y} &= \int_0^s+\int_s^1\bigg(a\Big(\frac{s}{y}\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\bigg)\frac{dy}{y} \\ &= I+II \end{aligned}$$

To estimate II we simply use that a is bounded and hence

$$II\lesssim\int_s^1\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\frac{dy}{y}\lesssim\Big(\log_1\frac{1}{s}\Big)^{n}\log_3\frac{1}{s},$$

and to estimate I we proceed as follows:

$$\begin{aligned} I &= \int_0^s a\Big(\frac{s}{y}\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\frac{dy}{y} = \int_1^\infty a(u)\Big(\log_1\frac{u}{s}\Big)^{n-1}\log_3\frac{u}{s}\frac{du}{u} \\ &\lesssim \Big(\log_1\frac{1}{s}\Big)^{n-1}\log_3\frac{1}{s}\int_1^\infty a(u)\Big(\log_1 u\Big)^{n-1}\log_3 u\frac{du}{u} \\ &\lesssim \Big(\log_1\frac{1}{s}\Big)^{n-1}\log_3\frac{1}{s}. \end{aligned}$$

Finally, if $s>1$,

$$III=\int_0^1 a\Big(\frac{s}{y}\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\frac{dy}{y}\le\int_0^1 a\Big(\frac{1}{y}\Big)\Big(\log_1\frac{1}{y}\Big)^{n-1}\log_3\frac{1}{y}\frac{dy}{y}$$

and we argue as in the estimation of I to conclude that $III\le C<\infty$. Consequently,

$$||(T_1\circ T_2)f||_{R_n(\mu)}\lesssim\int_0^\infty f^*_\mu(s)\Big(\log_1\frac{1}{s}\Big)^{n}\log_3\frac{1}{s}ds,$$

as we wanted to see. □

Remark 3.9. Observe that if we consider the Hardy-Littlewood maximal operator M on $(0,1)$, $M: L^{p,\infty}(0,1)\to L^{p,\infty}(0,1)$ is bounded with norm less than or equal to $C/(p-1)$ and hence, if we apply the previous theorem, we can conclude that

$$M\circ M: L\,(\log_1 L)\log_3 L(0,1)\longrightarrow L^{1,\infty}(0,1)$$

is bounded, and, except for the $\log_3 L$ term, this would be the best result that can be obtained in the sense that $(M \circ M)f = M(Mf) \in L^{1,\infty}$ if and only if $Mf \in L^1(0,1)$ and this happens if and only if $f \in L(\log_1 L)(0,1)$.

Even though the condition of $a \in \mathcal{A}$ being bounded is satisfied for many operators, we have already seen that this condition implies that such operators are of weak type $(1,1)$. If we want to include the cases of operators which are not bounded on L^1, we have to remove the boundedness assumption for the functions in $\mathcal{A}$. Then looking again at the proof where this property has been used we also have the following result, which shall be useful for the applications. For simplicity we state it for a single measure μ, but the same result holds for operators acting between different measure spaces.

Theorem 3.10. *Let $n \geq 1$ and let us assume that*

$$(T_2 f)^*_\mu(t) \lesssim \int_0^\infty a(s) f^*_\mu(st)\,ds$$

where a is decreasing and such that

$$\int_1^\infty a(s)(\log_1 s)^{n-1} \log_3 s \,\frac{ds}{s} < \infty.$$

If

$$T_1 : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant $C/(p-1)^n$, then

$$T_1 \circ T_2 : D \longrightarrow R_n(\mu)$$

is bounded, where D is the set of functions f such that

$$||f||_D = ||T_2 f||_{L^{1,\infty}} + \int_0^1 f^*_\mu(s) a(s) \Big(\log_1 \frac{1}{s}\Big)^n \log_3 \frac{1}{s} ds + \int_1^\infty f^*_\mu(s)\,ds$$

is finite.

In particular:

Corollary 3.11. *Let μ and ν be finite measures. Let T_1 be such that for $1 < p < p_0$,*

$$T_1 : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\mu)$$

is bounded with constant $1/(p-1)$ and let T_2 be such that

$$(T_2 f)^*_\mu(t) \leq \int_0^\infty a(s) f^*_\mu(st)\,ds$$

where a is a decreasing function. Then,

$$T_1 \circ T_2 : D \longrightarrow L^{1,\infty}(\mu)$$

is bounded where

$$||f||_D = \int_0^1 f^*_\mu(s) a(s) \log_1 \frac{1}{s} \log_3 \frac{1}{s} ds.$$

Proof. Let us assume that μ and ν are probability measures. By Theorem 3.10, we only have to study for which functions f we have that $T_2 f \in L^{1,\infty}(\mu)$. Now, since a is decreasing,

$$||T_2 f||_{L^{1,\infty}(\mu)} = \sup_{0<t\leq 1} t(T_2 f)^*_\mu(t) \leq \sup_{0<t\leq 1} \int_0^1 a\Big(\frac{s}{t}\Big) f^*_\mu(s)ds \lesssim \int_0^1 f^*_\mu(s)a(s)ds,$$

and the result follows. □

3.1. **Applications.** For a general weight u, it is known (see [22]) that

$$(Mf)^*_u(t) \lesssim \int_0^1 \Phi_u(s) f^*_u(st)ds, \tag{3.4}$$

where

$$\Phi_u(s) = \sup_Q \frac{u(Q)}{|Q|}(u^{-1}\chi_Q)^*_u(u(Q)s).$$

As a consequence, we have the following applications of our previous results.

Corollary 3.12. *Let u be a weight such that for $1 < p < p_0$ (with p_0 fixed)*

$$\int_0^1 \Phi_u(s)s^{-1/p}ds \lesssim \frac{1}{(p-1)^m}. \tag{3.5}$$

Then

$$M : [L(\log_1 L)^{m-1}\log_3 L(u)]_1 \longrightarrow R_m(u)$$

is bounded.

Proof. It is enough to observe that, by (3.4) and (3.5), we have that

$$||Mf||_{L^{p,\infty}(u)} \lesssim ||f||_{L^{p,\infty}(u)} \int_0^1 \Phi_u(s)s^{-1/p}ds \lesssim ||f||_{L^{p,\infty}(u)} \frac{1}{(p-1)^m},$$

and the result follows by Theorem 3.5. □

Also, we have the following result for an integrable weight u.

Corollary 3.13. *Let u be an integrable weight satisfying (3.5) and let T be a sublinear operator bounded on $L^{p,\infty}(u)$ with constant less than or equal to $1/(p-1)$. Then*

$$T \circ M : D \longrightarrow L^{1,\infty}(u)$$

is bounded where

$$||f||_D = \int_0^1 f^*_u(s)\Phi_u(s)\log_1\frac{1}{s}\log_3\frac{1}{s}ds.$$

Proof. It is an immediate consequence of Corollary 3.11. □

A similar result could also be stated for a non-integrable weight u using Theorem 3.10.

Remark 3.14. If $u \in A_p$, it was proved in [22] that
$$\Phi_u(s) \leq \frac{||u||_{A_p}^{1/p}}{s^{1/p'}},$$
where
$$||u||_{A_p} = \sup_Q \Big(\frac{1}{|Q|}\int_Q u\Big)\Big(\frac{1}{|Q|}\int_Q u^{1-p'}\Big)^{p-1}.$$
Hence, we have the following examples to which the previous corollaries could be applied:
i) If $u \in \cap_{p>1} A_p \setminus A_1$ and $||u||_{A_p} \lesssim \frac{1}{(p-1)}$, then
$$\Phi_u(s) \leq \frac{1}{s} \inf_{p>1} \frac{s^{1/p}}{p-1} \lesssim \log_1 \frac{1}{s}.$$
ii) If $u \in \cap_{p>1} A_p \setminus A_1$ and $||u||_{A_p} \lesssim \log \frac{1}{(p-1)}$, then one can easily see that
$$\Phi_u(s) \lesssim \log_2 \frac{1}{s}.$$

Concerning the case of the Hilbert transform, it was proved in [6] that if $u \in A_\infty = \cup_{p>1} A_p$, then
$$(Hf)_u^*(t) \lesssim \int_{t/4}^{\infty} (Mf)_u^*(s)\frac{ds}{s}$$
which combined with (3.4) gives us
$$\begin{aligned}
(Hf)_u^*(t) &\lesssim \int_{t/4}^{\infty}\int_0^1 \Phi_u(v) f_u^*(vs)dv\frac{ds}{s} = \int_0^1 \Phi_u(v)\int_{t/4}^{\infty} f_u^*(vs)\frac{ds}{s}dv \\
&= \int_0^1 \Phi_u(v)\int_{vt/4}^{\infty} f_u^*(s)\frac{ds}{s}dv = \int_0^{\infty} f_u^*(s)\int_0^{\min(4s/t,1)} \Phi_u(v)dv\frac{ds}{s} \\
&= \int_0^{\infty} f_u^*(st)\int_0^{\min(4s,1)} \Phi_u(v)dv\frac{ds}{s} = \int_0^{\infty} f_u^*(st)\Psi_u(s)ds
\end{aligned}$$
with
$$\Psi_u(s) = \frac{1}{s}\int_0^{\min(4s,1)} \Phi_u(v)dv.$$

As a consequence, we also have the following results:

Corollary 3.15. *Let u be a weight in A_∞ such that for every $1 < p < p_0$ (with p_0 fixed)*
$$\int_0^1 \Psi_u(s)s^{-1/p}ds \lesssim \frac{1}{(p-1)^m}. \tag{3.6}$$
Then
$$H : [L(\log_1 L)^{m-1}\log_3 L(u)]_1 \longrightarrow R_m(u)$$
is bounded.

Since a weight in A_∞ is not integrable we cannot have the analogue to Corollary 3.13, but using Theorem 3.10 we can conclude the following result.

Corollary 3.16. *Let u be an A_∞ weight satisfying (3.6) and let T be a sublinear operator bounded on $L^{p,\infty}(u)$ with constant less than or equal $1/(p-1)$. Then*

$$T \circ H : D \longrightarrow L^{1,\infty}(u)$$

is bounded where

$$||f||_D = ||Hf||_{L^{1,\infty}(u)} + \int_0^1 f_u^*(s)\Psi_u(s)\log_1 \frac{1}{s} \log_3 \frac{1}{s} ds + \int_1^\infty f_u^*(s) ds.$$

We should mention here that, in the case $u \in A_\infty$, it is known that $Hf \in L^{1,\infty}(u)$ if for example $f \in L^1(Mu)$.

Remark 3.17. All the above results lead us to the following considerations: It is known that, if u is a weight in the Muckenhoupt class $u \in A_p$ (see [12], [21]), then

$$M : L^p(u) \longrightarrow L^p(u)$$

is bounded with constant $\frac{C}{p-1}||u||_{A_p}^{1/p-1}$ and

$$M : L^p(u) \longrightarrow L^{p,\infty}(u) \tag{3.7}$$

is bounded with constant $C||u||_{A_p}^{1/p}$. But, which is the best bound for the norm $||M||_{L^{p,\infty}(u)\to L^{p,\infty}(u)}$ in terms of p and u?

We could ask the same question in the context of Calderón-Zygmund operators T. See [18], [27], [28] and the references there quoted for papers dealing with the behavior of $||T||_{L^p(u)\to L^p(u)}$. But again, which is the best bound for $||T||_{L^{p,\infty}(u)\to L^{p,\infty}(u)}$?

The same question is of interest concerning the norm of the commutator $[T, b]$ on $L^{p,\infty}(u)$ with T a Calderón-Zygmund operator and $b \in BMO$.

4. Extrapolation on $L^{p,\infty}$ with $p > p_0 > 1$

Our next goal is to obtain boundedness properties as those given above for operators T satisfying an estimate that blows up when p tends to $p_0 > 1$. This happens for the example III) in Section 2 taking into account Proposition 2.4.

Theorem 4.1. *Let T be a sublinear operator such that*

$$T : L^{p,\infty}(\mu) \longrightarrow L^{p,\infty}(\nu)$$

is bounded with constant less than or equal to $1/(p-p_0)^m$ for every $p > p_0$. Then:

$$T : [L(\log_1 L)^{m-1}(\mu)]_{p_0} \longrightarrow R_m^{p_0}(\nu)$$

where

$$||g||_{R_m^{p_0}(\nu)} = \sup_{t>0} \frac{\left(\int_0^t g_\nu^*(s)^{p_0} ds\right)^{1/p_0}}{(1+\log^+ t)^m}.$$

Proof. We have to follow the same steps as in the proof of the case $p_0 = 1$ using the following facts:
i) Now, the operators S have to be taken satisfying

$$\frac{1}{B}\bigg(\frac{1}{t}\int_0^t f_\mu^*(s)^{p_0}ds\bigg)^{1/p_0} \leq (Sf)_\mu^*(t) \leq B\bigg(\frac{1}{t}\int_0^t f_\mu^*(s)^{p_0}.ds\bigg)^{1/p_0} \tag{4.1}$$

ii) We have to use (see [14]) that if

$$T : L^p(\mu) \longrightarrow L^{p,\infty}(\nu)$$

with constant $1/(p-p_0)$ then

$$||Tf||_{R_m^{p_0}(\nu)} \lesssim ||f||_{L^{p_0}(\mu)} + \int_0^1 \bigg(\int_0^t f_\mu^*(s)^{p_0}ds\bigg)^{1/p_0}\bigg(\log_1 \frac{1}{t}\bigg)^{m-1}\frac{dt}{t}.$$

In fact, we have to mention here that although the above boundedness is stated for operators

$$T : L^p(\mu) \longrightarrow L^p(\nu)$$

with constant $1/(p-p_0)$, the proof only uses the fact that these operators satisfy $T : L^p(\mu) \longrightarrow \Gamma^{p,\infty}(\nu)$ with the same behavior of the constant, where

$$||f||_{\Gamma^{p,\infty}(\nu)} = \sup_{t>0} f_\mu^{**}(t)t^{1/p}.$$

But since p is far from $p = 1$, this space coincides with the space $L^{p,\infty}(\mu)$ and the constant in the equivalence does not blow up when $p \to p_0$.

Then, with the same proof than in Proposition 3.2, we have that, for every $B > 0$,

$$T : E_{p_0}^B \cap \underline{L\,(\log_1 L)^{m-1}(\mu)} \longrightarrow R_{p_0}(\nu) \tag{4.2}$$

is bounded, where

$$E_{p_0}^B = \bigg\{g \in L^{p_0,\infty}(\mu); \exists h,\ \frac{1}{B}g_\mu^*(t) \leq \bigg(\frac{1}{t}\int_0^t h_\mu^*(s)^{p_0}ds\bigg)^{1/p_0} \leq Bg_\mu^*(t), \bigg\},$$

and

$$||f||_{L^{p_0,\infty}(\mu)\cap\underline{L\,(\log_1 L)^{m-1}(\mu)}} = ||f||_{L^{p_0,\infty}(\mu)} + \int_0^1 f_\mu^*(t)t^{1/p_0-1}\bigg(\log_1 \frac{1}{t}\bigg)^{m-1} dt.$$

Let $g \in [L(\log_1 L)^{m-1}(\mu)]_{p_0}$ and let $H(y) = \sup_{t\leq y} t^{1/p_0}g_\mu^*(t)$. Then $g_\mu^*(y) \leq H(y)/y^{1/p_0}$ and hence there exists $k \in L^\infty$ with $||k||_\infty \leq 1$ such that $g_\mu^*(y) = k(y)\frac{H(y)}{y^{1/p_0}}$. Let σ be the measure preserving transformation such that $g_\mu^*(\sigma(x)) = g(x)$. Then

$$g(x) = k(\sigma(x))\frac{H(\sigma(x))}{\sigma(x)^{1/p_0}}.$$

Let us define $T_k(f) = T((k\circ\sigma)f)$. Then clearly T_k satisfies the same hypothesis as T_k. Now

$$Tg(x) = T_k\left(\frac{H(\sigma(x))}{\sigma(x)^{1/p_0}}\right)$$

and $H_\sigma(x) = \frac{H(\sigma(x))}{\sigma(x)^{1/p_0}}$ satisfies that $(H_\sigma)^*_\mu(y) = H(y)/y^{1/p_0}$ with H^{p_0} quasi-concave. Hence, there exists a concave function G such that

$$\frac{1}{2^{1/p_0}}H(y) \leq G(y)^{1/p_0} \leq 2^{1/p_0}H(y)$$

Now, $G(y) = \int_0^y g(s)ds$ with g a decreasing function and since $g(s) = h^*_\mu(y)^{p_0}$ for some h, we have that $H_\sigma \in E^B_{p_0}$ with $B = 2^{1/p_0}$. Therefore, by (4.2) we have that

$$\begin{aligned} ||Tg||_{R^{p_0}_m} &= ||T_k(H_\sigma)||_{R^{p_0}_m} \lesssim ||H_\sigma||_{L^{p_0,\infty}(\mu)\cap \underline{L(\log_1 L)^{m-1}(\mu)}} \\ &= ||H_\sigma||_{L^{p_0,\infty}} + \int_0^1 (H_\sigma)^*_\mu(t)t^{1/p_0-1}\left(\log_1\frac{1}{t}\right)^{m-1} dt \\ &= ||g||_{[L(\log_1 L)^{m-1}(\mu)]_{p_0}}, \end{aligned}$$

and we are done.

□

5. Functional properties of the spaces $[X]_p$

Lemma 5.1. *Let X be a quasi-Banach function space with the weak-Fatou property. Then $[X]_p$ is also a quasi-Banach function space with the weak-Fatou property for every $p \geq 1$.*

Remark 5.2. Recall that a quasi-Banach function space X has the *weak-Fatou* property when for every increasing sequence (f_n) in X_+ such that $\sup_n \|f_n\|_X < \infty$, there exists $f \in X$ such that $f_n \uparrow f$ almost everywhere.

Proof. It is easy to check that

$$\|f\|_{[X]_p} = \|f\|_{L^{p,\infty}} + \left\|\sup_{t\leq y}\frac{t^{1/p}f^*(t)}{y}\right\|_{\tilde{X}}$$

defines a quasi-norm.

Let now (f_n) in $[X]_p$, such that $f_n \geq 0$ and

$$\sup_n \|f_n\|_{[X]_p} = \sup_n\left(\|f_n\|_{L^{p,\infty}} + \left\|\sup_{t\leq y}\frac{t^{1/p}f_n^*(t)}{y}\right\|_{\tilde{X}}\right) < \infty.$$

Since $L^{p,\infty}$ is weak-Fatou, it follows that there is $f \in L^{p,\infty}$ such that $f_n \uparrow f$ almost everywhere. Now, let

$$g_n(y) = \sup_{t\leq y}\frac{t^{1/p}f_n^*(t)}{y}.$$

Clearly, (g_n) is an increasing sequence in $\tilde{X}_+$ which by hypothesis satisfies $\sup_n \|g_n\|_{\tilde{X}} < \infty$. Therefore, $g_n(y) \uparrow g(y)$ almost everywhere to some $g \in \tilde{X}_+$. Moreover, since $f_n \uparrow f$, we have that

$$\sup_{t\leq y} \frac{t^{1/p} f_n^*(t)}{y} \uparrow \sup_{t\leq y} \frac{t^{1/p} f^*(t)}{y}.$$

This means that $\sup_{t\leq y} \frac{t^{1/p} f^*(t)}{y} = g(y)$ almost everywhere, so it belongs to $\tilde{X}$. This shows that $f \in [X]_p$ and hence this space has weak-Fatou.

In particular, under these conditions, $[X]_p$ is a quasi-Banach space (cf. [26, 2.35]). □

Remark 5.3. (i) Trivially,

$$[L \log_3 L]_1 \subset L^{1,\infty} \cap \underline{L \ \log_3 L}$$

and the embedding is also strict. To see this, we observe that if both spaces coincide, then

$$\int_0^1 \frac{\sup_{t\leq y} t g^*(t)}{y} \log_3 \frac{1}{y} dy \lesssim ||g||_{L^{1,\infty}} + \int_0^1 g^*(t) \log_3 \frac{1}{t} dy$$

and taking $g^*(t) = \chi_{(0,r)}(t)$, we need to have that, for every $0 < r < 1$

$$\int_0^1 \min\left(1, \frac{r}{y}\right) \log_3 \frac{1}{y} dy \lesssim r + \int_0^r \log_3 \frac{1}{t} dy$$

which implies

$$r \int_r^1 \frac{1}{y} \log_3 \frac{1}{y} dy \lesssim r + \int_0^r \log_3 \frac{1}{t} dy$$

and this is clearly false by making r tends to zero.

(ii) Observe also that the function defined in (3.3) is neither in the space $L \log_2 L$. In fact, taking g_m such that

$$g_m^*(t) = \frac{1}{t \log_1 \frac{1}{t} \log_2 \frac{1}{t} \left(\log_3 \frac{1}{t}\right)^2 \log_4 \frac{1}{t} \cdots \log_{m-1} \frac{1}{t} \left(\log_m \frac{1}{t}\right)^3}$$

one can see that $g_m \in [L \log_3 L]_1$ but $g_m \notin L \log_3 L \log_k L$, for any $k \neq 3$, $k < m$.

Proposition 5.4.
1) If $X \subset Y$, then $[X]_p \subset [Y]_p$ for every $p \geq 1$.
2) $[[X]_1]_1 = [X]_1$

Proof. 1) Clear.
2) Notice that in general we have

$$\begin{aligned} \|f\|_{[[X]_p]_q} &= \|f\|_{L^{q,\infty}} + \|f\|_{L^{p,\infty}} + \left\| \sup_{y\leq x} \frac{y^{\frac{1}{p}}}{x} \sup_{t\leq y} \frac{t^{\frac{1}{q}} f^*(t)}{y} \chi_{(0,1)}(x) \right\|_{\tilde{X}} \\ &= \|f\|_{L^{q,\infty}} + \|f\|_{L^{p,\infty}} + \left\| \sup_{t\leq y} \frac{t^{\frac{1}{p}+\frac{1}{q}-1} f^*(t)}{y} \right\|_{\tilde{X}}. \end{aligned}$$

So, in particular

$$[[X]_1]_1 = [X]_1.$$

□

Let us now restrict ourselves to the probability measure case. In this situation we have better properties.

Proposition 5.5. *Let X be an r.i. space over a probability space (Ω, Σ, μ):*
1) For every $1 \leq q \leq p$ it holds that $[X]_p \subset [X]_q$
2) $[X]_p \subset X$ for every $p \geq 1$.
3) For a quasi-concave function φ on $[0,1]$ we consider the Marcinkiewicz space

$$M_\varphi = \{f \in L^0(\mu) : \sup_t \varphi(t) f^*_\mu(t) < \infty\}.$$

It holds that $[M_\varphi]_1 = M_\varphi$ isometrically.
4) If the upper Boyd index $\overline{\alpha_X} < 1$, then there is a constant $C > 0$ such that

$$\|f\|_X \leq \|f\|_{[X]_1} \leq C\|f\|_X.$$

Proof. Notice that in the probability case we always have $\tilde{X} \subset L^{1,\infty}[0,1]$, this fact together with the inequality

$$f^*_\mu(y) \leq \sup_{t \leq y} \frac{t f^*_\mu(t)}{y} \tag{5.1}$$

yield that

$$\|f\|_{[X]_1} \approx \Big\| \sup_{t \leq y} \frac{t f^*_\mu(t)}{y} \Big\|_{\tilde{X}}.$$

1) This is clear.
2) It follows from inequality (5.1) that

$$\|f\|_X \leq \|f\|_{[X]_1}.$$

Now, by (1), we obtain that, for any $p \geq 1$,

$$[X]_p \subset [X]_1 \subset X.$$

3) Let φ be an increasing function with $\frac{\varphi(t)}{t}$ decreasing. We have that

$$\begin{aligned} \|f\|_{[M_\varphi]} &= \sup_y \varphi(y) \sup_{t \leq y} \frac{t f^*_\mu(t)}{y} = \sup_t t f^*_\mu(t) \sup_{y \geq t} \frac{\varphi(y)}{y} \\ &= \sup_t \varphi(t) f^*_\mu(t) = \|f\|_{M_\varphi}. \end{aligned}$$

4) Let M denote the Hardy-Littlewood maximal operator. It follows that

$$\|f\|_{[X]_1} = \Big\| \sup_{t \leq y} \frac{t f^*(t)}{y} \Big\|_{\tilde{X}} \leq \Big\| \frac{1}{y} \int_0^y f^*(t) dt \Big\|_{\tilde{X}} \lesssim \|M\|_{X \to X} \|f\|_X.$$

It is well-known that $M : X \to X$ is bounded if and only if $\overline{\alpha_X} < 1$ (cf. [8, Chapter 3]). □

As an immediate application, by property (3) in the last proposition, we have $[L^{p,\infty}(0,1)]_1 = L^{p,\infty}(0,1)$.

Notice that in general, the properties described in this last proposition are no longer true for the infinite measure case.

6. Appendix

With a completely similar proof than in Theorem 3.3, we can show the following result, for which we first need to recall the following definition due to Kalton [20].

Definition 6.1. A space X is said to be logconvex if, for every $a_n \in X$,

$$\left\| \sum_{n=0}^{\infty} a_n \right\|_X \lesssim \sum_{n=0}^{\infty} \log_1 n \, ||a_n||_X.$$

The classical example is $X = L^{1,\infty}$ and the following result is interesting since on many occasions we may have operators for which the unique information that we have is that

$$T : L^{p,\infty}(\mu) \longrightarrow L^{1,\infty}$$

is bounded with constant less than or equal to $1/(p-1)$. Observe that if the measure is finite $L^{p,\infty}(\mu) \subset L^{1,\infty}(\mu)$ and hence this condition is weaker than the one assumed in Theorem 3.3 but as we see from the following result the conclusion is the same (the proof is completely similar to Theorem 3.3 and we omit it).

Theorem 6.2. *Let T be a sublinear operator such that for $1 < p < p_0$ (with p_0 fixed)*

$$T : L^{p,\infty}(\mu) \longrightarrow X$$

is bounded with constant less than or equal to $1/(p-1)$. Then:
i) if X is a Banach space,

$$T : [L^1(\mu)]_1 \longrightarrow X$$

is bounded.
ii) if X is logconvex,

$$T : [L \log_3 L(\mu)]_1 \longrightarrow X$$

is bounded.

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