# MAUREY-ROSENTHAL FACTORIZATION FOR $p$-SUMMING OPERATORS AND DODDS-FREMLIN DOMINATION

CARLOS PALAZUELOS, ENRIQUE A. SÁNCHEZ PÉREZ, AND PEDRO TRADACETE

Abstract. We characterize by means of a vector norm inequality the space of operators that factorize through a $p$-summing operator from an $L_r$-space to an $L_s$-space. As an application, we prove a domination result in the sense of Dodds-Fremlin for $p$-summing operators on Banach lattices with cotype 2, showing moreover that this cannot hold in general for spaces with higher cotype. We also present a new characterization of Banach lattices satisfying a lower 2-estimate in terms of the order properties of 2-summing operators.

## 1. Introduction

Factorization of operators on Banach spaces through $L_p$-spaces is a fundamental tool for obtaining results in operator theory. The Maurey-Rosenthal factorization technique provides such a factorization for operators between Banach lattices when the adequate concavity/convexity requirements on the spaces and the operator are fulfilled. In [5, 6], it was proved that the principles that lie under the arguments that prove these theorems can also be generalized in order to include in the same scheme other factorization theorems that can be considered as independent. Essentially, a separation argument based on the Hahn-Banach Theorem (or equivalently, Ky Fan's Lemma) applied to bilinear forms defined by the operators provides a domination result (in the sense of Pietsch) that can be translated in order to obtain the factorization result. A relevant result that can be proved as an application of this technique is the one that relates an inequality of the form

$$\Big\|\Big(\sum_{k=1}^{n}|T(\lambda_k x_k)|^s\Big)^{1/s}\Big\|_F \leq K\|(\lambda_k)\|_t\Big\|\Big(\sum_{k=1}^{n}|x_k|^r\Big)^{1/r}\Big\|_E$$

2000 *Mathematics Subject Classification.* Primary 47B10, 47B65, Secondary 47B38.

*Key words and phrases.* $p$-summing operator, positive operator, Banach lattice, factorization, Dodds-Fremlin domination.

for an operator $T : E \to F$ (where $E$ is an $r$-convex Banach lattice, $T$ is $s$-concave and $1/s = 1/t + 1/r$) with the factorization

$$\begin{array}{ccc} E(\mu) & \xrightarrow{T} & F(\nu) \\ {\scriptstyle M_f}\downarrow & & \uparrow{\scriptstyle M_g} \\ L_r(\mu) & \xrightarrow{R} & L_s(\nu) \end{array}$$

(see Theorem 3.1 in [6]). Notice that in this case, the operator $R$ that appears in the scheme is clearly bounded and carries the concavity properties of the original operator $T$.

Let us assume now that the original operator $T$ is $p$-summing. We want to factor $T$ through a scheme as the one given above but with the additional requirement that the operator that provides the factorization also carries the $p$-summability. We shall show which is the inequality that must be fulfilled by an operator $T$ for this to happen. Since by the ideal property of $p$-summing operators the converse is always true, the inequality must be stronger than the one that characterize $p$-summing operators. Thus, the first part of this paper is devoted to characterizing the space of operators that factorize through a $p$-summing operator defined between an $L_r$-space and an $L_s$-space.

In the second part of the paper we use the results in the second section to study the domination problem (in Dodds-Fremlin's sense) for $p$-summing operators. Recall that Dodds-Fremlin Theorem [8] asserts that given positive operators $0 \leq R \leq T : E \to F$ between Banach lattices such that $E$ and $F^*$ are order continuous then $R$ is compact whenever $T$ is. This problem has also been studied for the classes of weakly compact [17], Dunford-Pettis [12], and strictly singular operators ([9], [10]) among others. However, the same problem for non-closed operator ideals (such as the ideal of $p$-summing operators) does not seem to have been studied in the literature. The main reason for this appears to be the fact that, in general, "local properties" of Banach spaces, such as summability, just do not fit properly within the lattice structure. So, in general we cannot hope to get a statement of the kind: if $0 \leq R \leq T : E \to F$ with $T$ $p$-summing, then $R$ is $p$-summing (see Proposition 3.6). Hence, it might be even more surprising that such a statement holds if $E$ and $F$ have cotype 2. Namely, if $\pi_p(T)$ denotes the $p$-summing constant of $T$, that is

$$\pi_p(T) = \sup\Big\{ \Big( \sum_{i=1}^m \|Tx_i\|^p \Big)^{\frac{1}{p}} : \sup_{\|x^*\|_{E^*} \leq 1} \Big( \sum_{i=1}^m |\langle x^*, x_i\rangle|^p \Big)^{\frac{1}{p}} \leq 1 \Big\},$$

we show in Theorem 3.3 that for some fixed constant $C < \infty$, $\pi_p(R) \leq C\pi_p(T)$ whenever $0 \leq R \leq T : E \to F$ and both $E$ and $F$ have cotype 2.

In Section 4 we also present some remarks concerning the constant involved in the domination results, showing that we cannot expect it to be one even in the simplest cases. More precisely, we show that this is not the case for absolutely summing operators between $\ell_1^2$ and $\ell_2^2$, nor for 2-summing operators between $\ell_1^3$ and $\ell_2^2$. Notice that in these cases the domination theorems hold trivially due to Grothendieck's inequality, moreover these spaces have cotype 2.

Our notation regarding Banach lattices and operators is standard. Our fundamental references on Banach lattices and $p$-summing operators are [13] and [7], respectively.

## 2. Factorization theorems for concave-summing operators between Banach lattices

Let us start by recalling the definitions of convexity/concavity and lower/upper estimates for Banach lattices. The connections among these notions and type/cotype of Banach lattices can be found in [13, 1.d-1.f].

Given a Banach lattice $E$ and a Banach space $X$, an operator $T : E \to X$ is q-concave for $1 \leq q \leq \infty$, if there exists a constant $M < \infty$ so that

$$\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} \leq M \Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|, \qquad \text{if } 1 \leq q < \infty,$$

or

$$\max_{1\leq i\leq n} \|Tx_i\| \leq M \Big\|\bigvee_{i=1}^n |x_i|\Big\|, \qquad \text{if } q = \infty,$$

for every choice of vectors $(x_i)_{i=1}^n$ in $E$. The smallest possible value of $M$ is denoted by $M_{(q)}(T)$.

Similarly, an operator $T : X \to E$ is p-convex for $1 \leq p \leq \infty$, if there exists a constant $M < \infty$ such that

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq M \Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}}, \qquad \text{if } 1 \leq p < \infty,$$

or

$$\Big\|\bigvee_{i=1}^n |Tx_i|\Big\| \leq M \max_{1\leq i\leq n} \|x_i\|, \qquad \text{if } p = \infty,$$

for every choice of vectors $(x_i)_{i=1}^n$ in $X$. The smallest possible value of $M$ is denoted by $M^{(p)}(T)$. Recall that a Banach lattice is $q$-concave (resp. $p$-convex) whenever the identity operator is $q$-concave (resp. $p$-convex).

A Banach lattice $E$ satisfies a lower (resp. upper) p-estimate whenever there exists a constant $M < \infty$ such that

$$\Big\| \sum_{i=1}^n x_i \Big\| \geq M^{-1} \Big( \sum_{i=1}^n \|x_i\|^p \Big)^{\frac{1}{p}} \quad \Big( \text{resp. } \Big\| \sum_{i=1}^n x_i \Big\| \leq M \Big( \sum_{i=1}^n \|x_i\|^p \Big)^{\frac{1}{p}} \Big)$$

for every choice of pairwise disjoint elements $(x_i)_{i=1}^n$ in $E$.

**Definition 2.1.** Given a Banach lattice $E$, a subset $A \subset E$ is called $r$-convex (for $1 \leq r < \infty$) if it is solid and for any $x_1, \ldots, x_n \in A$ and scalars $(a_i)_{i=1}^n$ such that $\sum_{i=1}^n |a_i|^r \leq 1$, we have that $\Big( \sum_{i=1}^n |a_i x_i|^r \Big)^{1/r} \in A$.

Recall that a subset $A$ of a Banach lattice is solid if $x \in A$ whenever $|x| \leq |y|$ and $y \in A$. It is easy to see that if $A \subset E$ is $r$-convex, then it is $s$-convex for every $1 \leq s \leq r$. In particular, every $r$-convex set is convex.

Recall that given a Banach space $X$, the prepolar of a set $A \subset X^*$ is the set

$$A_0 = \{x \in X : \langle \varphi, x \rangle \leq 1 \ \forall \varphi \in A\}.$$

The following fact is based on a standard construction (see for instance [15]).

**Lemma 2.2.** *Given a Banach lattice $E$ and $A \subseteq B_{E^*}$ an $r$-convex set, then the prepolar set $A_0$ generates an $r'$-concave Banach lattice (where $\frac{1}{r} + \frac{1}{r'} = 1$) which we denote by $E(A)$, whose norm is given by the Minkowski functional of $A_0$ and such that there is a (continuous) extended quotient mapping $Q_A : E \to E(A)$ with dense range.*

*Proof.* Let $\phi$ denote the Minkowski functional corresponding to $A_0$. Hence, for $x \in E$ we have

$$\phi(x) = \inf\{\lambda > 0 : x \in \lambda A_0\} = \inf\{\lambda > 0 : \langle \varphi, x \rangle \leq \lambda \ \forall \varphi \in A\}.$$

Since $A$ is solid, it follows that $A_0$ is a convex, balanced and solid subset of $E$ with $0 \in A_0$. Hence, $\phi(\cdot)$ defines a lattice semi-norm on $E$. Now, notice that since $A \subseteq B_{E^*}$, it follows immediately that $B_E = (B_{E^*})_0 \subseteq A_0$, so in particular $\phi(x) \leq \|x\|$ for every $x \in E$.

Now, let $I_\phi = \{x \in E : \phi(x) = 0\}$, which is clearly a closed ideal of $E$. Let $E(A)$ denote the completion of the quotient $E/I_\phi$ endowed with the norm induced by $\phi$. It is a Banach lattice. Moreover, since $\phi(x) \leq \|x\|_E$, the quotient mapping extends to a lattice homomorphism $Q_A : E \to E(A)$, whose range is clearly dense.

Let us see now that $E(A)$ is an $r'$-concave Banach lattice. First notice that by the definition of $I_\phi$, the duality between an element $\varphi$ of $A$ and any element $x + Ker\phi \in E/I_\phi$ is well defined by $\langle \varphi, x + Ker\phi \rangle = \langle x, \varphi \rangle$.

Let us take $\varepsilon > 0$. Given $x_1, \dots, x_n$ in $E_+$, by the definition of $\phi$, there are $\varphi_1, \dots, \varphi_n \in A$ such that $\langle x_i, \varphi_i \rangle \geq \varepsilon/n + \phi(x_i)$. Hence, since $\big(\sum_{i=1}^n |a_i\varphi_i|^r\big)^{1/r} \in A$, by [13, Prop. 1.d.2(iii)], for any positive $(a_i)_{i=1}^n$ with $\sum_{i=1}^n a_i^r \leq 1$, we have

$$\begin{aligned} \sum_{i=1}^n a_i\phi(x_i) &\leq \sum_{i=1}^n a_i\langle x_i, \varphi_i\rangle + \varepsilon \leq \Big\langle \Big(\sum_{i=1}^n |x_i|^{r'}\Big)^{1/r'}, \Big(\sum_{i=1}^n |a_i\varphi_i|^r\Big)^{1/r} \Big\rangle + \varepsilon \\ &\leq \phi\Big(\Big(\sum_{i=1}^n |x_i|^{r'}\Big)^{1/r'}\Big) + \varepsilon. \end{aligned}$$

Therefore, taking suprema over all $(a_i)_{i=1}^n$ with $\sum_{i=1}^n a_i^r \leq 1$, we get

$$\Big(\sum_{i=1}^n \phi(x_i)^{r'}\Big)^{\frac{1}{r'}} \leq \phi\Big(\Big(\sum_{i=1}^n x_i^{r'}\Big)^{\frac{1}{r'}}\Big).$$

Since this inequality holds for all $x_1, \dots, x_n$ in $E_+$ and $Q_A$ is a lattice homomorphism whose image is dense in $E(A)$, this implies that $E(A)$ is $r'$-concave. $\square$

**Example 2.3.** Let $E = L_2(0,1)$ and consider

$$A = \{f \in L_2(0,1) : \|f\|_{L_2} \leq 1,\ f\chi_{[\frac{1}{2},1]} = 0\}.$$

Clearly $A$ is 2-convex in $E$, and the construction of Lemma 2.2 in this case yields that $E(A) = L_2(0, \frac{1}{2})$, and $Q_A : L_2(0,1) \to L_2(0, \frac{1}{2})$ is the corresponding band projection.

**Example 2.4.** Let $E = L_p(0,1)$ and $A = B_{L_{q'}}$ for $q < p$ ($\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$). It follows that $A$ is $q'$-convex in $L_{p'}(0,1)$, and in this case we have $E(A) = L_q(0,1)$ and $Q_A : L_p(0,1) \hookrightarrow L_q(0,1)$ is the formal inclusion.

The following result follows the lines of Theorem 3.1 in [6] (see also [5]). It is a specialized version of this result in which the factorizing operator $R$ is required to be $p$-summing. Recall that if $E(\mu)$ is a Banach function space, we write $E'$ for its Köthe dual and $E^*$ for its dual. Also recall that the $p$-power of a Banach function space $E(\mu)$ (also called $p$-concavification cf. [13]) is the space of elements $E(\mu)_{[p]} = \{f \in L_0(\mu) : |f|^{1/p} \in E(\mu)\}$. This is a quasi-Banach lattice endowed with the quasi-norm $\|f\|_{E(\mu)_{[p]}} = \||f|^{1/p}\|_E^p$, which is equivalent to a norm whenever the space $E(\mu)$ is $p$-convex (see [14] for details).

In the following results we assume for the aim of simplicity that $M^{(p)}(E) = 1$ and $M_{(s)}(F) = 1$; by Proposition 1.d.8 in [13], this is not a restriction on the lattices $E$ and $F$.

**Theorem 2.5.** *Let $1 < s \leq p < \infty$ and $t$ such that $1/s = 1/p + 1/t$. Let $1 \leq r \leq \infty$. Let $E(\mu)$ be a Banach function space and $F(\nu)$ be an $s$-concave Banach function space. Let $T : E(\mu) \to F(\nu)$ be an operator. Then the following are equivalent:*

(1) *There is an $r'$-convex weak* closed set $A \subseteq B_{E^*}$ and a constant $K > 0$ such that for every $x_1, ..., x_n \in E(\mu)$ and $\lambda_1, ..., \lambda_n \in \mathbb{K}$,*
$$\left\| \left( \sum_{k=1}^{n} |T(\lambda_k x_k)|^s \right)^{\frac{1}{s}} \right\|_F \leq K \|(\lambda_k)\|_t \sup_{x^* \in A} \left( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^p \right)^{\frac{1}{p}}.$$

(2) *There exist an $r$-concave Banach function space $E(A)$, a lattice homomorphism $Q_A$ (of norm one), a positive function $g \in B_{M(L_s(\nu),F(\nu))}$ (i.e. defining a multiplication operator) and a $p$-summing operator $R : E(A) \to L_s(\nu)$ such that the following diagram commutes.*

$$\begin{array}{ccc} E(\mu) & \xrightarrow{\quad T \quad} & F(\nu) \\ {\scriptstyle Q_A}\downarrow & & \uparrow{\scriptstyle M_g} \\ E(A) & \xrightarrow{\quad R \quad} & L_s(\nu) \end{array}$$

*Moreover, in the factorization $\pi_p(R) \leq K$.*

*Proof.* Let us prove (1) $\Rightarrow$ (2). It is a direct consequence of Theorem 3.2 in [6], that is based in Theorem 1 in [5]; however, since the assumptions in the definition of Banach function space in these papers are more restrictive than the ones that we assume here (a version of the Fatou property is assumed there) we give a sketch of the proof for showing that this requirement is not needed. For doing this we consider two cases.

(a) *$F$ is $q$-convex for some $q > 1$.* By duality the condition in (1) is equivalent to
$$\left( \sum_{k=1}^{n} |\langle T(\lambda_k x_k), y_k^* \rangle| \right) \leq K \|(\lambda_k)\|_t \sup_{x^* \in A} \left( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^p \right)^{\frac{1}{p}} \left\| \left( \sum_{k=1}^{n} |y_k^*|^{s'} \right)^{1/s'} \right\|_{F^*}$$
for every $x_1, ..., x_n \in E$, $\lambda_1, ..., \lambda_k \in \mathbb{R}$ and $y_1^*, ..., y_k^* \in F^*$. Also by duality, this is equivalent to
$$\left( \sum_{k=1}^{n} |\langle T(x_k), y_k^* \rangle|^{t'} \right)^{\frac{1}{t'}} \leq K \sup_{x^* \in A} \left( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^p \right)^{\frac{1}{p}} \left\| \left( \sum_{k=1}^{n} |y_k^*|^{s'} \right)^{1/s'} \right\|_{F^*}$$
for every $x_1, ..., x_n \in E$ and $y_1^*, ..., y_k^* \in F^*$. Now, notice that the map $x \rightsquigarrow \langle x, \cdot \rangle \in \ell_\infty(A)$ defines an homogeneous representation of $E$ in the $r$-convex lattice $\ell_\infty(A)$, in the sense of [5]. In the same way the map $y^* \rightsquigarrow y^* \in F^*$ is the trivial homogeneous representation of $F^*$ in $F^*$, and so the argument follows the lines of the proof of [5, Theorem 1]; note that since the space $F^*$

is $s'$-convex we obtain that $(F^*)_{[s']}$ is a Banach function space. Moreover, $F^*$ is also $q'$-concave with $q' < \infty$, since $F$ is $q$-convex. This implies that $F^*$ is order continuous (cf. [13, 1.a]). Using all this, we can define the convex family of concave functions $\Phi : \mathcal{P}(A) \times B_{((F^*)_{[s']})^*} \to \mathbb{R}$, where $\mathcal{P}(A)$ is the set of probability measures on the compact set $A$, each $\Phi$ depending on finite families of vectors $x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$, as

$$\Phi(\delta, \psi) := K\frac{t'}{p} \int_A \big( \sum_{k=1}^n |\langle x_k, x^* \rangle|^p \big)\, d\delta + K\frac{t'}{s'} \big( \langle \sum_{k=1}^n |y_k^*|^{s'}, \psi \rangle \big) - \sum_{k=1}^n |\langle T(x_k), y_k^* \rangle|^{t'}.$$

Notice that the Dirac deltas of points of $A$, when considered in the integral, attain the maximum in the expression $\sup_{x^* \in A} \big( \sum_{k=1}^n |\langle x_k, x^* \rangle|^p \big)^{\frac{1}{p}}$ and the norm in $F^*$ is attained by the elements of $B_{((F^*)_{[s']})^*}$. Ky Fan's Lemma gives a probability measure $\eta \in B_{\ell_\infty(A)^*}$ and other positive functional $\varphi \in B_{((F^*)_{[s']})^*}$ such that for all $x \in E$ and $y^* \in F^*$,

$$|\langle T(x), y^* \rangle|^{t'} \le K\frac{t'}{p} ( \int_A |\langle x, x^* \rangle|^p \, d\eta) + K\frac{t'}{s'} \varphi(|y^*|^{s'}).$$

A simple trick using the homogeneity of these expressions (see the end of the proof of Theorem 1 in [5]) gives also

$$|\langle T(x), y^* \rangle| \le K ( \int_A |\langle x, x^* \rangle|^p \, d\eta)^{1/p} \varphi(|y^*|^{s'})^{1/s'}.$$

Now, we take into account that $F$ and $F^*$ are order continuous, as has been already mentioned above. Then clearly $(F^*)_{[s']}$ is order continuous too. Therefore $((F^*)_{[s']})' = ((F^*)_{[s']})^*$ and $M(L_s(\mu), F) = M(F^*, L_{s'}(\mu)) = (((F^*)_{[s']})')_{[1/s']}$ (isometrically).

Thus, we obtain that there is a probability measure $\eta$ and a function $0 \le \omega \in L_0(\mu)$ in $B_{M(L_s(\mu), F)}$ such that

$$|\langle T(x), y^* \rangle| \le K \Big( \int_A |\langle x, x^* \rangle|^p d\eta \Big)^{1/p} \Big( \int |y^*|^{s'} \omega^{s'} d\nu \Big)^{1/s'}$$

for every $x \in E$ and $y^* \in F^*$.

Consequently, we obtain the inequality

$$\Big( \int \big( \frac{|T(x)|}{\omega^{s'}} \big)^s \omega^{s'} d\nu \Big)^{1/s} = \Big( \int \frac{|T(x)|^s}{\omega^s} \, d\nu \Big)^{1/s} \le K \Big( \int_A |\langle x, x^* \rangle|^p d\eta \Big)^{1/p}. \quad (*)$$

(b) *$F$ is not $q$-convex for any $q > 1$.* In this case a convexification procedure must be used, exactly as it is described in the proof of [6, Theorem 3.2]: since $F$ is a Banach function space, the $1/2$-th power $F_{[1/2]}$ of $F$ is 2-convex. Then, changing the multiplication by scalars defined in both spaces $\ell_\infty(A)_{[1/2]r} = \ell_\infty(A)$ and $F_{[1/2]}$, it is possible to define a homogeneous form $u_T : \ell_\infty(A) \times (F_{[1/2]})' \to \mathbb{R}$ such that the arguments in (a) can also be applied (notice that the bilinearity of the map $(x, y^*) \rightsquigarrow \langle T(x), y^* \rangle$ has not been used in the arguments above: only homogenity is needed). Therefore, the inequality (*) obtained in case (a) also holds for this case.

By Pietsch's domination theorem, the inequality (*) can be understood as $p$-summability of the operator $T : E(A) \to L_s(d\nu/\omega^s)$, at least for the elements of $E(A)$ that are the image by $Q_A$ of elements in $E$, for which the operator is defined. But notice that we have shown that the operator $T$ can be considered as taking values in $L_s(d\nu/\omega^s)$, and by Lemma 2.2 the operator $T$ can be extended to $E(A)$, since the image of $E$ by $Q_A$ is dense in $E(A)$. Let us denote this extension by $T_0 : E(A) \to L_s(\mu/\omega^s)$, for which we have $\pi_p(T_0) \leq K$. Now, if we consider the multiplication isometry $M_{1/\omega} : L_s(\mu/\omega^s) \to L_s(\mu)$, then the map $M_{1/\omega} \circ T_0$ is also $p$-summing. Therefore, we have that the composition $R = M_{1/\omega} \circ T_0$ is a $p$-summing operator from $E(A)$ to $L_s(\nu)$ with $\pi_p(R) \leq \|M_{1/\omega}\| \pi_p(T_0) \leq K$. Considering the multiplication operator $M_g$ given by $g = \omega$, this provides the desired factorization.

The converse is given by the following straightforward computations. For every $x_1, ..., x_n \in E(\mu)$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}$,

$$
\begin{aligned}
\Big\| \Big( \sum_{k=1}^n |\lambda_k T(x_k)|^s \Big)^{1/s} \Big\|_F &= \Big\| \Big( \sum_{k=1}^n |\lambda_k M_g(R(Q_A(x_k)))|^s \Big)^{1/s} \Big\|_F \\
&\leq \Big\| g \Big( \sum_{k=1}^n |\lambda_k R(Q_A(x_k))|^s \Big)^{1/s} \Big\|_F \\
&\leq \|M_g\| \Big( \sum_{k=1}^n \int |\lambda_k R(Q_A(x_k))|^s \, d\mu \Big)^{1/s} \\
&\leq \|M_g\| \, \|(\lambda_k)\|_t \Big( \sum_{k=1}^n \|R(Q_A(x_k))\|^p \Big)^{1/p} \\
&\leq \|M_g\| \, \|(\lambda_k)\|_t \, \pi_p(R) \sup_{y^* \in U} \Big( \sum_{k=1}^n |\langle x_k, y^* \rangle|^p \Big)^{1/p}
\end{aligned}
$$

where $U = Q_A^*(B_{E(A)^*})$ is an $r'$-convex weak*-closed set included in $B_{E^*}$. This proves the result. □

**Remark 2.6.** A key observation for the next section, which follows from the previous proof, is the fact that the factorization given in Theorem 2.5 behaves well with respect to the Banach lattices order. That is, if $0 \leq S \leq T : E \to F$ and $T$ satisfies the conditions of the Theorem, then there is a similar factorization for $S$. Namely,

$$\begin{tikzcd} E \arrow[r, "S"] \arrow[d, "Q_A"'] & F \\ E(A) \arrow[r, "P"] & L_s \arrow[u, "M_g"'] \end{tikzcd}$$

where $0 \leq P \leq R : E(A) \to L_s$. Notice that this does not follow directly from Pietsch's Factorization theorem (see also Remark 3.4).

**Corollary 2.7.** *Let $1 < s \leq p < \infty$ and let $t$ be such that $1/s = 1/p + 1/t$. Let $1 \leq r \leq \infty$. Let $E(\mu)$ and $F(\nu)$ be $r$-convex and $s$-concave Banach function spaces, respectively. Suppose also that $E$ is $\sigma$-order continuous and let $T : E(\mu) \to F(\nu)$. Then the following are equivalent:*

(1) *There is an $r'$-convex weak* closed set $A \subseteq B_{E^*}$ such that for every $x_1, ..., x_n \in E(\mu)$ and $\lambda_1, ..., \lambda_n \in \mathbb{K}$,*

$$\Big\| \Big( \sum_{k=1}^{n} |T(\lambda_k x_k)|^s \Big)^{\frac{1}{s}} \Big\|_F \leq \|(\lambda_k)\|_t \sup_{x^* \in A} \Big( \sum_{k=1}^{n} |\langle x_k, x^* \rangle|^p \Big)^{\frac{1}{p}}.$$

(2) *There exist positive functions $f \in M(E(\mu), L_r(\mu))$ and $g \in M(L_s(\nu), F(\nu))$, as well as a $p$-summing operator $R : L_r(\mu) \to L_s(\nu)$ such that*

$$\begin{tikzcd} E(\mu) \arrow[r, "T"] \arrow[d, "M_f"'] & F(\nu) \\ L_r(\mu) \arrow[r, "R"] & L_s(\nu) \arrow[u, "M_g"'] \end{tikzcd}$$

*Moreover, in this case $\pi_p(R) \leq K$.*

*Proof.* Let us see (1) $\Rightarrow$ (2). An application of Theorem 2.5 gives a factorization $T = M_g \circ R_0 \circ Q_A$ where $R_0 : E(A) \to L_s(\nu)$ is $p$-summing. Therefore, the *positive* operator $Q_A$ is defined from an $r$-convex Banach function space into an $r$-concave Banach lattice, so by Krivine's theorem [13, Theorem 1.d.11] $Q_A$ factorizes through a scheme $Q_A = S \circ M_f$, where $M_f : E(\mu) \to L_r(\mu)$ is a multiplication operator and $S : L_r(\mu) \to E(A)$ (see also [5, Corollary 5] or [14, Corollary 6.17]). Notice also that $R = R_0 \circ S$ is $p$-summing, since $R_0$ is. The desired factorization is thus obtained. For the converse, just adapt the final computations in the proof of Theorem 2.5. □

**Remark 2.8.** A simple duality argument shows that in the case that $E$ is not $\sigma$-order continuous the result remains valid whenever $T'(F'(\nu)) \subset E'(\mu)$.

**Remark 2.9.** Notice that a simple argument similar to the one that proves Theorem 2.5 gives the equivalence between the following statements for an operator $T$ between Banach function spaces $E(\mu)$ and $F(\nu)$.

(1) There exist an $r$-convex set $A \subseteq B_{E^*}$ and a constant $K > 0$ such that for every $x_1, \cdots, x_n \in E$,
$$\Big(\sum_{k=1}^n \|T(x_k)\|^p\Big)^{1/p} \le K \sup_{\varphi\in A} \Big(\sum_{k=1}^n |\langle x_k, \varphi\rangle|^p\Big)^{1/p}.$$
(2) There is an $r$-concave Banach lattice $E(A)$, a (norm one) lattice homomorphism $Q_A : E \to E(A)$ with dense range and a $p$-summing map $R$ such that the following diagram commutes.

$$\begin{tikzcd} E(\mu) \arrow[rr, "T"] \arrow[dr, "Q_A"'] & & F(\nu) \\ & E(A) \arrow[ur, "R"'] & \end{tikzcd}$$

Moreover, if this holds, $\pi_p(R) \le K$. Note that $r$-convexity/$s$-concavity conditions for $E$ or $F$ respectively are not needed in this case.

## 3. Dodds-Fremlin domination for $p$-summing operators

As an application of the factorization results given in the previous section, we present several results regarding the domination problem for $p$-summing operators. We are interested in finding out conditions on Banach lattices $E$ and $F$ such that whenever $0 \le R \le T : E \to F$ and $T$ is $p$-summing, then $R$ is also $p$-summing. Precisely, if $\pi_p(T)$ denotes the $p$-summing norm of $T$, then we would like to know whether or not there exists a constant $C < \infty$ such that $\pi_p(R) \le C\pi_p(T)$ whenever $0 \le R \le T : E \to F$. In the next section, we present some remarks concerning the constant $C$ involved, in particular we show that this constant cannot be avoided (i.e. $C = 1$) even in the simplest finite dimensional cases.

Notice that this problem is not trivial in general. Indeed, [9, Examples 3.12 and 3.14] provide positive operators which are $p$-summing and dominate operators which are not strictly singular. A direct application of Dvoretzky-Rogers Theorem (cf. [7, p. 2]) shows these dominated operators are not $p$-summing as well.

Recall that a sequence $(x_n)$ in a Banach space $X$ is weakly $p$-summable if $\sup_{x^*\in B_{X^*}} \big(\sum_n |\langle x^*, x_n\rangle|^p\big)^{\frac{1}{p}}$ is bounded, equivalently $x_n = T(e_n)$ for some $T : \ell_{p'} \to X$ (where $(e_n)$ is the unit vector basis of $\ell_{p'}$ and $\frac{1}{p} + \frac{1}{p'} = 1$). Similarly, $(x_n)$ is called strongly $p$-summable when $\big(\sum_n \|x_n\|^p\big)^{\frac{1}{p}}$ converges. Hence, an operator $T : E \to F$ is $p$-summing if it maps weakly $p$-summable sequences into strongly $p$-summable. The main obstruction that avoids a general domination result for $p$-summing operators stems from the fact that weak summability is not a lattice property. Namely, the sequence $(|x_n|)$ need not be weakly $p$-summing although $(x_n)$ was. This is just because an operator $T : \ell_{p'} \to E$ need not have a bounded modulus (cf. [1]).

There is a connection with the restricted class of "positive $p$-summing" operators, introduced in [3], which are exactly those for which the aforementioned problem is no longer an obstruction. In particular, a domination theorem holds trivially for this class. Notice that this terminology might be misleading: a $p$-summing operator which is positive need not be a "positive $p$-summing" operator.

Our analysis mainly focuses on the class of 2-summing operators, whereas many of the results given here can be easily extended to the general case of $p$-summing operators.

In order to motivate the first positive results, we consider operators on a Hilbert space. In finite dimension, an operator $T : \ell_2^n \to \ell_2^n$ can be considered as an $n \times n$ matrix $(a_{ij})_{i,j=1}^n$. Clearly, $\ell_2^n$ with the coordinate-wise ordering becomes a Banach lattice, where two operators $T = (a_{ij})$ and $R = (b_{ij})$ satisfy $0 \leq R \leq T$ whenever $0 \leq b_{ij} \leq a_{ij}$ for every $i, j = 1, \ldots, n$. Notice that for operators on Hilbert space, the classes of 2-summing operators and Hilbert-Schmidt operators coincide. Moreover

$$\pi_2(a_{ij}) = \Big( \sum_{i=1}^n \sum_{j=1}^n a_{ij}^2 \Big)^{\frac{1}{2}},$$

Hence, it is clear that $\pi_2(R) \leq \pi_2(T)$ whenever $0 \leq R \leq T : \ell_2^n \to \ell_2^n$. Note that the same proof works for operators on the infinite dimensional $\ell_2$. We present below a more general argument for operators into a general Hilbert space (see Proposition 3.2).

However, there exist other simple cases in which a domination theorem holds. Recall that a Banach lattice $E$ is an AM-space if $\|x \vee y\| = \|x\| \vee \|y\|$ for any $x, y \in E_+$. Typically, these spaces are of the form $C(K)$ or $L_\infty(\mu)$.

**Proposition 3.1.** *Let $0 \leq R \leq T : E \to F$ be positive operators from an AM-space $E$ to Banach lattice $F$. If $T$ is $p$-summing for some $1 \leq p < \infty$, then $R$ is $p$-summing.*

*Proof.* By [13, Theorem 1.d.10], every positive operator from a $C(K)$ is $p$-summing if and only if it is $p$-concave. The result follows from the fact that

$$\Big( \sum_{k=1}^n \|R(x_k)\|^p \Big)^{\frac{1}{p}} \leq \Big( \sum_{k=1}^n \|T(|x_k|)\|^p \Big)^{\frac{1}{p}} \leq M_{(p)}(T) \Big\| \Big( \sum_{k=1}^n |x_k|^p \Big)^{\frac{1}{p}} \Big\|.$$

□

The following proposition will be useful.

**Proposition 3.2.** *Let $0 \leq R \leq T : E \to L_2(\mu)$. If $T$ is absolutely summing, then so is $R$. Moreover, $\pi_1(R) \leq C\pi_1(T)$.*

*Proof.* If $T$ is absolutely summing, then it is 1-concave (see [13, p. 56]). In particular, $R$ is also 1-concave, and by Krivine's theorem [13, Thm. 1.d.11] $R$ can be factored through an $L_1$ space. Thus, by Grothendieck's theorem (cf. [7, Theorem 3.4]) $R$ is absolutely summing. $\square$

As we will see in Section 4, we cannot remove the constant $C$ in Proposition 3.2. This can be shown even considering operators from $\ell_1^2$ to $\ell_2^2$.

We present next our main result on domination. Recall that a Banach lattice has cotype 2 if and only if it is 2-concave. In a certain sense, Theorem 2.5 allows us to reduce to the case of operators into $L_2(\mu)$, and then apply Proposition 3.2.

**Theorem 3.3.** *Let $1 \leq p < \infty$. Given Banach lattices $E$ and $F$ with cotype 2 and $0 \leq S \leq T : E \to F$, if $T$ is a $p$-summing operator, then $S$ is also $p$-summing with $\pi_p(S) \leq C_p\pi_p(T)$. Where here $C_p$ is a universal constant (depending only on $p$ and the cotype constants of $E$ and $F$).*

*Proof.* First of all, since both $E$ and $F$ have cotype 2, it follows that for every $1 < p < \infty$, the class of $p$-summing operators coincides with that of absolutely summing operators (cf. [7, Corollary 11.16]). In particular, $T$ is 2-summing.

Now, using Pietsch Domination Theorem (cf. [7, 2.12]), together with Khintchine's Inequality, we can deduce that for every $x_1, \cdots, x_n \in E$, the following inequality holds

$$\Big(\int_0^1 \Big\|\sum_{i=1}^n r_i(t)T(x_i)\Big\|^2 dt\Big)^{\frac{1}{2}} \leq C' \sup\Big\{\Big(\sum_{i=1}^n |\langle x^*, x_i\rangle|^2\Big)^{\frac{1}{2}} : x^* \in B_{E^*}\Big\}.$$

Actually, $C'$ can be taken equal to $= B_2\pi_2(T)$, where $B_2$ is the constant appearing in Khintchine's inequality for $L_2$ (cf. [7, 12.5]).

Furthermore, since $F$ is a 2-concave Banach lattice, Maurey-Khinchine's inequality (cf. [13, Theorem 1.d.6]) yields that for every $x_1, \cdots, x_n \in E$, we have

$$\Big\|\Big(\sum_{i=1}^n |T(x_i)|^2\Big)^{\frac{1}{2}}\Big\| \leq C'' \sup\Big\{\Big(\sum_{i=1}^n |\langle x^*, x_i\rangle|^2\Big)^{\frac{1}{2}} : x^* \in B_{E^*}\Big\}.$$

In particular, this means that for every $x_1, \ldots, x_n$ in $E$, and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ we have

$$\Big\|\Big(\sum_{i=1}^n |T(\lambda_i x_i)|^2)^{\frac{1}{2}}\Big\|_F \leq \|(\lambda_i)\|_\infty \sup_{x^* \in A}\Big(\sum_{i=1}^n |\langle x^*, x_i\rangle|^2\Big)^{\frac{1}{2}},$$

where $A = B_{E^*}$ is a 2-convex set because $E$ is 2-concave. Therefore, Theorem 2.5 with $s = r = 2$ (and $t = \infty$) implies that we can factor $T$ in the following way

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "R"'] & & F \\ & L_2(\mu) \arrow[ur, "M_g"'] & \end{tikzcd}$$

where $R : E \to L_2(\mu)$ is 2-summing and $M_g : L_2(\mu) \to F$ is a multiplication operator for some $g \in M(L_2(\mu), F)$. Moreover, it can be seen from the proof of Theorem 2.5 that if $0 \le S \le T$, then $S$ also factors as

$$\begin{tikzcd} E \arrow[rr, "S"] \arrow[dr, "U"'] & & F \\ & L_2(\mu) \arrow[ur, "M_g"'] & \end{tikzcd}$$

with $0 \le U \le R : E \to L_2(\mu)$.

Now, since $R : E \to L_2(\mu)$ is absolutely summing (cf. [7, Corollary 11.16]), Proposition 3.2 implies that $U : E \to L_2(\mu)$ is absolutely summing. In particular, we get that $S$ is $p$-summing (for $1 \le p < \infty$) and the proof is finished. □

**Remark 3.4.** An important step in the previous proof was the fact that a positive operator $T : E \to F$ (where $E$ and $F$ have cotype 2 and $F'$ is order continuous) which is 2-summing can be factored as

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "R"'] & & F \\ & L_2(\mu) \arrow[ur, "M_g"'] & \end{tikzcd}$$

where $R : X \to L_2(\mu)$ is a positive 2-summing operator and $g$ is a positive function. This fact, which is a consequence of Theorem 2.5, is still true without any condition on $E$.

It is worth noting here that this factorization cannot be obtained as a particular case of the canonical factorization for 2-summing operators given by Pietsch's Theorem, since this factorization cannot be expected to respect positivity (cf. [7]). Notice that this would imply, in particular, that any positive 2-summing operator factorizes through an $L_\infty(\mu)$ by positive operators. We provide an easy counterexample showing this cannot happen in general.

**Example 3.5.** Given $n \in \mathbb{N}$, consider the identity operator $id_n : \ell_1^n \to \ell_1^n$. It is well known that $\pi_2(id_n) \leq \sqrt{n}$ (see [4]). On the other hand, it is easy to see that $\iota(id_n) = n$, where $\iota$ denotes the integral norm of the operator. Now, it can be proved that any positive operator $T : L_\infty(\mu) \to L_1(\mu)$ is integral with $\|T\| = \iota(T)$ (see [4]). This fact, together with the ideal property of integral operators, tells us that $\|u\|\|v\| \geq n$ for every pair of positive operators $u : \ell_1^n \to L_\infty(\mu)$, $v : L_\infty(\mu) \to \ell_1^n$ such that $id = v \circ u$. A standard argument from local theory shows that there exist 2-summing positive operators from $L_1(\nu)$ into $L_1(\nu)$ which do not factorize through any $L_\infty(\mu)$ by positive operators.

The following result tells us that we cannot expect a positive answer to the domination problem when the range space has cotype greater than 2. Recall that a Banach lattice $E$ satisfies a lower 2-estimate whenever there is a constant $M < \infty$ such that, for every choice of pairwise disjoint elements $(x_i)_{i=1}^n$ in $E$, we have

$$\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}} \leq M\Big\|\sum_{i=1}^n x_i\Big\|.$$

It is well-known that a Banach lattice $E$ which has cotype 2 must also satisfy a lower 2-estimate, and if $E$ satisfies a lower 2-estimate, then $E$ has cotype $q$ for every $q > 2$ [13]. However, there exist Banach lattices which satisfy a lower 2-estimate but are not of cotype 2 (cf. [13, Example 1.f.19]).

**Proposition 3.6.** *Let $E$ be a Banach lattice with finite cotype. Suppose that for some constant $C < \infty$, the 2-summing operators from $E$ to a Banach lattice $F$ satisfy $\pi_2(R) \leq C\pi_2(T)$ whenever $0 \leq R \leq T : E \to F$. Then $F$ satisfies a lower 2-estimate.*

*Proof.* We proceed by contradiction. Suppose that $F$ does not satisfy a lower 2-estimate. Hence, for every $N \in \mathbb{N}$ there exist $x_1^N, \dots, x_m^N$ disjoint in $F_+$ such that

$$\Big(\sum_{i=1}^m \|x_i^N\|^2\Big)^{\frac{1}{2}} > N\Big\|\sum_{i=1}^m x_i^N\Big\|.$$

Now, since $E$ has finite cotype, it can be represented as a Banach lattice of measurable functions on some $(\Omega, \Sigma, \mu)$ such that, for some $p < \infty$, the inclusions $L_p(\mu) \hookrightarrow E \hookrightarrow L_1(\mu)$ are bounded, say with norm one (cf. [11, p.14]). We can therefore consider a family $(r_k)$ of Rademacher functions on $\Omega$ which are weakly 2-summable with $\sup\{\sum_{k=1}^n \langle x^*, r_k\rangle^2 : x^* \in B_{L_p}\} \leq B_p$, where $B_p$ is the constant appearing in Khintchine's inequality for $L_p(\mu)$ (hence independent of $n$).

Let us now define for $N \in \mathbb{N}$ the operator $T_N : E \to F$ given by

$$T_N(f) = \int f d\mu \sum_{i=1}^{m} x_i^N,$$

and $R_N : E \to F$ given by

$$R_N(f) = \sum_{i=1}^{m} \Big( \int f r_i^+ d\mu \Big) x_i^N.$$

Clearly, these operators satisfy $0 \leq R_N \leq T_N$. Moreover, since $T_N$ is a rank one operator it holds that $\pi_2(T_N) = \|T_N\| \leq \Big\| \sum_{i=1}^m x_i^N \Big\|$. Meanwhile, since the Rademacher functions are weakly 2-summable in $E$, we have

$$\begin{aligned} \pi_2(R_N) &\geq \tfrac{1}{B_p} \Big( \sum_{k=1}^m \|R_N(r_k)\|^2 \Big)^{\frac{1}{2}} \\ &= \tfrac{1}{B_p} \Big( \sum_{k=1}^m \| \sum_{j=1}^m \int r_k r_j^+ d\mu x_j^N \|^2 \Big)^{\frac{1}{2}} \\ &= \tfrac{1}{B_p} \Big( \sum_{k=1}^m \| \tfrac{1}{2} x_k^N \|^2 \Big)^{\frac{1}{2}} \\ &> \tfrac{1}{2B_p} N \Big\| \sum_{k=1}^m x_k^N \Big\|. \end{aligned}$$

Since this holds for every $N \in \mathbb{N}$ there cannot be a constant $C < \infty$ such that $\pi_2(R) \leq C\pi_2(T)$ holds whenever $0 \leq R \leq T : E \to F$. Hence, we have reached a contradiction. □

**Remark 3.7.** Notice that, according to Proposition 3.1, the hypothesis of finite cotype in the previous construction cannot be removed.

A particular case of the proof of Theorem 3.3 together with Proposition 3.6 actually yield the following characterization of Banach lattices satisfying a lower 2-estimate.

**Theorem 3.8.** *Let $F$ be a Banach lattice. Then $F$ satisfies a lower 2-estimate if and only if for every 2-concave Banach lattice $E$, domination holds for the ideal $\Pi_2(E, F)$ of 2-summing operators.*

*Proof.* Suppose $F$ satisfies a lower 2-estimate, and let $0 \leq R \leq T : E \to F$ be such that $\pi_2(T) < \infty$. Hence, for every $x_1, \cdots, x_n \in E$, we have

$$\Big( \sum_{i=1}^n \|T(x_i)\|^2 \Big)^{\frac{1}{2}} \leq \pi_2(T) \sup \Big\{ \Big( \sum_{i=1}^n |\langle x^*, x_i \rangle|^2 \Big)^{\frac{1}{2}} : x^* \in B_{E^*} \Big\}.$$

As in the proof of Theorem 3.3, Pietsch Domination Theorem, together with Khinchine's Inequality, imply that

$$\Big( \int_0^1 \Big\| \sum_{i=1}^n r_i(t) T(x_i) \Big\|^2 dt \Big)^{\frac{1}{2}} \leq B_2 \pi_2(T) \sup \Big\{ \Big( \sum_{i=1}^n |\langle x^*, x_i \rangle|^2 \Big)^{\frac{1}{2}} : x^* \in B_{E^*} \Big\}.$$

Now, since $F$ satisfies a lower 2-estimate, in particular it is $q$-concave for every $q > 2$, so we can also use Maurey-Khinchine's inequality ([13, Theorem 1.d.6]) and we get

$$\Big\|\Big(\sum_{i=1}^{n}|T(x_i)|^2\Big)^{\frac{1}{2}}\Big\| \leq C\sup\Big\{\Big(\sum_{i=1}^{n}|\langle x^*, x_i\rangle|^2\Big)^{\frac{1}{2}} : x^* \in B_{E^*}\Big\}$$

for every $x_1, \cdots, x_n \in E$, and some constant $C < \infty$. Since $E$ is 2-concave, the set $B_{E^*}$ is clearly 2-convex, and we can apply Theorem 2.5. The rest of the argument follows the one given in the proof of Theorem 3.3.

The converse implication follows directly from Proposition 3.6. □

**Question 3.9.** What can we say about the case of operators from a Banach lattice without cotype 2? Is there a domination theorem if the range space has cotype 2? Note that there is a domination theorem for absolutely summing operators, so we cannot expect a counterexample in the form of 3.6.

## 4. A remark on the constant involved in the domination theorem

The aim of this section is to show that we can not remove the constant appearing in Theorem 3.3. We will show that this constant is necessary even in the simplest cases. In the first example, we will show the existence of two positive operators $0 \leq S \leq T : \ell_1^2 \to \ell_2^2$ such that $\pi_1(S) > \pi_1(T)$. Note that these spaces satisfy the conditions of Theorem 3.3. Furthermore, the domination is trivial here because of the Grothendieck's Theorem. It is also interesting to note that we can not expect a similar example for the 2-summing norm on these spaces. This is a consequence of the non trivial fact that every operator $T$ from $\ell_1^2$ into $\ell_2^2$ satisfies $\|T\| = \pi_2(T)$ (see [2]) and the easy fact that the operator norm is monotone with respect to domination. However, we will show that if we consider $\ell_1^3$, there exist operators $0 \leq S \leq T : \ell_1^3 \to \ell_2^2$ such that $\pi_2(S) > \pi_2(T)$.

We begin with the example for the 1-summing norm.

**Example 4.1.** Let $0 < \varepsilon < \varepsilon_0$ and consider the operator $T_\varepsilon : \ell_1^2 \to \ell_2^2$ defined by

$$T_\varepsilon(e_1) = e_1 + \varepsilon e_2$$
$$T_\varepsilon(e_2) = \tfrac{1}{\sqrt{2}}(e_1 + e_2).$$

It is clear that $0 \leq T_\varepsilon \leq T_{\varepsilon'}$ whenever $0 \leq \varepsilon \leq \varepsilon'$. We will see, however, that $\pi_1(T_\varepsilon)$ is not an increasing function of $\varepsilon$. Indeed, recall that the application $A : \ell_\infty^2 \to \ell_1^2$ defined by

$$A(e_1) = \tfrac{1}{2}(e_1 + e_2)$$
$$A(e_2) = \tfrac{1}{2}(e_1 - e_2)$$

is a linear isometry. Thus, by the injectivity of the ideal of 1-summing operators, it suffices to compute the 1-summing norm, or equivalently, the integral norm (cf. [4]) of the operator $\tilde{T}_\varepsilon : \ell_\infty^2 \to \ell_2^2$ defined by

$$\begin{aligned}\tilde{T}_\varepsilon(e_1) &= \tfrac{1}{2}((1+\tfrac{1}{\sqrt{2}})e_1 + (\varepsilon + \tfrac{1}{\sqrt{2}})e_2)\\ \tilde{T}_\varepsilon(e_2) &= \tfrac{1}{2}((1-\tfrac{1}{\sqrt{2}})e_1 + (\varepsilon - \tfrac{1}{\sqrt{2}})e_2).\end{aligned}$$

An easy computation shows that

$$\pi_1(T_\varepsilon) = \frac{1}{2}[(2+\varepsilon^2+\sqrt{2}(1+\varepsilon))^{\frac{1}{2}} + (2+\varepsilon^2-\sqrt{2}(1+\varepsilon))^{\frac{1}{2}}].$$

It is easy to see that this function is decreasing in a certain interval $[0,\xi)$, for some $\xi > 0$. Namely, the function $\pi_1(T_\varepsilon)$ has the following form (see Figure 4.1).

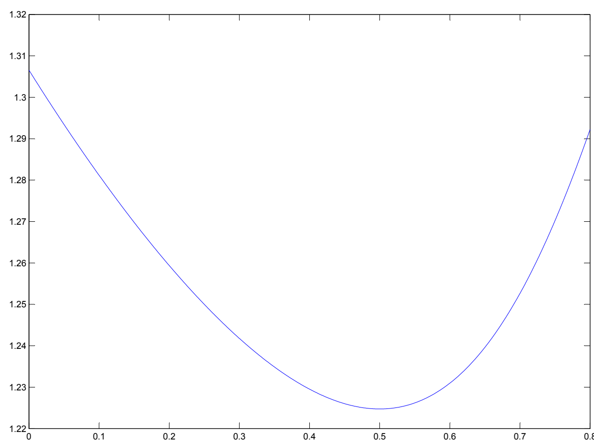


FIGURE 1. $\pi_1(T_\varepsilon)$ as a function of $\varepsilon$

Next, we show an example for the 2-summing norm:

**Example 4.2.** Consider now the operator $T_\varepsilon : \ell_1^3 \to \ell_2^2$ defined by

$$\begin{aligned}T_\varepsilon(e_1) &= e_1 + \varepsilon e_2\\ T_\varepsilon(e_2) &= \tfrac{1}{\sqrt{2}}(e_1+e_2)\\ T_\varepsilon(e_3) &= \varepsilon e_1 + e_2.\end{aligned}$$

Due to the simplicity of the operator we are able to compute the exact value of $\pi_2(T_\varepsilon)$. Obviously, we have $T_\varepsilon \geq T_{\varepsilon'}$ if $\varepsilon \geq \varepsilon'$. We will see that $\pi_2(T_\varepsilon)$ does not respect this order.

It is well known (see for instance [16, Proposition 9.7]) that the 2-summing norm of this operator can be obtained as

$$\pi_2(T) = \sup\{\pi_2(Tu) : u : \ell_2^2 \to \ell_1^3, \|u\| \leq 1\}.$$

We will calculate $\pi_2(T_\varepsilon)^2$ just to avoid the square root. Then, we have to solve the following problem:

$$\max\{(x_1+\frac{1}{\sqrt{2}}x_2+\varepsilon x_3)^2+(\varepsilon x_1+\frac{1}{\sqrt{2}}x_2+x_3)^2+(y_1+\frac{1}{\sqrt{2}}y_2+\varepsilon y_3)^2+(\varepsilon y_1+\frac{1}{\sqrt{2}}y_2+y_3)^2\},$$

subject to the restrictions

$$\begin{array}{c} (x_1+x_2+x_3)^2+(y_1+y_2+y_3)^2 \leq 1, \\ (x_1+x_2-x_3)^2+(y_1+y_2-y_3)^2 \leq 1, \\ (x_1-x_2+x_3)^2+(y_1-y_2+y_3)^2 \leq 1, \\ (-x_1+x_2+x_3)^2+(-y_1+y_2+y_3)^2 \leq 1. \end{array}$$

Now, by mean of several changes of variable and because of the "simplicity" of the geometry of the problem, the previous optimization problem can be reduced to the following one:

$$\max_{0\leq u\leq 1} f_\varepsilon(u) = \frac{1}{2}\{[C(\varepsilon)\sqrt{u}+(1+\varepsilon)]^2+(1-u)D(\varepsilon)^2\},$$

where $C(\varepsilon) = \sqrt{2}-(1+\varepsilon)$ and $D(\varepsilon) = 1-\varepsilon$ (we save the reader against the tedious calculations).

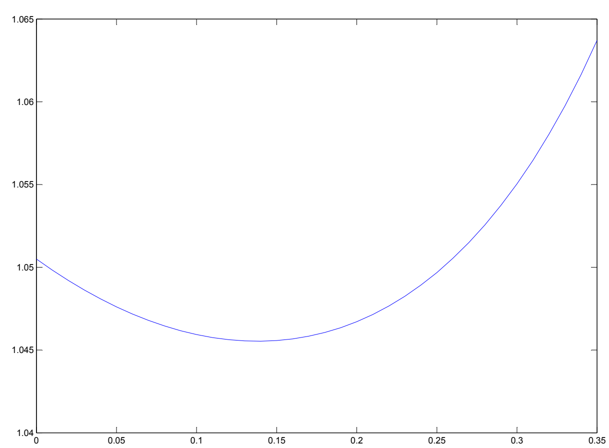


FIGURE 2. $\pi_2(T_\varepsilon)$ as a function of $\varepsilon$

Now it is easy to see that this function is decreasing in a certain $[0,\xi)$, for some $\xi > 0$. Actually, we can represent the function $\varepsilon \mapsto \sqrt{f_\varepsilon(u_\varepsilon)}$ or, which is the same

$$\varepsilon \mapsto \pi_2(T_\varepsilon),$$

in an interval $[0,\xi)$ (see Figure 4.2).

**Acknowledgements** First author partially supported by Spanish grants I-MATH, MTM2008-01366 and CCG08- UCM/ESP-4394. Second author supported by the Ministerio de Educación y Ciencia, under project MTM2009-14483-C02-02 (Spain), and FEDER. Third author partially supported by Spanish MICINN through Juan de la Cierva program and grant MTM2008-02652, Santander/Complutense PR34/07-15837 and Generalitat Valenciana grant Prometeo/2008/101.

## References

[1] C. D. Aliprantis and O. Burkinshaw, Positive Operators. Springer, Dordrecht, 2006.

[2] A. Arias, T. Figiel, W. B. Johnson and G. Schechtman, Banach spaces with the 2-summing property, Trans. Amer. Math. Soc. 347 (1995), 3835–3858.

[3] O. Blasco, Positive $p$-summing operators on $L_p$-spaces, Proc. Amer. Math. Soc., 100 2 (1987) 275–280.

[4] A. Defant and K. Floret, Tensor Norms and Operator Ideals. North-Holland, 1993.

[5] A. Defant, Variants of the Maurey-Rosenthal Theorem for quasi-Köthe function spaces, Positivity 5 (2001) 153–175.

[6] A. Defant and E.A. Sánchez Pérez, Maurey-Rosenthal factorizations of positive operators and convexity. J. Math. Anal. Appl. 297 (2004) 771-790.

[7] J. Diestel, H. Jarchow and A. Tonge, Absolutely Summing Operators, Cambridge Studies in Advanced Mathematics 43, Cambridge, 1995.

[8] P. G. Dodds, and D. H. Fremlin, Compact operators in Banach lattices. Israel J. Math. 34 (1979), 287–320.

[9] J. Flores, F. L. Hernández, Domination by positive strictly singular operators. J. London Math. Soc. (2) 66, (2002) 433–452.

[10] J. Flores, F. L. Hernández, P. Tradacete, Powers of operators dominated by strictly singular operators, Q. J. Math. 59 (2008), 321–334.

[11] W. B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, Symmetric structures in Banach spaces. Mem. Amer. Math. Soc. 217, (1979).

[12] N. Kalton, P. Saab, Ideal properties of regular operators between Banach lattices. Illinois J. Math 29, (1985) 382-400.

[13] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II, Springer, Berlin, 1979.

[14] S. Okada, W. J. Ricker and E. A. Sánchez Pérez, Optimal Domain and Integral Extension of Operators acting in Function Spaces, Operator Theory: Adv. Appl., vol. 180, Birkhäuser, Basel, 2008.

[15] Y. Raynaud and P. Tradacete, Interpolation of Banach lattices and factorization of p-convex and q-concave operators, Integral Equations Operator Theory 66 (2010), 79–112.

[16] N. Tomczak-Jaegermann, Banach-Mazur distances and finite-dimensional operator ideals, Longmann, (1988).

[17] A. W. Wickstead, Extremal structure of cones of operators. Quart. J. Math. Oxford Ser (2) 32, (1981) 239–253.

Carlos Palazuelos, Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040 Madrid. Spain.
*E-mail address*: carlospalazuelos@mat.ucm.es

Enrique A. Sánchez Pérez, Instituto Universitario de Matemática Pura y Aplicada, Universidad Politécnica de Valencia, 46071 Valencia. Spain.
*E-mail address*: easancpe@mat.upv.es

Pedro Tradacete, Departamento de Matemática Aplicada y Análisis, Universidad de Barcelona, 08007 Barcelona. Spain.
*E-mail address*: tradacete@ub.edu