STRICTLY SINGULAR AND POWER-COMPACT OPERATORS ON BANACH LATTICES

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Abstract. Compactness of the iterates of strictly singular operators on Banach lattices is analyzed. We provide suitable conditions on the behavior of disjoint sequences in a Banach lattice, for strictly singular operators to be Dunford-Pettis, compact or have compact square. Special emphasis is given to the class of rearrangement invariant function spaces (in particular, Orlicz and Lorentz spaces). Moreover, examples of rearrangement invariant function spaces of fixed arbitrary indices with strictly singular non power-compact operators are also presented.

1. Introduction

A classical result of V. Milman [26] states that every strictly singular endomorphism on an L_p-space has a compact square. This fact provides a closer connection on L_p-spaces between strictly singular and compact operators, which have in general a very different behavior. Recall that an operator between Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite dimensional subspace. This class forms a closed operator ideal that contains the compact operators and was introduced in connection with the perturbation theory of Fredholm operators [19]. In particular, the sum of a strictly singular operator and a Fredholm operator is again Fredholm with the same index (cf. [22]), and as a consequence, the spectra of strictly singular operators resembles that of compact operators. However, notice that, unlike compact operators, strictly singular operators are not stable under duality (cf. [27], [34]) and fail to have invariant subspaces ([29]).

The aim of this paper is to study extensions of Milman's result on L_p-spaces to wider classes of Banach lattices. Hence, we are looking for conditions on a Banach lattice ensuring that the square (or higher powers) of a strictly singular operator to be compact. Moreover, we also present conditions which imply that the class of strictly singular operators coincides with that of compact or Dunford-Pettis operators. Our main applications will focus on rearrangement invariant function spaces, especially classical Lorentz and Orlicz spaces.

The approach here is mainly based on the analysis of disjoint sequences in a Banach lattice arising from Kadeč-Pełczyński's dichotomy; according to this every subspace of an order continuous Banach lattice is either strongly embedded in some $L_1(\mu)$ or has an almost disjoint sequence (cf. [10]). This and other related facts have consistently proved the importance of disjoint sequences for understanding the geometry of Banach lattices and the operators on them (cf. [7], [9]). In particular, under some

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natural assumptions, it can be seen that an operator on a Banach lattice which is not strictly singular must be invertible on the span of some disjoint sequence or on a Hilbert subspace (see [9]).

A key ingredient here is the notion of *disjointly homogeneous* Banach lattice, introduced recently in [8]. This class of Banach lattices is defined by the property that every couple of disjoint normalized sequences share an equivalent subsequence. In particular, this class contains $L_p(\mu)$-spaces ($1 \leq p \leq \infty$), Lorentz function spaces ($L_{p,q}$ and $\Lambda(W,q)$), certain classes of Orlicz function spaces and also some discrete spaces such as Tsirelson space.

Let us mention that a different extension of Milman's result has been recently studied in [2], where the authors introduce the classes of Schreier S_ξ-singular operators. These are particular kinds of strictly singular operators which fill the existing gap between the ideal of compact and strictly singular operators. Thus, it is proved that on a Banach space with a finite number of non-equivalent classes of Schreier-spreading sequences, the composition of Schreier S_ξ-singular operators is compact (see also [28]). However, this is a rather restricted class of spaces which does not include, for instance, $L_p(\mu)$-spaces with $1 \leq p < 2$, so the corresponding results for such spaces require also some duality arguments. Our approach here does not rely on these techniques nor stability under duality for strictly singular operators on similar Banach lattices.

The paper is structured as follows. In Section 2, general results are given for the square of strictly singular operators defined on disjointly homogeneous Banach lattices with finite cotype to be compact (see Theorem 2.9). A partial result in this direction has been obtained in [8] for the class of regular operators. Moreover, the interesting case of p-disjointly homogeneous Banach lattices is also studied, i.e. those where every disjoint normalized sequence has a subsequence equivalent to the unit vector basis of ℓ_p (see Theorem 2.11). In particular, for 1-disjointly homogeneous Banach lattices we show that every strictly singular operator is Dunford-Pettis while in the case of 2-disjointly homogeneous, every strictly singular operator is already compact. Thus, this well-known fact for Hilbert spaces (due to Kato) extends to this Banach lattice class (f.i. for Orlicz spaces $L^{x^2 log^\alpha(1+x)}$). This same property holds also for *discrete* disjointly homogenous Banach lattices, f.i. Tsirelson type spaces (see Theorem 2.13).

In Section 3 we give applications to *Lorentz function spaces* $\Lambda(W,q)$ and $L_{p,q}$. These spaces are disjointly homogeneous and thus the square of strictly singular operators on them are compact. We also consider the spaces $L_{p,\infty}$ and the order continuous part $L^o_{p,\infty}$. While a similar result for $L^o_{p,\infty}$ spaces also holds, we show that this is not the case for $L_{p,\infty}$ spaces where the connection between strict singularity and compactness of iterations is more elusive. Section 4 is devoted to *Orlicz function spaces.* First, we provide a characterization of when Orlicz function spaces L^φ are disjointly homogeneous: namely, the associated set E^∞_φ, in the sense of [21], has to be the function $\{t^p\}$, for some $1 \leq p < \infty$, up to equivalence. Notice that, this holds for all regular Orlicz functions φ satisfying $\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p$. The condition $E^\infty_\varphi \cong \{t^p\}$ implies, in particular, that the upper and lower indices of the Orlicz space must coincide. However, a more general result for this bigger class does not hold: we provide examples of *minimal* Orlicz function spaces L^φ (in the sense of [22], see also [13]), with equal indices, on which strictly singular operators with non-compact squares exist. Moreover, using the existence of different ℓ_p-complemented copies, we also provide examples of Orlicz spaces with different lower and upper indices where strictly singular non power-compact operators exist (see Proposition 4.5).

Finally, Section 5 deals with strictly singular non power-compact operators on general rearrangement invariant spaces having equal indices. Using real interpolation methods we show two different constructions of rearrangement invariant function spaces E, one with equal Boyd indices and the other one with equal lattice indices, such that strictly singular non power-compact operators exist on E (see Propositions 5.3 and 5.4). The latter example is based on scales of Tsirelson type spaces instead of different ℓ_p-complemented copies. We thank prof. N. J. Kalton for his helpful remarks regarding these questions.

2. Compact squares of strictly singular operators

Let us first fix some terminology used in the sequel. By an operator we always mean a bounded linear operator. Given a Banach space E, we will denote by $\mathcal{S}(E)$ (respectively $\mathcal{K}(E)$) the space of all strictly singular (resp. compact) endomorphisms on E. Two elements x, y in a Banach lattice E are said to be disjoint whenever $|x| \wedge |y| = 0$, and a sequence (x_n) is called disjoint whenever its elements are pairwise disjoint.

Recall that every order continuous Banach lattice E with a weak unit can be represented as a Banach lattice of functions over some probability space (Ω, Σ, μ) in such a way that the formal inclusions $L_\infty(\mu) \hookrightarrow E \hookrightarrow L_1(\mu)$ are bounded (cf. [23, Theorem 1.b.14]). Also recall the Kadec-Pełczyński's dichotomy for a normalized sequence (x_n) in an order continuous Banach lattice E (see [10], [23]):

(1) either $(\|x_n\|_{L_1})$ is bounded away from zero,
(2) or there exist a subsequence (x_{n_k}) and a disjoint sequence (z_k) in E such that $\|z_k - x_{n_k}\| \longrightarrow 0$ as $k \to \infty$.

A Banach lattice E has finite cotype (or finite concavity) if there exist $M, q < \infty$ such that for any $(x_i)_{i=1}^n$ in E it holds that

$$\Big(\sum_{i=1}^n \|x_i\|^q\Big)^{\frac{1}{q}} \leq M\Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|.$$

This is equivalent to the fact that E does not contain copies of ℓ_∞^n uniformly (cf. [23]). Moreover, every Banach lattice E with finite cotype satisfies the *subsequence splitting property* ([33]). This means that every bounded sequence (x_n) in E has a subsequence that can be written as $x_{n_k} = g_k + h_k$, with $|g_k| \wedge |h_k| = 0$, the sequence (g_k) being equi-integrable and (h_k) disjoint. Recall that a bounded sequence (g_n) in a Banach lattice of measurable functions over a measure space (Ω, Σ, μ) is equi-integrable if $\sup_n \|g_n \chi_A\| \to 0$ as $\mu(A) \to 0$. Since every Banach lattice with finite cotype is order continuous, and hence representable as a Banach lattice of functions, this definition is general enough for our purposes.

Given $1 \leq p < \infty$, a Banach lattice E satisfies an upper (resp. lower) p-estimate if for certain constant $M > 0$

$$\Big\|\sum_{i=1}^n x_i\Big\| \leq M\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}} \qquad \Big(\text{resp. } \Big\|\sum_{i=1}^n x_i\Big\| \geq M\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}}\Big)$$

for every sequence of disjoint elements (x_i) in E. We recall the definition of the *indices* of a Banach lattice E:

$$s(E) = \sup\{p \geq 1 : E \text{ satisfies an upper } p\text{-estimate}\}$$

$$\sigma(E) = \inf\{p \geq 1 : E \text{ satisfies a lower } p\text{-estimate}\}$$

It holds that $1 \leq s(E) \leq \sigma(E) \leq \infty$. For a rearrangement invariant (r.i.) function space X on $[0,1]$ we will also use the *Boyd indices* which are given by

$$p_X = \lim_{s\to\infty} \frac{\log s}{\log \|D_s\|} \qquad\qquad q_X = \lim_{s\to 0^+} \frac{\log s}{\log \|D_s\|},$$

where $D_s : X \to X$ is the dilation operator given by $(D_s f)(t) = f(t/s)$ when $t \leq \min(1,s)$ and zero otherwise (see [23, Section 2.b]). Notice that in general $1 \leq s(X) \leq p_X \leq q_X \leq \sigma(X) \leq \infty$, but this inequalities can be strict (see [23, p. 132]).

Definition 2.1. A Banach lattice E has *property* (C) if it is order continuous, and there exist $q < \infty$ and a probability space (Ω, Σ, μ) such that the inclusions $L_q(\mu) \hookrightarrow E \hookrightarrow L_1(\mu)$ hold.

Notice that condition (C) is a very mild assumption. Indeed, every Banach lattice with a weak order unit (f.i. separable) and finite cotype satisfies property (C) (see [15, p. 14]). Moreover, every order continuous rearrangement invariant function space on $[0,1]$ with upper Boyd index $q_X < \infty$ also has property (C) (though it may have trivial cotype, [23, Proposition 2.b.3]).

Given a Banach lattice E and a Banach space X, an operator $T : E \to X$ is *AM-compact* whenever $T([-x,x])$ is a relatively compact set in X for every $x \in E_+$ (recall that the order interval $[-x,x]$ is the set $\{y \in E : |y| \leq x\}$). An operator $T : E \to X$ is called *M-weakly compact* if it maps disjoint sequences in B_E to sequences converging to zero. Notice that if an operator is AM-compact and M-weakly compact then it is compact ([25, Proposition 3.7.4]). Also recall that an operator $T : X \to Y$ is *Dunford-Pettis* if it maps weakly null sequences to sequences converging to zero.

Definition 2.2. An operator $T : X \to X$ is *power-compact* if there exists some $k \in \mathbb{N}$ such that T^k is compact.

We refer the reader to [1], [23] and [25] for unexplained notation and terminology regarding Banach lattices and operator theory, as well as to [6] for an overview on Orlicz and Lorentz spaces.

We will make use of the following fact for strictly singular operators.

Lemma 2.3. *Let E be a Banach lattice with property (C). If an operator $T \in \mathcal{S}(E)$, then every equi-integrable sequence (g_n) has a subsequence such that $(T(g_{n_k}))$ converges for the $L_1(\mu)$ norm.*

Proof. Since E has property (C), there is a probability space (Ω, Σ, μ) such that $L_q(\mu) \hookrightarrow E \hookrightarrow L_1(\mu)$. Now, let (g_n) be an equi-integrable sequence in E. Then for every $\varepsilon > 0$ there exists $M_\varepsilon < \infty$, such that $\|g_n \chi_{\{|g_n| \geq M_\varepsilon\}}\| < \varepsilon$ for all $n \in \mathbb{N}$. Therefore, for every $\varepsilon > 0$ we have

$$(g_n) \subset [-M_\varepsilon, M_\varepsilon] + \varepsilon B_E.$$

Since $T : E \to E$ is strictly singular, $L_q(\mu)$ has finite cotype and $L_1(\mu)$ satisfies a lower 2 estimate, by [9, Proposition 2.5], it follows that $T : L_q(\mu) \hookrightarrow E \to E \hookrightarrow L_1(\mu)$ is AM-compact. Since

$$(T(g_n)) \subset T([-M_\varepsilon, M_\varepsilon]) + \varepsilon \|T\| B_E,$$

for every $\varepsilon > 0$, and $T([-M_\varepsilon, M_\varepsilon])$ is a relatively compact set in the norm of $L_1(\mu)$, we get that $(T(g_n))$ is also relatively compact. In particular, there is a subsequence such that $(T(g_{n_k}))$ converges in the norm of $L_1(\mu)$. □

The following concept, which was introduced in [8], plays a key role throughout the paper.

Definition 2.4. A Banach lattice E is *disjointly homogeneous* whenever for every pair of disjoint normalized sequences (x_n), (y_n) in E there exist equivalent subsequences. That is, for some increasing natural number sequence (n_k), some constant $C > 0$ and every scalar sequence $(a_k)_{k=1}^{\infty}$ it holds

$$C^{-1} \Big\| \sum_{k=1}^{\infty} a_k x_{n_k} \Big\| \leq \Big\| \sum_{k=1}^{\infty} a_k y_{n_k} \Big\| \leq C \Big\| \sum_{k=1}^{\infty} a_k x_{n_k} \Big\|.$$

The class of disjointly homogeneous Banach lattices includes of course L_p-spaces ($1 \leq p \leq \infty$), but also other rearrangement invariant function spaces such as Lorentz spaces and certain Orlicz spaces (see Sections 3 and 4). In addition, other now classical spaces such as Tsirelson's space and its modifications also belong to this class (cf. [4]).

Definition 2.5. Given $1 \leq p \leq \infty$, a Banach lattice is called *p-disjointly homogeneous* if every disjoint normalized sequence has a subsequence equivalent to the unit vector basis of ℓ_p (c_0 when $p = \infty$).

It is clear that every p-disjointly homogeneous Banach lattice is disjointly homogeneous. However p-disjointly homogeneous spaces for $1 \leq p \leq \infty$ do not exhaust the class of disjointly homogeneous Banach lattices. For instance, Tsirelson's space is an example of a disjointly homogeneous Banach lattice which is not p-disjointly homogeneous for any $1 \leq p \leq \infty$ (cf. [8]). Before giving the main result for disjointly homogeneous Banach lattices we consider the following special case:

Theorem 2.6. *Let E be a 1-disjointly homogeneous Banach lattice with finite cotype. Every operator $T \in \mathcal{S}(E)$ is Dunford-Pettis.*

Proof. Let (x_n) be a weakly null sequence in E. We claim that (x_{n_k}) is equi-integrable for some subsequence (n_k). Indeed, passing to a subsequence we may assume that $x_n = g_n + h_n$ where as usual (g_n) is equi-integrable, (h_n) disjoint and $|g_n| \wedge |h_n| = 0$. If (h_n) were seminormalized then, since E is 1-disjointly homogeneous, (h_n) would have a subsequence equivalent to the unit vector basis of ℓ_1. However, since (g_n) must have some subsequence converging weakly to $g \in E$, it follows that $(h_n = x_n - g_n)$ has a subsequence converging weakly to g. This is impossible since the unit vector basis of ℓ_1 is not weakly Cauchy. Therefore, (x_{n_k}) is equi-integrable and Lemma 2.3 implies that, passing to a further subsequence, $(T(x_{n_k}))$ tends to zero in $L_1(\mu)$. Now, if $(T(x_{n_k}))$ does not tend to zero in the norm of E, then, by Kadec-Pełczyński's dichotomy $(T(x_{n_k}))$ has a subsequence, still denoted (n_k), which is equivalent to a disjoint sequence, hence equivalent to the unit vector basis of ℓ_1. This yields that for scalars $(a_k)_{k=1}^{\infty}$ and some constant $C > 0$ we have

$$C \sum_{k=1}^{\infty} |a_k| \leq \Big\| \sum_{k=1}^{\infty} a_k T(x_{n_k}) \Big\| \leq \|T\| \Big\| \sum_{k=1}^{\infty} a_k x_{n_k} \Big\| \leq \|T\| \big(\sup_k \|x_{n_k}\| \big) \sum_{k=1}^{\infty} |a_k|.$$

Therefore, T is an isomorphism on a subspace isomorphic to ℓ_1, in contradiction with the fact that T is strictly singular. □

Corollary 2.7. *Let E be a 1-disjointly homogeneous Banach lattice with finite cotype. Every operator $T \in \mathcal{S}(E)$ has compact square.*

Proof. It is clear that a 1-disjointly homogeneous Banach lattice E cannot contain a subspace isomorphic to c_0 ([23, p. 35]), in particular E is weakly sequentially complete [23, Theorem 1.c.4]. Since, by Rosenthal's Theorem [22, Theorem 2.e.5], every non weakly compact operator into a weakly sequentially complete Banach space is an isomorphism on a subspace isomorphic to ℓ_1, we therefore conclude that every $T \in \mathcal{S}(E)$ is weakly compact. By Theorem 2.6, every $T \in \mathcal{S}(E)$ is Dunford-Pettis. Notice that the composition of a weakly compact with a Dunford-Pettis operator gives a compact operator. Hence, T^2 is compact. □

We pass to study the compactness of the square of strictly singular operators on disjointly homogeneous Banach lattices.

Lemma 2.8. *Let E be a disjointly homogeneous Banach lattice with property (C). If $T \in \mathcal{S}(E)$ then T^2 is AM-compact.*

Proof. Fix $x \in E_+$; since E is order continuous it is enough to show that every weakly null sequence (g_k) in $[-x, x]$ has a subsequence such that $\|T^2(g_{k_j})\|$ converges to zero. Hence, if $\|T^2(g_k)\|$ does not tend to zero in E then, passing to a subsequence, we can assume that $(T(g_k))$ and $(T^2(g_k))$ are seminormalized. Since T is strictly singular, by Lemma 2.3 we have that both sequences $(T(g_k))$ and $(T^2(g_k))$ tend to zero in the norm of $L_1(\mu)$. Now, by the Kadec-Pełczyński's dichotomy, either $(T^2(g_k))$ tends to zero in the norm of E or both $(T(g_k))$ and $(T^2(g_k))$ have subsequences equivalent to disjoint sequences in E. Since E is disjointly homogeneous, the latter case would imply that T is an isomorphism on the span of some subsequence $[T(g_{k_j})]$, which is a contradiction with the fact that T is strictly singular. Hence, we must have that $T^2(g_k) \to 0$ in the norm of E, which proves that T^2 is AM-compact. □

Theorem 2.9. *Let E be a disjointly homogeneous Banach lattice with finite cotype and an unconditional basis. Every operator $T \in \mathcal{S}(E)$ has a compact square.*

Proof. By Lemma 2.8 and [25, Proposition 3.7.4], it is enough to show that $T \in \mathcal{S}(E)$ implies that T^2 is M-weakly compact.

Let (h_k) be a disjoint sequence in B_E. First notice that a disjointly homogeneous Banach lattice whose dual is not order continuous is 1-disjointly homogeneous (cf. [25, Theorem 2.4.14]). Thus, by Corollary 2.7, we can suppose that E^* is order continuous, which, in particular, implies that (h_k) is weakly null. Let us prove that the sequence $(T^2(h_k))$ tends to zero in norm.

By the subsequence splitting property, passing to a subsequence we can write $T(h_k) = u_k + v_k$ with (u_k) equi-integrable, (v_k) disjoint and $|v_k| \wedge |u_k| = 0$. If (v_k) is seminormalized then, since E is disjointly homogeneous, there exist a subsequence (k_j) and a constant $C > 0$ such that for any $n \in \mathbb{N}$ and scalars $(a_j)_{j=1}^n$

$$C^{-1}\Big\|\sum_{j=1}^n a_j v_{k_j}\Big\| \leq \Big\|\sum_{j=1}^n a_j h_{k_j}\Big\| \leq C\Big\|\sum_{j=1}^n a_j v_{k_j}\Big\|.$$

Moreover, since E has an unconditional basis and $(T(h_k))$ is weakly null, we can assume, passing to a further subsequence, that $(T(h_k))$ is an unconditional basic sequence. Hence, using that $|v_k| \leq |T(h_k)|$ and [23, Theorem 1.d.6] we would have

$$\begin{aligned}\Big\|\sum_{j=1}^n a_j h_{k_j}\Big\| &\leq C\Big\|\sum_{j=1}^n a_j v_{k_j}\Big\| = C\Big\|\Big(\sum_{j=1}^n |a_j v_{k_j}|^2\Big)^{\frac{1}{2}}\Big\| \\ &\leq C\Big\|\Big(\sum_{j=1}^n |a_j T(h_{k_j})|^2\Big)^{\frac{1}{2}}\Big\| \leq CB\Big\|\sum_{j=1}^n a_j T(h_{k_j})\Big\|\end{aligned}$$

where B is a constant involving the unconditional basis constant of $(T(h_k))$ and the cotype constant of E. This means that T is invertible on the span of $[h_{k_j}]$ in contradiction with the fact that T is strictly singular. Therefore, $(T(h_k))$ is equi-integrable, and by Lemma 2.3, the sequence $(T^2(h_k))$ tends to zero in the norm of L_1. Now, by Kadec-Pełczyński's dichotomy, the sequence $(T^2(h_k))$ is either equivalent to a disjoint sequence or convergent to zero in the norm of E. Note that since E is disjointly homogeneous, the first case would lead to a contradiction with the strict singularity of T. This finishes the proof. □

In most applications, the existence of an unconditional basis on the space is granted. However in several cases we can safely avoid this assumption:

Lemma 2.10. *Let E be an order continuous p-disjointly homogeneous Banach lattice for some $2 \leq p \leq \infty$. Every operator $T \in \mathcal{S}(E)$ is M-weakly compact.*

Proof. Let (h_n) be a normalized disjoint sequence in E. By hypothesis, we can assume, passing to a subsequence, that it is equivalent to the unit vector basis of ℓ_p. By [23, Proposition 1.a.9] we can consider a closed ideal X of E containing $(T(h_n))$ which can be represented as an order dense ideal in $L_1(\mu)$ over some probability space (Ω, Σ, μ) (cf. [23, Theorem 1.b.14]). Since $p \geq 2$ the operator $R : \ell_2 \to L_1(\mu)$ defined by $R(\sum_k a_k e_k) = \sum_k a_k T(h_k)$ is well defined and strictly singular. Hence, by [9, Proposition 2.1], R is compact and so the sequence $(T(h_n))$ tends to zero in the norm of $L_1(\mu)$. If $(T(h_n))$ were not convergent to zero in the norm of E, then by Kadec-Pełczyński's dichotomy we could extract a subsequence equivalent to a disjoint sequence, hence equivalent to the unit vector basis of ℓ_p. This would imply that T is an isomorphism on a subspace isomorphic to ℓ_p, in contradiction with the fact that T is strictly singular. This proves that $(T(h_n))$ converges to zero in E, and so T is M-weakly compact. □

Theorem 2.11. *If E is a p-disjointly homogeneous Banach lattice ($2 \leq p \leq \infty$) with property (C), then every operator $T \in \mathcal{S}(E)$ has a compact square.*

Proof. Lemmas 2.8 and 2.10 yield that every operator $T \in \mathcal{S}(E)$ is M-weakly compact and has a AM-compact square. In particular, T^2 is compact [25, Proposition 3.7.4]. □

It is well known that strictly singular operators on $L_2(\mu)$ spaces are compact. This property is shared by 2-disjointly homogeneous Banach lattices as the following result shows:

Theorem 2.12. *Let E be a 2-disjointly homogeneous Banach lattice with property (C). Every operator $T \in \mathcal{S}(E)$ is compact.*

Proof. It is clear that E does not contain any sublattice isomorphic to c_0 nor ℓ_1, hence, E is reflexive (cf. [1, Theorem 14.23]).

First, let us show that T is AM-compact. Let (g_n) be a sequence in $[-x, x]$ for some $x \in E_+$. Since E is reflexive, passing to a subsequence and taking differences, we can assume that (g_n) is weakly null. We will prove that (Tg_n) converges to zero in E. Let us suppose that $\|T(g_n)\| \geq \alpha > 0$, for every $n \in \mathbb{N}$. Since $|g_n| \leq x$ and E is order continuous, for every $\varepsilon > 0$ there exists $M < \infty$, such that $\|g_n \chi_{\{|g_n| \geq M\}}\| < \varepsilon$ for all $n \in \mathbb{N}$. Since the sequence $g_n^M = g_n \chi_{\{|g_n| < M\}}$ is contained in the order interval $[-M, M]$, passing to some subsequence we can assume that (g_n^M) converges weakly to a certain $g \in [-M, M]$. Moreover, since $\|g_n \chi_{\{|g_n| \geq M\}}\| \leq \varepsilon$, and (g_n) is weakly null, it follows that $\|g\| \leq \varepsilon$.

Let $z_n = g_n^M - g$. Clearly, (z_n) is weakly null. Moreover, it holds that

$$\|Tz_n\| \geq \|Tg_n^M\| - \|Tg\| \geq \|Tg_n\| - (\|T(g_n\chi_{\{|g_n|\geq M\}})\| + \|Tg\|) \geq \alpha - 2\|T\|\varepsilon,$$

which is bounded below for ε small enough. Since $|z_n| \leq 2M$ and (z_n) is weakly null in E, passing to a subsequence we can assume that it is an unconditional basic sequence in $L_q(\mu)$ with unconditional constant K. Hence, using [23, Theorem 1.d.6], for some constants $C, D < \infty$, we have

$$\begin{aligned}\Big\|\sum_{n=1}^k a_n z_n\Big\|_E &\leq C\Big\|\sum_{n=1}^k a_n z_n\Big\|_{L_q} \leq CK\int_0^1 \Big\|\sum_{n=1}^k a_n r_n(t) z_n\Big\|_{L_q} dt \\ &\leq CKD\Big\|\Big(\sum_{n=1}^k |a_n z_n|^2\Big)^{\frac{1}{2}}\Big\|_{L_q} \leq 2CKDM\Big(\sum_{n=1}^k |a_n|^2\Big)^{\frac{1}{2}},\end{aligned}$$

for any sequence of scalars $(a_n)_{n=1}^k$. Therefore, by [9, Proposition 2.1], $\|T(z_n)\|_{L_1} \to 0$. Hence, if (Tz_n) is not convergent to zero in E, then by Kadec-Pełczyński's dichotomy we can extract a subsequence (still denoted (Tz_n)) which is disjoint and equivalent to the unit vector basis of ℓ_2. However, this would yield the following estimation

$$\Big(\sum_{n=1}^k |a_n|^2\Big)^{\frac{1}{2}} \leq A\Big\|\sum_{n=1}^k a_n T(z_n)\Big\| \leq A\|T\|\Big\|\sum_{n=1}^k a_n z_n\Big\| \leq B\Big(\sum_{n=1}^k |a_n|^2\Big)^{\frac{1}{2}},$$

for certain constants A and B. But this means that T is an isomorphism on a subspace isomorphic to ℓ_2, in contradiction with the strict singularity of T.

Therefore, we can assume that (Tz_n) is contained in some Kadec-Pełczyński set, and since $\|T(z_n)\|_{L_1} \to 0$, this means that $\|T(z_n)\|_E \to 0$. In particular, we get

$$\|T(g_n)\| \leq \|T(z_n)\| + \|T(g)\| + \|T(g_n - g_n^M)\| \leq \varepsilon + 2\|T\|\varepsilon$$

for n large enough. Since ε was arbitrary this shows that $(T(g_n))$ tends to zero in the norm of E. Therefore, T is AM-compact.

Finally, Lemma 2.10 yields that T is M-weakly compact. Since the operator T is AM-compact and M-weakly compact, we conclude, by [25, Proposition 3.7.4], that T is compact. □

When it comes to *discrete* Banach lattices some additional remarks are in order. In the discrete case the class of disjointly homogeneous Banach lattices E with a basis of disjoint vectors is a rather small class, since "most" basic sequences in E are equivalent to disjoint sequences. Among the examples, apart from the spaces ℓ_p or c_0, and ℓ_p-sums of finite dimensional Banach lattices, we can also consider the Tsirelson space and some of its generalizations, such as Tsirelson-like spaces T_θ (see [4]), and Baernstein spaces B_p (see [4, Chapter 0]). However, Lorentz and Orlicz sequence spaces (distinct from spaces ℓ_p) are not disjointly homogenous.

In this discrete setting we can improve Theorem 2.9, obtaining, as a particular case, the known result which establishes that strictly singular endomorphisms on ℓ_p must be compact (cf. [22, p. 76]).

Theorem 2.13. *Let E be a discrete Banach lattice with a disjoint basis. If E is disjointly homogeneous then every operator $T \in \mathcal{S}(E)$ is compact.*

Proof. Suppose first that E^* is not order continuous, then by [25, Theorem 2.4.14], this is equivalent to E being 1-disjointly homogeneous, so in particular E does not contain a subspace isomorphic to c_0 and is weakly sequentially complete (cf. [23, Theorem 1.c.4]). Assume T is not compact and let

(x_n) be a bounded sequence in E such that $(T(x_n))$ has no convergent subsequence. By Rosenthal's Theorem, passing to a subsequence, we have that either $(T(x_{n_k}))$ is equivalent to the unit vector basis of ℓ_1 or it is weakly Cauchy. In the first case, for some constant $C > 0$ and scalars (a_i) we have

$$C\sum_{i=1}^n |a_i| \leq \Big\|\sum_{i=1}^n a_i T(x_{n_i})\Big\| \leq \|T\|\Big\|\sum_{i=1}^n a_i x_{n_i}\Big\| \leq \|T\|(\sup_n \|x_n\|)\sum_{i=1}^n |a_i|,$$

which contradicts that T is strictly singular.

Alternatively, if $(T(x_{n_k}))$ is weakly Cauchy, we have, since E is weakly sequentially complete, that $(T(x_{n_k}))$ is weakly convergent. Let us consider the sequence $(y_k = x_{n_{2k}} - x_{n_{2k+1}})$. It follows that $(T(y_n))$ is weakly null and also seminormalized because $(T(x_n))$ has no convergent subsequence. Hence, since E is discrete, we obtain from [22, Proposition 1.a.12] that the sequence $(T(y_n))$ is equivalent to a disjoint sequence, and so, passing to a further subsequence, it must be equivalent to the unit vector basis of ℓ_1. Therefore, for some constant $M > 0$ and scalars (a_i) we have

$$M\sum_{i=1}^n |a_i| \leq \Big\|\sum_{i=1}^n a_i T(y_{n_i})\Big\| \leq \|T\|\Big\|\sum_{i=1}^n a_i y_{n_i}\Big\| \leq 2\|T\|(\sup_n \|x_n\|)\sum_{i=1}^n |a_i|,$$

which is again a contradiction with the fact of T be strictly singular.

Let us consider now the case that E^* is order continuous. Since order intervals on a discrete Banach lattice with a disjoint basis are compact, every operator $T : E \to E$ is AM-compact. Hence, by [25, Proposition 3.7.4] , it suffices to show that every strictly singular operator on E is M-weakly compact. Let (x_n) be a bounded disjoint sequence which, by [25, Theorem 2.4.14], is weakly null. Suppose that for some $\alpha > 0$ and some sequence (n_k) we have $\|T(x_{n_k})\| > \alpha$ for every $k \in \mathbb{N}$. Then, since the sequence $(T(x_{n_k}))$ is also weakly null, by [22, Proposition 1.a.12], and passing to a further subsequence, we can assume that $(T(x_{n_k}))$ is equivalent to a disjoint sequence in E. Since E is disjointly homogeneous, this implies that T is an isomorphism on $[x_{n_j}]$ for some subsequence. Therefore, it cannot happen that $\|T(x_{n_k})\| > \alpha$ for any $\alpha > 0$, that is $(T(x_n))$ tends to zero, and so T is M-weakly compact. This finishes the proof. □

It would be interesting to know under which conditions the converses of Theorems 2.6, 2.9 and 2.12 hold. For instance, in the case of rearrangement invariant function spaces E on $[0, 1]$, does $\mathcal{S}(E) = \mathcal{K}(E)$ imply that E is 2-disjointly homogeneous?

3. Strictly singular operators on Lorentz spaces

We consider now rearrangement invariant function spaces on $[0, 1]$. We refer to [23] for the basic definitions and properties of these spaces.

Recall that given $1 \leq q < \infty$ and W a positive, non-increasing function in $[0, 1]$, such that $\lim_{t\to 0} W(t) = \infty$, $W(1) > 0$ and $\int_0^1 W(t)dt = 1$, the *Lorentz function space* $\Lambda(W, q)$ ([24]) is the space of all measurable functions f on $[0, 1]$ such that

$$\|f\| = \Big(\int_0^1 f^*(t)^q W(t)dt\Big)^{1/q} < \infty,$$

where f^* denotes the decreasing rearrangement of a function f.

Let us also recall that for $1 < p < \infty$ and $1 \leq q \leq \infty$, the Lorentz space $L_{p,q}$ is the space of all measurable functions f in $[0,1]$ such that

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} < \infty & \text{for } 1 \leq q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t) < \infty, & \text{if } q = \infty. \end{cases}$$

In order to use the results of the previous section we need the following important result ([10, Proposition 5.1],[3]):

Proposition 3.1. *Let $(f_n)_n$ be a disjoint normalized sequence in $\Lambda(W,q)$ (resp. $L_{p,q}$). For each $\varepsilon > 0$, there exists a subsequence (f_{n_k}) which is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_q, whose span is a complemented subspace of $\Lambda(W,q)$ (resp. $L_{p,q}$) with projection constant smaller than $(1+\varepsilon)$.*

Thus Lorentz function spaces $\Lambda(W,q)$ and $L_{p,q}$ are q-disjointly homogeneous. And as a consequence of Theorems 2.9, 2.12, and Corollary 2.7 in Section 2 we get the following:

Proposition 3.2. *Given $1 < p < \infty$ and $1 \leq q < \infty$, every operator $T \in \mathcal{S}(\Lambda(W,q))$ or $T \in \mathcal{S}(L_{p,q})$ has a compact square. Moreover, if $q = 2$ then T is already compact, while if $q = 1$, then T is Dunford-Pettis.*

It is well-known that strictly singular non-compact operators exist on $L_{p,q}$ spaces for $q \neq 2$. For instance, consider a complemented subspace isomorphic to ℓ_q and the span of the Rademacher functions which is isomorphic to ℓ_2. Denote $P_1 : L_{p,q} \to \ell_q$ and $P_2 : L_{p,q} \to \ell_2$ the corresponding projections, $i_{s,t} : \ell_s \to \ell_t$ the canonical inclusion, and $Q : \ell_q \hookrightarrow L_{p,q}$ and $R : \ell_2 \hookrightarrow L_{p,q}$ the corresponding embeddings. When $q < 2$, we can consider $T = R i_{q,2} P_1 \in \mathcal{S}(L_{p,q}) \backslash \mathcal{K}(L_{p,q})$, and when $q > 2$, take $S = Q i_{2,q} P_2 \in \mathcal{S}(L_{p,q}) \backslash \mathcal{K}(L_{p,q})$ (cf. [12]).

Note also that these results do not hold for Lorentz spaces on the unbounded interval $(0,\infty)$ and that the spaces $L_{p,q}(0,\infty)$ $(p \neq q)$ are not disjointly homogeneous (notice that $L_{p,q}(0,\infty)$ contains disjoint sequences whose span is isomorphic to $\ell_{p,q}$).

The behavior of the extreme Lorentz spaces $L_{p,\infty}$ is different:

Proposition 3.3. *There exists an operator $T \in \mathcal{S}(L_{p,\infty})$, for $p \neq 2$, whose cube T^3 is not compact.*

In order to show this we first give a precise way of embedding ℓ_p as a complemented subspace in $L_{p,\infty}$ (notice that for $p < 2$ even L_p can be embedded as a complemented subspace of $L_{p,\infty}$, this follows from work of N. Kalton, see [17, Prop. 3.4]).

Proposition 3.4. *Given $1 < p < \infty$, there exists a disjoint normalized sequence (f_n) in $L_{p,\infty}$ whose span is complemented and isomorphic to ℓ_p.*

Proof. Let $(t_n) \subset [0,1]$ with $t_n \downarrow 0$, and for $n \geq 1$ consider the functions on $[0,1]$ defined by

$$f_n(t) = \frac{p-1}{p}(t - t_n)^{-\frac{1}{p}} \chi_{(t_{n+1},t_n)}(t).$$

We claim that the closed linear span $[f_n]$ is isomorphic to ℓ_p. Indeed, since $\|f\|_{L_{p,\infty}} = \sup_{s>0} s(\mu_f(s))^{\frac{1}{p}}$, where $\mu_f(s) = \mu\{|f(t)| > s\}$, and for each $n \in \mathbb{N}$ we have

$$
\begin{aligned}
\mu_{f_n}(s) &= \mu\Big\{t \in (t_{n+1}, t_n) : t < t_n + \Big(\frac{p-1}{p}\Big)^p \frac{1}{s^p}\Big\} \\
&= \begin{cases} t_n - t_{n+1} & \text{if } s \leq \frac{p-1}{p(t_n - t_{n+1})^{\frac{1}{p}}}, \\ \left(\frac{p-1}{p}\right)^p \frac{1}{s^p} & \text{if } s > \frac{p-1}{p(t_n - t_{n+1})^{\frac{1}{p}}}, \end{cases}
\end{aligned}
$$

it holds that $\|f_n\|_{L_{p,\infty}} = \frac{p-1}{p}$. Let us see that $\|\sum_{i=1}^{n} a_i f_i\|_{L_{p,\infty}} \sim (\sum_{i=1}^{n} |a_i|^p)^{\frac{1}{p}}$ for any scalars $(a_i)_{i=1}^n$. Indeed, since $(f_i)_{i=1}^n$ are disjoint, we have

$$
\begin{aligned}
\|\sum_{i=1}^{n} a_i f_i\|_{L_{p,\infty}} &= \sup_{s>0} s\Big(\sum_{i=1}^{n} \mu_{f_i}(\frac{s}{|a_i|})\Big)^{\frac{1}{p}} \geq s_0\Big(\sum_{i=1}^{n} \mu_{f_i}(\frac{s_0}{|a_i|})\Big)^{\frac{1}{p}} \\
&= s_0\Big[\Big(\frac{p-1}{p}\Big)^p \sum_{i=1}^{n} \frac{|a_i|^p}{s_0^p}\Big]^{\frac{1}{p}} = \frac{p-1}{p}\Big(\sum_{i=1}^{n} |a_i|^p\Big)^{\frac{1}{p}}
\end{aligned}
$$

where s_0 is any number greater than $\max_i \left\{ \frac{|a_i|p-1}{p(t_i - t_{i+1})^{\frac{1}{p}}} \right\}$. And, since $L_{p,\infty}$ satisfies an upper p-estimate ([5]), we also have $\big\|\sum_{i=1}^{n} a_i f_i\big\|_{L_{p,\infty}} \leq C\big(\sum_{i=1}^{n} |a_i|^p\big)^{\frac{1}{p}}$ for certain constant $C > 0$.

Now, to construct a projection onto $[f_n]$, let us consider a Banach limit $B \in \ell_\infty^*$. The operator $R : L_{p,\infty} \to \ell_p(0,1)$ given by

$$
Rx(t) = B\Big(n^{(1-\frac{1}{p})} \int_t^{t+\frac{1}{n}} x(s)ds\Big)
$$

for $x \in L_{p,\infty}$ and $0 < t < 1$, is bounded and has norm one [31]. We define now the operator $P : L_{p,\infty} \to L_{p,\infty}$ by

$$
Px(t) = \sum_{k=1}^{\infty} Rx(t_k) f_k(t).
$$

Notice that P can be seen as the composition

$$
\begin{array}{ccc}
L_{p,\infty} & \xrightarrow{\ P\ } & L_{p,\infty} \\
{\scriptstyle R}\downarrow & & \uparrow{\scriptstyle J} \\
\ell_p(0,1) & \xrightarrow{\ T\ } & \ell_p
\end{array}
$$

where $Tx(t) = x(t)\chi_{\{t_k\}}(t)$ and $J((a_k)) = \sum_{k=1}^{\infty} a_k f_k$. Hence P is bounded. Moreover, we have

$$
Rf_k(t) = \begin{cases} 0 & \text{if } t \neq t_k \\ 1 & \text{if } t = t_k \end{cases}
$$

Indeed, for $t \neq t_k$ we have

$$
n^{(1-\frac{1}{p})} \int_t^{t+\frac{1}{n}} f_k(s)ds \leq n^{(1-\frac{1}{p})} \frac{p-1}{p} \int_t^{t+\frac{1}{n}} (t - t_k)^{-\frac{1}{p}} ds \leq n^{-\frac{1}{p}} (t - t_k)^{-\frac{1}{p}} \underset{n\to\infty}{\longrightarrow} 0.
$$

While for $t = t_k$, and $n > \frac{1}{t_k}$, we have

$$n^{(1-\frac{1}{p})} \int_{t_k}^{t_k+\frac{1}{n}} f_k(s)ds = n^{(1-\frac{1}{p})} \frac{p-1}{p} \int_0^{\frac{1}{n}} s^{-\frac{1}{p}} ds = 1.$$

Therefore, $P(f_k) = f_k$, which yields that P is a projection onto $[f_k]$ as claimed. □

Proof of Proposition 3.3. It is well-known that the Lorentz sequence space $\ell_{p,\infty}$ embeds as a complemented sublattice into $L_{p,\infty}$ (see [20]), and the Rademacher functions span a complemented subspace of $L_{p,\infty}$ isomorphic to ℓ_2 (cf. [23, Theorem 2.b.4]). We can also consider a complemented subspace isomorphic to ℓ_∞ in $L_{p,\infty}$ (cf. [23, Proposition 1.a.7, and p. 105]). Therefore, Proposition 3.4 provides a complemented subspace of $L_{p,\infty}$ isomorphic to $\ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty$. Hence, for $p < 2$ we can define the operator T given by

$$\begin{array}{ccc} L_{p,\infty} & \xrightarrow{\quad T \quad} & L_{p,\infty} \\ {\scriptstyle P}\downarrow & & \uparrow{\scriptstyle J} \\ \ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty & \xrightarrow{\quad S \quad} & \ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty \end{array}$$

where P is a projection, J an isomorphic embedding, and for $(x, y, z, w) \in \ell_p \oplus \ell_{p,\infty} \oplus \ell_2 \oplus \ell_\infty$, the operator $S(x, y, z, w) = (0, x, y, z)$ is a "shift-like" operator. Clearly, T is strictly singular, and T^3 is not compact.

Similarly, for $p > 2$ we consider the operator T given by

$$\begin{array}{ccc} L_{p,\infty} & \xrightarrow{\quad T \quad} & L_{p,\infty} \\ {\scriptstyle P}\downarrow & & \uparrow{\scriptstyle J} \\ \ell_2 \oplus \ell_p \oplus \ell_{p,\infty} \oplus \ell_\infty & \xrightarrow{\quad S \quad} & \ell_2 \oplus \ell_p \oplus \ell_{p,\infty} \oplus \ell_\infty \end{array}$$

where now the shift operator S is defined $S(x, y, z, w) = (0, x, y, z)$. Thus, we get as before that T is strictly singular, and T^3 is not compact. □

Similarly we can define strictly singular operators on $L_{2,\infty}$ whose square is not compact. However, we do not know whether there exists some $n \in \mathbb{N}$ such that T^n is compact whenever $T \in \mathcal{S}(L_{p,\infty})$; we do not even know whether every operator $T \in \mathcal{S}(L_{p,\infty})$ is power-compact.

When it comes to the order continuous part $L^o_{p,\infty}$ of $L_{p,\infty}$, every strictly singular operator on $L^o_{p,\infty}$ has compact square. This follows from Theorem 2.11, since $L^o_{p,\infty}$ is ∞-disjointly homogeneous and his Boyd indices equal p.

A similar statement also holds for order continuous Marcinkiewicz spaces $M(\varphi)$ with finite upper Boyd index (since they are also ∞-disjointly homogeneous Banach lattices, cf. [30]).

4. Strictly singular operators on Orlicz spaces

Recall that given an Orlicz function φ, the *Orlicz function space* L_φ on $[0, 1]$ is the space of all measurable functions f on $[0, 1]$ such that $\int_0^1 \varphi\Big(\frac{|f(t)|}{r}\Big) d\lambda < \infty$ for some $r > 0$. The norm is defined by

$$\|f\| = \inf \left\{ r > 0 : \int_0^1 \varphi\Big(\frac{|f(t)|}{r}\Big) d\lambda \leq 1 \right\}.$$

Let us consider the associated sets to the Orlicz function φ in the space $C(0, \infty)$ ([21]):

$$E^\infty_{\varphi,s} = \overline{\Big\{\frac{\varphi(rt)}{\varphi(r)} : r \geq s\Big\}}, \qquad E^\infty_\varphi = \bigcap_{s>0} E^\infty_{\varphi,s}\,, \text{ and } C^\infty_\varphi = \overline{conv}(E^\infty_\varphi).$$

As usual, we will write $E^\infty_\varphi \cong \{F\}$ whenever every function in E^∞_φ is equivalent to certain function F at 0. It turns out that this condition actually characterizes disjointly homogeneous Orlicz spaces. Namely, we have the following:

Theorem 4.1. *An Orlicz space L_φ is disjointly homogeneous if and only if $E^\infty_\varphi \cong \{F\}$ for certain function F. Moreover, L_φ is disjointly homogeneous if and only if it is p-disjointly homogenous for some $1 \leq p \leq \infty$, and in this case $E^\infty_\varphi \cong \{t^p\}$.*

Proof. Suppose first that there are two functions F and G in E^∞_φ, which are not equivalent at 0. Since $F \in E^\infty_\varphi$, there exists an increasing sequence (t_n) such that $\frac{\varphi(t_n x)}{\varphi(t_n)}$ converge to $F(x)$ uniformly for $x \in [0,1]$. We can actually take (t_n) such that $\varphi(t_n) \geq 2^n$ and

$$\Big|\frac{\varphi(t_n x)}{\varphi(t_n)} - F(x)\Big| < \frac{1}{2^n}$$

for all $x \in [0,1]$ and every $n \in \mathbb{N}$. Let us consider $f_n = t_n \chi_{A_n}$ where (A_n) is a sequence of disjoint sets with $\mu(A_n) = \frac{1}{\varphi(t_n)}$. Notice that

$$\int_0^1 \varphi(\sum_{n=1}^\infty \lambda_n f_n) = \sum_{n=1}^\infty \varphi(\lambda_n t_n)\mu(A_n) = \sum_{n=1}^\infty \frac{\varphi(\lambda_n t_n)}{\varphi(t_n)}$$

is convergent if and only if $\sum_{n=1}^\infty F(\lambda_n) < \infty$. Therefore, (f_n) is equivalent to the unit vector basis of ℓ_F. Similarly, we can construct a disjoint sequence in L_φ equivalent to the unit vector basis of ℓ_G.

Now, assuming L_φ is disjointly homogeneous, and using the fact that the unit vector basis of any Orlicz sequence space is symmetric, and in particular subsymmetric, we would get that F and G are equivalent at 0. This contradiction proves that $E^\infty_\varphi \cong \{F\}$ for some F whenever L_φ is disjointly homogeneous.

For the converse, assume $E^\infty_\varphi \cong \{F\}$, so we also have that $C^\infty_\varphi = \overline{conv}(E^\infty_\varphi) \cong \{F\}$. Thus, given a sequence (f_n) in L_φ of normalized disjoint functions, by [21, Proposition 3], there exists a subsequence (f_{n_k}) equivalent to the unit vector basis of ℓ_G for some $G \in C^\infty_\varphi$. Therefore, G and F are equivalent at 0 and so every disjoint sequence in L_φ has a subsequence equivalent to the unit vector basis of ℓ_F. This proves that an Orlicz space is disjointly homogeneous if and only if the set E^∞_φ reduces to one function (up to equivalence at 0).

To prove the second statement, assume that L_φ is disjointly homogeneous. Notice that we can assume that L_φ is reflexive, otherwise L_φ would contain ℓ_1 or c_0 and as a consequence it would be 1- or ∞-disjointly homogeneous (cf. [1, Theorem 14.23]). Now, by the previous part of the proof, there is F such that $E^\infty_\varphi \cong \{F\}$. We want to prove that $F(t)$ is equivalent to t^p for some $1 < p < \infty$. By a classical result of Polya, it suffices to show that F is quasi-multiplicative, that is, there exists $C > 0$ such that

$$\frac{1}{C} < \frac{F(xy)}{F(x)F(y)} < C$$

for every $x, y \in [0,1]$.

To this end, since $F \in E^\infty_\varphi$, for any increasing sequence (t_n) tending to ∞, the sequence $(\frac{\varphi(t_n\cdot)}{\varphi(t_n)})_{n=1}^\infty$ converges uniformly on $[0,1]$ to F. Then, for $x, y \in [0,1]$ we have

$$\frac{F(xy)}{F(x)F(y)} = \lim_n \frac{\frac{\varphi(t_n xy)}{\varphi(t_n)}}{\frac{\varphi(t_n x)}{\varphi(t_n)}\frac{\varphi(t_n y)}{\varphi(t_n)}} = \lim_n \frac{\varphi(t_n xy)}{\varphi(t_n y)}\frac{\varphi(t_n)}{\varphi(t_n x)} = \frac{F_y(x)}{F(x)}$$

where $F_y(x) := \lim_n \frac{\varphi(t_n yx)}{\varphi(t_n y)}$. Now, using that $E^\infty_\varphi \cong \{F\}$, which in particular implies that this set is compact, it can be seen that for certain constant $C > 0$, it holds that $\frac{1}{C} \leq \frac{F_y(x)}{F(x)} \leq C$ for every $x, y \in [0,1]$. This completes the proof. □

Notice that the second statement of the previous theorem can also be deduced from the first part together with the fact that Orlicz spaces are stable [11].

Corollary 4.2. *Let φ be an Orlicz function such that $E^\infty_\varphi \cong \{t^p\}$, for some $1 \leq p < \infty$, then every operator $T \in \mathcal{S}(L_\varphi)$ has a compact square. Furthermore for $p = 2$, the operator T is already compact, while for $p = 1$, T is Dunford-Pettis.*

Proof. By Theorem 4.1 L_φ is p-disjointly homogeneous. Now, for $1 \leq p < \infty$ and $p \neq 2$, by Theorem 2.9 and Corollary 2.7, if $T \in \mathcal{S}(L_\varphi)$ then the square T^2 is compact. Clearly, for $p = 2$, Theorem 2.12 yields that every $T \in \mathcal{S}(L_\varphi)$ is compact, and for $p = 1$, Theorem 2.6 yields that T is Dunford-Pettis. □

For example, the spaces L_φ with $\varphi(x) = x^p \log^\alpha(1+x)$ for $-\infty < \alpha < \infty$ are p-disjointly homogeneous. Notice that, the condition $E^\infty_\varphi \cong \{t^p\}$ implies equality of the indices $s(L_\varphi) = \sigma(L_\varphi) = p$. Recall that for Orlicz function spaces the associated lattice indices and Boyd indices coincide, i.e. $s(L_\varphi) = p_{L_\varphi}$ and $\sigma(L_\varphi) = q_{L_\varphi}$ (cf. [23, p. 139]). Many regular Orlicz functions satisfy the condition $E^\infty_\varphi \cong \{t^p\}$, for example the class of all Orlicz functions such that

$$\lim_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} = p.$$

In general we cannot weaken this condition on E^∞_φ, as the following shows:

Proposition 4.3. *There exist Orlicz spaces L_φ with indices $s(L_\varphi) = \sigma(L_\varphi) = p$, and an operator $T \in \mathcal{S}(L_\varphi)$ whose square T^2 is not compact.*

Proof. Let $2 < p < \infty$. Consider the Orlicz spaces L_φ defined by the functions

$$\varphi(t) \equiv \varphi_{p,q}(t) = t^p \exp\{q \sum_{k=1}^\infty \Big(1 - \cos\big(\frac{\pi logt}{2^k}\big)\Big)\}$$

for $t > 0$, $\varphi(0) = 0$, where $|q| > 0$. These Orlicz functions, first introduced for $q = 1$ in [15], are minimal functions in the sense of [22] with indices p (see [14, Proposition 2]). Hence, by [13, Proposition 2], the spaces L_φ has a complemented subspace isomorphic to ℓ_φ.

Now, since $\varphi(t) \geq t^p$ the inclusion $\ell_\varphi \hookrightarrow \ell_p$ is bounded, and since for q big enough ℓ_φ has no complemented subspace isomorphic to ℓ_p (see [18, Theorem 3.4] and also [14, Corollary 1.7]), it follows that the inclusion $\ell_\varphi \hookrightarrow \ell_p$ is strictly singular.

Consider the decomposition $L_\varphi[0,1] = L_\varphi[0,\frac{1}{3}] \oplus L_\varphi[\frac{1}{3},\frac{2}{3}] \oplus L_\varphi[\frac{2}{3},1]$, and denote by $P_R : L_\varphi[0,\frac{1}{3}] \to [r_n]$ the projection onto the span of the Rademacher functions on $[0,\frac{1}{3}]$, and by $P_\varphi : L_\varphi[\frac{1}{3},\frac{2}{3}] \to \ell_\varphi$ the projection onto ℓ_φ.

Define the operator $T : L_\varphi \to L_\varphi$ by the following factorization diagram

$$\begin{array}{ccccccc}
L_\varphi[0,\frac{1}{3}] \oplus L_\varphi[\frac{1}{3},\frac{2}{3}] \oplus L_\varphi[\frac{2}{3},\frac{1}{3}] & \xrightarrow{T} & L_\varphi[0,\frac{1}{3}] \oplus L_\varphi[\frac{1}{3},\frac{2}{3}] \oplus L_\varphi[\frac{2}{3},\frac{1}{3}] \\
P_R \downarrow \qquad P_\varphi \downarrow & & \uparrow J_\varphi \qquad \uparrow J_p \\
\ell_2 \quad \oplus \quad \ell_\varphi & \xrightarrow{S} & \ell_\varphi \quad \oplus \quad \ell_p
\end{array}$$

where S denotes the formal inclusion, and J_φ and J_p are isomorphic embeddings. The operator T is well defined, and strictly singular since so are the inclusions $\ell_2 \hookrightarrow \ell_\varphi$ and $\ell_\varphi \hookrightarrow \ell_p$. However, T^2 is not compact, since it maps the Rademacher functions on $[0, \frac{1}{3}]$ to the canonical basis of ℓ_p. $\square$

Proposition 4.4. *There exist Orlicz spaces L_φ with indices $s(L_\varphi) = \sigma(L_\varphi) = 2$ and an operator $T \in \mathcal{S}(L_\varphi)$ such that T is not compact.*

Proof. Consider the function $\varphi = \varphi_{2,q}$ for $q > 0$ as defined in the above Proposition. Thus the space L_φ contains complemented copies of ℓ_φ and ℓ_2. Moreover, since $\varphi_{2,q}(x) \geq x^2$ at 0 and ℓ_φ has no complemented copy of ℓ_2, we deduce that the inclusion $\ell_\varphi \hookrightarrow \ell_2$ is strictly singular. Therefore, we can consider the operator

$$\begin{array}{ccc}
L_\varphi & \xrightarrow{T} & L_\varphi \\
P \downarrow & & \uparrow \\
\ell_\varphi \oplus \ell_2 & \xrightarrow{S} & \ell_\varphi \oplus \ell_2
\end{array}$$

where P is a projection and $S(x, y) = (0, x)$. Clearly, T is strictly singular but not compact. $\square$

Clearly the Orlicz spaces given above are not disjointly homogenous (this follows from Theorem 2.9). More generally, it holds that every *minimal* Orlicz function space L_φ (different from L_p) is not disjointly homogenous. Indeed, recall that in general for each $\psi \in C_\varphi^\infty$ there exists a sequence of normalized disjoint functions in L_φ equivalent to the symmetric canonical basis of ℓ_ψ ([21, Proposition 4]). Now, since φ is minimal, we have, by [13, Proposition 1], that $E_{\varphi,1}^\infty = E_\varphi^\infty = E_\varphi = E_{\varphi,1}$ and the set $E_{\varphi,1}^\infty$ contains uncountable many mutually non-equivalent Orlicz functions (see the proof of [22, Theorem 4.b.9]). Hence, using the symmetry, we deduce that in L_φ there are uncountable many sequences of normalized disjoint functions with no equivalent subsequence.

Notice also that in the class of all Orlicz spaces L_φ with different indices ($s(L_\varphi) \neq \sigma(L_\varphi)$) there are no disjointly homogenous spaces. This follows from the fact that for each $p \in [s(L_\varphi), \sigma(L_\varphi)]$ we have $t^p \in C_\varphi^\infty$ and there exist sequences of normalized disjoint functions in L_φ that are equivalent to the canonical basis of ℓ_p ([21, Proposition 4]).

Within this class there exist Orlicz spaces with strictly singular operators which are not power-compact:

Proposition 4.5. *Given $1 \leq p < q < \infty$, there exist Orlicz spaces L_φ with indices $\sigma(L_\varphi) = p$ and $s(L_\varphi) = q$ and an operator $T \in \mathcal{S}(L_\varphi)$ which is not power-compact.*

Proof. Given any strictly increasing sequence (p_n) contained in $[p, q]$, we can consider an Orlicz function space $L_\varphi[0, 1]$ with $\sigma(L_\varphi) = p$ and $s(L_\varphi) = q$ and which contains complemented copies of ℓ_{p_n} for every n (cf. [14]). Let us denote by $P_n : L_\varphi \to L_\varphi$, the projection onto each ℓ_{p_n}, which will be spanned by a sequence of functions supported in $[2^{-n}, 2^{-n+1}]$. Now, for every $k \in \mathbb{N}$, denote $m_k = \sum_{n=1}^k n = \frac{k(k+1)}{2}$.

Let us consider the operator T_k given by

$$\begin{array}{ccc} L_\varphi & \xrightarrow{\quad T_k \quad} & L_\varphi \\ R_k \Big\downarrow & & \Big\uparrow i_k \\ \bigoplus_{j=m_k+1}^{m_{k+1}} \ell_{p_j} & \xrightarrow{\quad S_k \quad} & \bigoplus_{j=m_k+1}^{m_{k+1}} \ell_{p_j} \end{array}$$

where $R_k(f) = (P_{m_k+1}(f), \dots, P_{m_{k+1}}(f))$, $S_k(f_1, \dots, f_k) = (0, f_1, \dots, f_{k-1})$ is a "shift" operator and i_k is just the isomorphic embedding. Clearly, T_k is a bounded operator in L_φ, acting only on functions supported in $[2^{-m_{k+1}}, 2^{-m_k}]$. In particular, $T_iT_j = 0$ unless $i = j$. Moreover, T_k is strictly singular but $(T_k)^{k-1}$ is not compact (although $(T_k)^k = 0$).

Let us consider the operator

$$T = \sum_{k=1}^{\infty} \frac{T_k}{2^k \|T_k\|}.$$

Clearly, T is bounded, and since

$$\Big\| T - \sum_{k=1}^{n} \frac{T_k}{2^k \|T_k\|} \Big\| \to 0$$

when $n \to \infty$, we have that T is strictly singular because so is $\sum_{k=1}^{n} \frac{T_k}{2^k\|T_k\|}$ for every $n \in \mathbb{N}$.

Let us see that T^k is not compact for any $k \in \mathbb{N}$. To this end, let $(e_n^k)_{n=1}^{\infty}$ denote the sequence of norm one functions in L_φ, supported in $[2^{-m_{k+2}}, 2^{-m_{k+1}}]$, which corresponds to the unit vector basis of $\ell_{p_{m_{k+1}}}$. Hence, by construction $((T_{k+1})^k(e_n^k))_{n=1}^{\infty}$ correspond to the unit vector basis of $\ell_{p_{m_{k+2}}}$, which is weakly null and of norm one.

Now, if $N > k$ is sufficiently large so that $\Big\| T^k - \Big(\sum_{n=1}^{N} \frac{T_n}{2^n\|T_n\|}\Big)^k \Big\| < \frac{1}{2(2^k\|T_{k+1}\|)^k}$, then, using the fact that T_n acts only on functions supported in $[2^{-m_{n+1}}, 2^{-m_n}]$, it follows that

$$\begin{aligned} \|T^k(e_n^k)\| &\geq \Big\| \Big(\sum_{n=1}^{N} \frac{T_n}{2^n\|T_n\|} \Big)^k (e_n^k) \Big\| - \Big\| T^k(e_n^k) - \Big(\sum_{n=1}^{N} \frac{T_n}{2^n\|T_n\|} \Big)^k (e_n^k) \Big\| \\ &\geq \frac{\|(T_{k+1})^k(e_n^k)\|}{(2^k\|T_{k+1}\|)^k} - \frac{1}{2(2^k\|T_{k+1}\|)^k} \\ &= \frac{1}{2(2^k\|T_{k+1}\|)^k}. \end{aligned}$$

This means that $(T^k(e_n^k))$ is bounded away from zero for every $n \in \mathbb{N}$, so T^k is not compact. □

We do not know whether strictly singular non power-compact endomorphisms can be given on Orlicz spaces L_φ with equal indices $s(L_\varphi) = \sigma(L_\varphi)$.

5. Strictly singular non power-compact operators on r. i. spaces

Let us start by showing that in rearrangement invariant spaces the results above on the behavior of the iterations of a given operator turn into results on the behavior of the composition of (different) operators.

Proposition 5.1. *Given a rearrangement invariant space X on $[0,1]$ and $n \in \mathbb{N}$, the following statements are equivalent:*

(1) *If an operator $T \in \mathcal{S}(X)$, then the power T^n is compact.*
(2) *If $T_1, \dots, T_n$ belong to $\mathcal{S}(X)$, then the composition $T_n \dots T_1$ is compact.*

Proof. Let us prove the non-trivial implication. Consider

$$X_i = \left\{x \in X : x = x\chi_{[\frac{i}{n+1}, \frac{i+1}{n+1}]}\right\}$$

for $i = 0, 1, \dots, n$. Clearly, each X_i is isomorphic to X, so we can denote these isomorphisms by $J_i : X \to X_i$, and we can decompose $X = \bigoplus_{i=0}^n X_i$. Now, if $T_1, \dots, T_n$ belong to $\mathcal{S}(X)$, then we can consider the operator $T : \bigoplus_{i=0}^n X_i \to \bigoplus_{i=0}^n X_i$ given by the following matrix

$$T = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ J_1T_1J_0^{-1} & 0 & 0 & \cdots & 0 \\ 0 & J_2T_2J_1^{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J_nT_nJ_{n-1}^{-1} & 0 \end{pmatrix}$$

Since the operators T_i are strictly singular, T is strictly singular. By hypothesis, T^n is compact, and in matrix form this operator is the following

$$T^n = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ J_nT_n\cdots T_1J_0^{-1} & 0 & \cdots & 0 \end{pmatrix}$$

This implies that the composition $T_n \cdots T_1$ is compact, because J_i are isomorphisms. □

In particular it follows from Theorem 2.9 that for disjointly homogeneous rearrangement invariant spaces the composition of two strictly singular endomorphisms is compact.

Remark 5.2. An inspection of the proofs in Section 2 (in particular Theorem 2.9) shows that, in order to obtain compactness for higher powers of a strictly singular operator, the condition of disjointly homogeneous can be weakened as follows: assume that given $n \in \mathbb{N}$, for any family of n disjoint normalized sequences $(y_i^1)_i, \dots, (y_i^n)_i$ on a Banach lattice E, there are two indices $j_1, j_2 \in \{1, \dots, n\}$ such that $(y_i^{j_1})_i$ and $(y_i^{j_2})_i$ share an equivalent subsequence, then every $T \in \mathcal{S}(E)$ satisfies that $T^n \in \mathcal{K}(E)$. This clearly allows us to apply the results on Section 2 to operators defined on direct sums of disjointly homogeneous spaces.

The examples of r.i. function spaces with strictly singular and non power-compact endomorphisms given in previous sections have different lattice indices. In fact, these are Orlicz spaces with different indices containing complemented sublattices isomorphic to ℓ_p for different values of p (Proposition 4.5). In this Section we provide examples of rearrangement invariant spaces showing that these conditions are far from necessary.

Let us first recall an interpolation construction which is useful for finding r.i. spaces with certain given properties (see [23, Section 2.g] for details). Let (X_1, X_2) be an interpolation pair of Banach spaces. For positive scalars a, b let $k(\cdot, a, b)$ denote the norm on $X_1 + X_2$ given by

$$k(z, a, b) = \inf\{a\|x_1\|_{X_1} + b\|x_2\|_{X_2} : z = x_1 + x_2\}.$$

Let Y be a Banach space with a normalized unconditional basis (y_n) whose unconditional constant is one, and let $(a_n), (b_n)$ be sequences of positive scalars such that $\sum_{n=1}^{\infty} \min(a_n, b_n) < \infty$. Now, we define the space $K(X_1, X_2, Y, (a_n), (b_n))$ as the space of all elements $z \in X_1 + X_2$ such that $\sum_{n=1}^{\infty} k(z, a_n, b_n) y_n$ converges in Y, endowed with the norm

$$\|z\|_{K(X_1,X_2)} = \sup_m \Big\|\sum_{n=1}^{m} k(z, a_n, b_n) y_n\Big\|_Y.$$

This construction defines a Banach space which is an interpolation space between X_1 and X_2 [23, Proposition 2.g.4].

Hence, if X_1 and X_2 are r.i. spaces, so is $Z = K(X_1, X_2, Y, (a_n), (b_n))$. Moreover, if the sequences (a_n) and (b_n) are chosen properly, then Z contains a complemented subspace isomorphic to Y. Namely, if (m_n) denotes an increasing sequence of numbers satisfying the lacunarity condition

$$\frac{1}{m_n} \sum_{i=1}^{n-1} m_i + m_n \sum_{i=n+1}^{\infty} \frac{1}{m_i} < \frac{1}{2^{n+1}}, \tag{1}$$

then, as in the proof of [23, Theorem 2.g.5], we can construct a subspace of $K(X_1, X_2, Y, (\frac{1}{m_n}), (m_n))$ isomorphic to Y (see also [22, Proposition 3.b.4]).

Now, we can give an example of an r.i. space with equal Boyd indices and a strictly singular non power-compact endomorphism.

Proposition 5.3. *Given $1 < p < \infty$, there exists a rearrangement invariant space E on $[0, 1]$ with Boyd indices $q_E = p_E = p$ and an operator $T \in \mathcal{S}(E)$ which is not power-compact.*

Proof. It is well-known that the universal space U of Pełczyński, which contains a complemented subspace isomorphic to any Banach space with unconditional basis, can be represented as an r.i. function space on $[0, 1]$ (see [23, Theorem 2.g.5]). In fact, we can consider the interpolated space

$$E = K(L_p, L_\varphi, U, (\frac{1}{m_n}), (m_n)),$$

where $\varphi(t) = t^p \log t$ and (m_n) satisfy the lacunarity condition (1). By the previous comments and the universality of U, it follows that U is isomorphic to the space E, which is an r.i. space with Boyd indices $p_E = q_E = p$ because $p_{L_p} = q_{L_p} = p_{L_\varphi} = q_{L_\varphi} = p$.

Now, we can proceed as in Proposition 4.5, since the space E has complemented subspaces isomorphic to ℓ_{p_n} for any increasing sequence (p_n). Hence, we can construct an operator $T \in \mathcal{S}(E)$ which is not power-compact. □

Observe that in this last Proposition, despite that the Boyd indices coincide, the space E contains subspaces isomorphic to ℓ_p' s for different values of p. This condition can be removed as it is shown in the following result.

Proposition 5.4. *Given $1 \le p < \infty$ there exists a rearrangement invariant space X on $[0, 1]$ with lattice indices $s(X) = \sigma(X) = p$ and an operator $S \in \mathcal{S}(X)$ which is not power-compact.*

Before giving the construction of this space we need to recall some facts concerning Tsirelson-like spaces (see [4]). For every $\theta \in (0, 1)$, let us denote T_θ^p the p-convexified Tsirelson-like space T_θ, which is defined as follows. Recall that the space T_θ is the completion of the space of eventually null sequence of real numbers under the norm $\|\cdot\| = \lim_m \|\cdot\|_m$, where the norms $\|\cdot\|_m$ are given as follows. First, for a sequence $x = (a_n)$ and a set of natural numbers $E \subset \mathbb{N}$ we denote $Ex = (b_n)$ with

$$b_n = \begin{cases} a_n & \text{if } n \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can define for $x = (a_n)$ and each $m \geq 0$

$$\begin{cases} \|x\|_0 = \max_n |a_n|, \\ \|x\|_{m+1} = \max\{\|x\|_m, \max \theta \sum_{i=1}^{k} \|E_i x\|_m\}, \end{cases}$$

where the inner max is taken over all choices of $k \leq E_1 < E_2 < \ldots < E_k$ (as in the definition of Tsirelson's space). Let (t_n^θ) denote the unit vector basis of T_θ. Now, for $1 < p < \infty$, the space T_θ^p is defined as the p-convexification of T_θ, thus the norm of an element $x = (a_n)$ is given by

$$\|x\|_{T_\theta^p} = \| \sum_n |a_n|^p t_n^\theta \|_{T_\theta}^{1/p}.$$

We need now a lemma following [4, Theorem X.a.3].

Lemma 5.5. *For any $0 < \theta, \varphi < 1$ with $\theta \neq \varphi$, the spaces T_θ^p and T_φ^p are totally incomparable.*

Proof. Let us denote $(t_n^{(\theta,p)})_n$ the unit vector basis of the space T_θ^p. Suppose that there exists some infinite dimensional space Z which embeds isomorphically both in T_θ^p and T_φ^p. Hence, using [4, Proposition X.e.2.(a)] and taking appropriate subsequence we get that $(t_{k_n}^{(\theta,p)})_n$ and $(t_{k_n}^{(\varphi,p)})_n$ are equivalent basic sequences. As in the proof of [4, Theorem X.a.3], using [4, Proposition IV.c.8] we reach a contradiction. □

Proof of Proposition 5.4. First, fix a strictly decreasing sequence of numbers (θ_n) inside $(0, 1)$, and let us consider the space

$$Y = \bigoplus_{\ell_p} T_{\theta_n}^p.$$

That is, if (y_{ij}) denotes the double indexed sequence formed by one at the coordinates (i, j) and zero in the remaining coordinates, then for a double indexed sequence (a_{ij}) we have

$$\| \sum_{i,j} a_{ij} y_{ij} \|_Y = \big(\sum_i \| \sum_j a_{ij} t_j^{(\theta_i,p)} \|_{T_{\theta_i}^p}^p \big)^{\frac{1}{p}}.$$

Notice that for every $k \in \mathbb{N}$ we can consider a "truncated shift" of order k, $S_k : Y \to Y$ in the following way. First, let us denote $m_k = \sum_{n=1}^k n$. If $x = \sum_{i,j} a_{ij} y_{ij}$ denotes an element of Y, that is $(a_{ij})_j \in T_{\theta_i}^p$ for every i, then we set

$$S_k(x) = \sum_{i=m_k+1}^{m_{k+1}-1} \sum_j a_{(i-1)j} y_{ij}.$$

Clearly, by Lemma 5.5, for every $k \in \mathbb{N}$ the operator S_k is strictly singular and, since $(S_k)^k(y_{m_k j}) = y_{(m_{k+1}-1)j}$, then $(S_k)^k$ is not compact. Hence, arguing as in Proposition 4.5, we can define a strictly singular operator $S : Y \to Y$ that is not power-compact.

Moreover, by construction, since every T_θ^p is p-convex and q-concave for every $q > p$, so is Y. Moreover, $(y_{ij})_{i,j=1}^\infty$ is clearly an unconditional basis of Y. Hence, we can consider the space $X =$

$K(L_p, L_\varphi, Y, \{m_n^{-1}\}, \{m_n\})$, where $\varphi(t) = t^p \log t$ is an Orlicz function (of index p) and the sequence (m_n) satisfies the lacunarity condition (1).

Finally, by [23, Proposition 2.g.4 and 2.g.6], X is a rearrangement invariant space which is p-convex and q-concave for every $q > p$, and such that Y is complemented in X. This allows us to extend the operator $S : Y \to Y$ to an operator on X with the required properties. □

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