c_0-SINGULAR AND ℓ_1-SINGULAR OPERATORS BETWEEN VECTOR-VALUED BANACH LATTICES

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Abstract. Given an operator $T : X \to Y$ between Banach spaces, and a Banach lattice E consisting of measurable functions, we consider the point-wise extension of the operator to the vector-valued Banach lattices $T_E : E(X) \to E(Y)$ given by $T_E(f)(\omega) = T(f(\omega))$. It is proved that for any Banach lattice E which does not contain c_0, the operator T is an isomorphism on a subspace isomorphic to c_0 if and only if so is T_E. An analogous result for invertible operators on subspaces isomorphic to ℓ_1 is also given.

1. Introduction

In the 1970's S. Kwapien [13] and G. Pisier [16] gave the following results for vector valued L_p spaces.

Theorem 1.1. *For $1 \leq p < \infty$, $L_p(\mu, X)$ contains an isomorphic copy of c_0 if and only if X does.*

Theorem 1.2. *For $1 < p < \infty$, $L_p(\mu, X)$ contains an isomorphic copy of ℓ_1 if and only if X does.*

These results were also proved in a different way by J. Bourgain in [3] and [4]. More generally, given a Banach space X, and a Köthe Function space E over a measure space (Ω, Σ, μ), we denote by $E(X)$ the space of X-valued μ-measurable functions f, such that the mapping $\omega \mapsto \|f(\omega)\|_X$ belongs to E. $E(X)$ equipped with the norm

$$\|f\|_{E(X)} = \| \, \|f(\omega)\|_X \|_E$$

forms a vector-valued Banach lattice. W. Hensgen proved in [9] that $E(X)$ contains a subspace isomorphic to c_0 if and only if E or X do.

The relation between X and $E(X)$ has been extensively studied by several authors, see for instance [6], [15], and [17].

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Our aim in this note is to study whether this kind of results can be extended to the operator setting. Precisely, given an operator $T : X \to Y$ between Banach spaces, we will consider the operator

$$T_E : E(X) \to E(Y),$$

defined by $T_E(f)(\omega) = T(f(\omega))$ for any $f \in E(X)$ and $\omega \in \Omega$. Thus, we want to study which invertibility properties are shared by T and T_E.

Given a Banach space Z, we say that an operator $T : X \to Y$ is *Z-singular* whenever the restriction $T|_M$ is not an isomorphism for any subspace $M \subset X$ isomorphic to Z. The notion of ℓ_p-singular operator are of particular importance, since these have been used recently to study certain properties of strictly singular operators (see [7] and [11]).

The main results of this note are the following:

Theorem 1.3. *Let E be a Banach lattice which does not contain a subspace isomorphic to c_0, and let $T : X \to Y$ be an operator between Banach spaces. The operator $T_E : E(X) \to E(Y)$ is c_0-singular if and only if so is $T : X \to Y$.*

Theorem 1.4. *Let E be an order continuous Banach lattice, such that E^* is also order continuous, and let $T : X \to Y$ be an operator between Banach spaces. The operator $T_E : E(X) \to E(Y)$ is ℓ_1-singular if and only if so is $T : X \to Y$.*

As an application of Theorem 1.3, we also give a version, in the context of operators, of a result by Hoffmann-Jorgensen [10] for sums of vector-valued random variables (see Theorem 3.5).

Notice that the hypothesis imposed on Theorems 1.3 and 1.4 in the Banach lattices involved, allows us to use the techniques considered in the second section, where order continuity plays a key role. Recall that a Banach lattice which does not contain c_0 is in particular order continuous [14, 1.a], while for a dual Banach lattice E^* being order continuous and not containing c_0 are equivalent statements. Remarks 3.2 and 3.4 will further clarify these requirements.

We refer the reader to [2], and [14], for unexplained terms and notation on Banach lattices, and to [5] for a detailed survey on vector-valued L_p spaces and related questions.

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2. Tools

Recall that an order continuous Banach lattice E with a weak unit can be considered as a Köthe Function space, that is, an (in general not closed) order ideal of $L_1(\Omega, \Sigma, \mu)$ for certain probability space (Ω, Σ, μ), such that $E \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is continuous with norm smaller than or equal to one (see [14, Prop. 1.b.14] for details). Recall also that in an order continuous Banach lattice every ideal is complemented by a positive projection (see [14, 1.b]).

For the proof of Theorem 1.3, we need the following vector valued version of Kadec-Pelczynski disjointness method (see [8] and [12] for the classical version of this result).

Proposition 2.1. *Given a Banach space X. Let E be an order continuous Banach lattice over a finite measure space (Ω, Σ, μ), and let M be a separable subspace of $E(X)$. If we consider the formal inclusion $i : E(X) \hookrightarrow L_1(X)$, then one of the following holds:*

1. *the restriction $i|_M$ is an isomorphic embedding,*
2. *or there exist a normalized sequence $(f_n)_{n=1}^\infty$ in M and a disjoint sequence $(g_n)_{n=1}^\infty$ in $E(X)$, such that $\|f_n - g_n\|_{E(X)} \to 0$ when $n \to \infty$.*

Proof. Given $\varepsilon > 0$, and $f \in E(X)$, let us consider the set

$$\sigma(f, \varepsilon) = \{\omega \in \Omega : \|f(\omega)\|_X \geq \varepsilon \|f\|_{E(X)}\}.$$

Now for $\varepsilon > 0$, let us also consider the Kadec-Pelczynski sets

$$\mathcal{KP}(\varepsilon) = \{f \in E(X) : \mu(\sigma(f, \varepsilon)) \geq \varepsilon\}.$$

Now, given a separable subspace M of $E(X)$, let us suppose first that $M \subset \mathcal{KP}(\varepsilon)$, for some $\varepsilon > 0$. Hence for every $f \in M$ we have

$$\|f\|_{E(X)} \geq \|f\|_{L_1(X)} = \int_\Omega \|f(\omega)\|_X d\mu \geq \int_{\sigma(f,\varepsilon)} \|f(\omega)\|_X d\mu \geq \varepsilon^2 \|f\|_{E(X)}.$$

Therefore, in this case, the inclusion $i : E(X) \hookrightarrow L_1(X)$ when restricted to the subspace M is an isomorphic embedding.

Suppose now that M is not contained in $\mathcal{KP}(\varepsilon)$, for any $\varepsilon > 0$. Therefore, there exists f_1 in M with $f_1 \notin \mathcal{KP}(4^{-2})$ and $\|f_1\|_{E(X)} = 1$. Thus,

$$\mu(\sigma(f_1, 4^{-2})) < 4^{-2},$$

and

$$\|\chi_{\Omega \setminus \sigma(f_1, 4^{-2})} f_1\|_{E(X)} \leq 4^{-2}.$$

Since E is order continuous, there exists $\delta_1 > 0$ such that $\|\chi_A f_1\|_{E(X)} < 4^{-3}$, whenever $\mu(A) < \delta_1$. Let $j_2 < 2 = j_1$ be such that $4^{-j_2} < \delta_1$. Hence, there exists $f_2 \in M$ with $\|f_2\|_{E(X)} = 1$ and $f_2 \notin \mathcal{KP}(4^{-j_2})$, which in turn means that

$$\mu(\sigma(f_2, 4^{-j_2})) < 4^{-j_2} < \delta_1,$$

and so

$$\|\chi_{\sigma(f_2, 4^{-j_2})} f_1\| \leq 4^{-(j_1+1)}.$$

Moreover,

$$\|\chi_{\Omega \setminus \sigma(f_2, 4^{-j_2})} f_2\|_{E(X)} \leq \|4^{-j_2} \chi_{\Omega \setminus \sigma(f_2, 4^{-j_2})}\|_{E(X)} \leq 4^{-j_2}.$$

We can continue this construction inductively and we get a normalized sequence $(f_n)_{n=1}^\infty$ in M, and a sequence $(j_n)_{n=1}^\infty$ of natural numbers such that

(1) $\mu(\sigma(f_n, 4^{-j_n})) < 4^{-j_n}$,
(2) $\|\chi_{\Omega\setminus\sigma(f_n,4^{-j_n})} f_n\|_{E(X)} \leq 4^{-j_n}$,
(3) $\|\chi_{\sigma(f_n,4^{-j_n})} f_i\|_{E(X)} \leq 4^{-(j_{n-1}+1)}$, for $i = 1, \dots, n-1$.

Now, if we consider

$$\sigma_n = \sigma(f_n, 4^{-j_n}) - \bigcup_{i=n+1}^{\infty} \sigma(f_i, 4^{-j_i}),$$

then $\sigma_n \cap \sigma_m = \emptyset$ whenever $n \neq m$. Let us define $g_n = \chi_{\sigma_n} f_n$, which is a disjoint sequence in $E(X)$. Moreover, it holds

$$\begin{aligned} \|f_n - g_n\|_{E(X)} &= \|\chi_{\Omega\setminus\sigma_n} f_n\|_{E(X)} \leq \|\chi_{\Omega\setminus\sigma(f_n,4^{-j_n})} f_n\|_{E(X)} + \|\chi_{\bigcup_{i=n+1}^{\infty}\sigma(f_i,4^{-j_i})} f_n\|_{E(X)} \\ &\leq 4^{-j_n} + \sum_{i=n+1}^{\infty} \|\chi_{\sigma(f_i,4^{-j_i})} f_n\|_{E(X)} \leq 4^{-j_n} + \sum_{i=n+1}^{\infty} 4^{-(j_{i-1}+1)} \\ &\leq \tfrac{1}{3} 4^{-(j_n-1)}. \end{aligned}$$

Therefore, $\|f_n - g_n\| \to 0$ when $n \to \infty$, as claimed. □

Notice that if E is an order continuous Banach lattice defined over an infinite measure space (Ω, Σ, μ), and M is a separable subspace of $E(X)$, then there exists a closed order ideal I of E, which can be considered as a function space over a finite measure space $(\Omega_1, \Sigma_1, \mu_1)$, such that M is a subspace of $I(X) \subset E(X)$ (see [14, Prop. 1.a.9]).

The following property of disjoint sequences in $E(X)$ will be useful.

Lemma 2.2. *Let E be a Köthe Function space over (Ω, Σ, μ), and X a Banach space. Suppose that $(f_n)_{n=1}^{\infty}$ is a normalized disjoint sequence in $E(X)$, and denote $\varphi_n(\omega) = \|f_n(\omega)\|$ which is also disjoint and normalized. Then $(f_n)_{n=1}^{\infty}$ and $(\varphi_n)_{n=1}^{\infty}$ are 1-equivalent unconditional basic sequences.*

Proof. For each natural number n, since $f_1, \dots, f_n$ are disjoint elements of $E(X)$, we can consider $B_1, \dots, B_n \in \Sigma$ such that $\bigcup_{i=1}^{n} B_i = \Omega$ and f_i is supported on B_i, for each $i = 1, \dots, n$. Hence for scalars $(a_i)_{i=1}^{n}$, we have:

$$\begin{aligned} \Big\|\sum_{i=1}^{n} a_i f_i\Big\|_{E(X)} &= \Big\|\Big\|\sum_{i=1}^{n} a_i f_i(\omega)\Big\|_X\Big\|_E = \Big\|\sum_{j=1}^{n} \chi_{B_j}(\omega)\Big\|\sum_{i=1}^{n} a_i f_i(\omega)\Big\|_X\Big\|_E \\ &= \Big\|\sum_{j=1}^{n}\Big\|\sum_{i=1}^{n} a_i \chi_{B_j}(\omega) f_i(\omega)\Big\|_X\Big\|_E = \Big\|\sum_{j=1}^{n} |a_j| \|f_j(\omega)\|_X\Big\|_E \\ &= \Big\|\sum_{j=1}^{n} a_j \varphi_j(\omega)\Big\|_E. \end{aligned}$$

Since this holds for every n, and scalars $(a_i)_{i=1}^{n}$, the proof is finished. □

The next ingredient for the proof of Theorem 1.3 is the following extension of Theorem 1.1:

Proposition 2.3. *Given an operator $T : X \to Y$, if T is c_0-singular, then so is $T_{L_1} : L_1(X) \to L_1(Y)$.*

The proof is based on the following version of J. Bourgain's theorem on c_0-sequences obtained by averaging of seminorms [3]. Recall that c_{00} denotes the space of sequences of real numbers which are eventually zero, equipped with the supremum norm.

Lemma 2.4. *Let (Ω, Σ, μ) be a probability measure space. For every $\omega \in \Omega$ we consider two seminorms in c_{00}, ρ_ω and σ_ω, such that the functions $\omega \mapsto \rho_\omega(x)$ and $\omega \mapsto \sigma_\omega(x)$ are integrable in (Ω, Σ, μ) for every $x \in c_{00}$, and there exists a finite constant $C > 0$ such that*

$$\rho_\omega(x) \leq C\sigma_\omega(x),$$

for all $x \in c_{00}$.

Let us define two seminorms on c_{00} by

$$\|x\|_1 = \int_\Omega \rho_\omega(x) d\mu(\omega)$$

$$\|x\|_2 = \int_\Omega \sigma_\omega(x) d\mu(\omega)$$

for every $x \in c_{00}$. If $(x_i)_{i=1}^\infty$ is a sequence in c_{00} which is equivalent to the c_0-basis for both $\|\cdot\|_1$ and $\|\cdot\|_2$, then there exists a set of positive measure $A \in \Sigma$ such that for every $\omega \in A$ there is a subsequence of $(x_i)_{i=1}^\infty$ which is equivalent to the c_0-basis for both ρ_ω and σ_ω.

Proof. First note that the set

$$A = \{\omega \in \Omega : \limsup \rho_\omega(x_i) > 0\}$$

is clearly measurable and has positive measure [3, Lemma 2]. It also follows from [3] (see also [5, p. 53] for a more detailed explanation) that

$$\sup_n \int_0^1 \sigma_\omega\Big(\sum_{i=1}^n r_i(t) x_i\Big) dt = B < \infty$$

for almost all $\omega \in \Omega$.

Thus, for every $\omega \in A$ we have

$$\limsup \rho_\omega(x_i) > 0, \qquad \text{and} \qquad \sup_n \int_0^1 \rho_\omega\Big(\sum_{i=1}^n r_i(t) x_i\Big) dt \leq CB < \infty.$$

Therefore, applying [3, Lemma 3] to the seminorm ρ_ω, we obtain a subsequence $(x_{i_k})_{k=1}^\infty$ which is a c_0 sequence for ρ_ω, and satisfies $\rho_\omega(x_{i_k}) > \alpha$ for some $\alpha > 0$

and every natural number k. It follows from the fact that $\rho_\omega(x) \leq C\sigma_\omega(x)$ and [5, Prop. 2.1.1. and Prop. 2.1.2.] that

$$\sigma_\omega(x_{i_k}) > C^{-1}\alpha > 0, \qquad \text{and} \qquad \sup_n \int_0^1 \sigma_\omega\Big(\sum_{k=1}^n r_k(t)x_{i_k}\Big)dt \leq B < \infty,$$

for every k. Hence, another application of [3, Lemma 3] gives a further subsequence (still denoted $(x_{i_k})_{k=1}^\infty$) which is a c_0-sequence for σ_ω. Thus, this sequence is a c_0-sequence for both ρ_ω and σ_ω, and the proof is finished. □

Now we can prove Proposition 2.3.

Proof of Proposition 2.3. Suppose that T_{L_1} is not c_0-singular. Let $(f_i)_{i=1}^\infty$ be a sequence in $L_1(X)$ such that for some $\delta > 0$ and $M > 0$, and for every $(a_i)_{i=1}^n \in c_{00}$:

$$\begin{aligned}\delta \max_{1\leq i\leq n}|a_i| \leq \Big\|\sum_{i=1}^n a_i T_{L_1}(f_i)\Big\|_{L_1(Y)} &= \int_\Omega \Big\|\sum_{i=1}^n a_i T(f_i(\omega))\Big\| d\mu(\omega)\\ &\leq \|T\| \int_\Omega \Big\|\sum_{i=1}^n a_i f_i(\omega)\Big\| d\mu(\omega) = \|T\| \Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_1(X)}\\ &\leq \|T\| M \max_{1\leq i\leq n}|a_i|.\end{aligned}$$

Let us define for each $\omega \in \Omega$, and each $x = \sum_{i=1}^n a_i e_i \in c_{00}$

$$\rho_\omega(x) = \Big\|\sum_{i=1}^n a_i T(f_i(\omega))\Big\|_Y,$$

and

$$\sigma_\omega(x) = \Big\|\sum_{i=1}^n a_i f_i(\omega)\Big\|_X.$$

Clearly, for every $x \in c_{00}$, the functions $\omega \mapsto \rho_\omega(x)$ and $\omega \mapsto \sigma_\omega(x)$ are integrable on (Ω, Σ, μ), and satisfy

$$\rho_\omega(x) \leq \|T\|\sigma_\omega(x).$$

Hence, we can consider the seminorms $\|\cdot\|_1$ and $\|\cdot\|_2$, as defined in Lemma 2.4. It follows that the unit vector sequence (e_i) in c_{00} is a c_0-sequence for both $\|\cdot\|_1$ and $\|\cdot\|_2$, because

$$\Big\|\sum_{i=1}^n a_i e_i\Big\|_1 = \int_\Omega \rho_\omega\Big(\sum_{i=1}^n a_i e_i\Big)d\mu(\omega) = \int_\Omega \Big\|\sum_{i=1}^n a_i T(f_i(\omega))\Big\| d\mu(\omega) = \Big\|\sum_{i=1}^n a_i T_{L_1}(f_i)\Big\|_{L_1(Y)}$$

$$\Big\|\sum_{i=1}^n a_i e_i\Big\|_2 = \int_\Omega \sigma_\omega\Big(\sum_{i=1}^n a_i e_i\Big)d\mu(\omega) = \int_\Omega \Big\|\sum_{i=1}^n a_i f_i(\omega)\Big\| d\mu(\omega) = \Big\|\sum_{i=1}^n a_i f_i\Big\|_{L_1(X)}$$

Now, Lemma 2.4 implies that the set of points $\omega \in \Omega$ such that $(e_i)_{i=1}^\infty$ has a subsequence which is a c_0-basis for both ρ_ω and σ_ω is a non null set. Thus, for every ω in this set, there exists an increasing sequence $(i_k)_{k=1}^\infty$ such that $(f_{i_k}(\omega))_{k=1}^\infty$ and

$(T(f_{i_k}(\omega)))_{k=1}^{\infty}$ are (non null) c_0-sequences. This implies that T is an isomorphism when restricted to the span of $(f_{i_k}(\omega))_{k=1}^{\infty}$ in X. □

Note that in fact, we proved more than it was claimed. It was shown that if $T_{L_1} : L_1(X) \to L_1(Y)$ is an isomorphism on the span of a sequence $[f_n]_{n=1}^{\infty} \subset L_1(X)$, which is isomorphic to c_0, then the set of all $\omega \in \Omega$ such that $T : X \to Y$ is an isomorphism on the span of a subsequence of $(f_n(\omega))_{n=1}^{\infty}$ (which is isomorphic to c_0) is a set of positive measure.

3. Main results

Now, we can give the proofs of our main results.

Theorem 3.1. *Let E be a Banach lattice not containing a subspace isomorphic to c_0. Let $T : X \to Y$ be an operator between Banach spaces. If the operator T is c_0-singular, then the same holds for $T_E : E(X) \to E(Y)$.*

Proof. Let $T : X \to Y$ be a bounded operator. And let $T_E : E(X) \to E(Y)$ be such that there exists a subspace M of $E(X)$, which is isomorphic to c_0, and the restriction $T_E|_M : M \to E(Y)$ is an isomorphic embedding.

Since c_0 is not contained in E, in particular E is order continuous [14, 1.a]. Hence, by Proposition 2.1 applied to $M \subset E(X)$, it follows that either $i : E(X) \hookrightarrow L_1(X)$ is an isomorphism when restricted to M or M contains a normalized sequence $(f_n)_{n=1}^{\infty}$, such that there exists a disjoint sequence $(g_n)_{n=1}^{\infty}$ in $E(X)$ with $\|f_n - g_n\|_{E(X)} \to 0$ when $n \to \infty$.

Suppose that $i : E(X) \hookrightarrow L_1(X)$ is not an isomorphism when restricted to M. Therefore, passing to a further subsequence we can assume that the basic sequences $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are equivalent. Since M is isomorphic to c_0, this means that $E(X)$ contains a disjoint sequence equivalent to the unit vector basis of c_0. Hence, by Lemma 2.2, E would also contain a disjoint sequence equivalent to the unit vector basis of c_0, which is a contradiction with the hypothesis on E.

Thus, we can assume that $i : E(X) \hookrightarrow L_1(X)$ is an isomorphism when restricted to M. The same argument shows that $i : E(Y) \hookrightarrow L_1(Y)$ is an isomorphism when restricted to $T_E(M)$ (which is also isomorphic to c_0).

Therefore, the operator $T_{L_1} : L_1(X) \to L_1(Y)$ is an isomorphism when restricted to M, which is isomorphic to c_0. Hence, by Proposition 2.3, we can conclude that $T : X \to Y$ is an isomorphism on a subspace isomorphic to c_0. □

Remark 3.2. Notice that if the Banach lattice E contains a subspace isomorphic to c_0, then the statement of Theorem 3.1 may fail to be true. Indeed, the identity $I : c_0 \to c_0$ can be seen as the extension T_{c_0} of the identity map on the scalar field $T : \mathbb{R} \to \mathbb{R}$, which clearly is not an isomorphism on a subspace isomorphic to c_0 (c_0 is just too big!).

This theorem has a natural analogue for ℓ_1-singular operators.

Theorem 3.3. *Let E be an order continuous Banach lattice, such that E^* is also order continuous. Let $T : X \to Y$ be an operator between Banach spaces. If T is ℓ_1-singular, then so is $T_E : E(X) \to E(Y)$.*

Proof. Let M be a subspace of $E(X)$ isomorphic to ℓ_1, such that $T_E : E(X) \to E(Y)$ is an isomorphism when restricted to M. Both M and $T_E(M)$ satisfy one of the alternatives of Proposition 2.1. Since E^* is order continuous, it follows that E cannot contain a sequence of disjoint elements whose closed linear span is isomorphic to ℓ_1 (cf. [2, Thm. 14.21]). Therefore, by Lemma 2.2, M cannot contain a normalized sequence equivalent to a disjoint sequence. Hence, the inclusion $i_X : E(X) \hookrightarrow L_1(X)$ is an isomorphism when restricted to M, and similarly $i_Y : E(Y) \hookrightarrow L_1(Y)$ is an isomorphism when restricted to $T_E(M)$.

Let $(f_n)_{n=1}^\infty$ in M be equivalent to the unit vector basis of ℓ_1. Since E and E^* are order continuous $(\|f_n(\cdot)\|_X)_{n=1}^\infty$ and $(\|T(f_n(\cdot))\|_Y)_{n=1}^\infty$ are uniformly integrable sequences in L_1 (cf. [2, Thm. 4.25] and [1, Thm. 5.2.9]). Hence, by [5, Thm. 2.2.1(a)], the set A of all $\omega \in \Omega$ such that $(T(f_n(\omega)))_{n=1}^\infty$ has a subsequence equivalent to the unit vector basis of ℓ_1 is a measurable set with positive measure.

Hence, $T(X)$ contains a subspace isomorphic to ℓ_1; thus, using [18, Prop. 1] we get that $T : X \to Y$ preserves a copy of ℓ_1.

Let $B \subset A$ be a set of positive measure, such that for some constant $C < \infty$, and for every $n \in \mathbb{N}$, $\|f_n(\omega)\|_X \leq C$ for $\omega \in B$. Therefore, for each $\omega \in B$, passing to a further subsequence, that may depend on ω, and for scalars $(a_k)_{k=1}^n$, we have:

$$\alpha(\omega)\sum_{k=1}^n |a_k| \leq \Big\|\sum_{k=1}^n a_k T(f_{n_k}(\omega))\Big\|_Y \leq \|T\| \Big\|\sum_{k=1}^n a_k (f_{n_k}(\omega))\Big\|_Y \leq C\|T\|\sum_{k=1}^n |a_k|,$$

where $\alpha(\omega) > 0$ for $\omega \in B$. Hence, $T : X \to Y$ is an isomorphism when restricted to a subspace isomorphic to ℓ_1. □

Remark 3.4. As for Theorem 3.1, the identity on ℓ_1, seen as the extension T_{ℓ_1} of the identity map $T : \mathbb{R} \to \mathbb{R}$ on the scalar field, shows that the hypothesis of order continuity on E^* cannot be removed from Theorem 3.3.

In connection with Theorem 3.1, we have a version for operators of Hoffmann-Jorgensen's result (see [10]).

Theorem 3.5. *Let $T : X \to Y$ be an operator between Banach spaces, and let (Ω, Σ, μ) be a probability space. The following are equivalent:*

1. *T is c_0-singular.*
2. *For every sequence $(X_n)_{n=1}^\infty$ of independent, symmetric, X-valued random variables on (Ω, Σ, μ), if the partial sums*

$$S_m = \sum_{n=1}^m X_n$$

are bounded almost everywhere, then $(T(S_m))_{m=1}^{\infty}$ *converges almost everywhere.*

Proof. (2) $\Rightarrow$ (1) is easy to see. For the implication (1) $\Rightarrow$ (2), let $(\epsilon_j)_{j=1}^{\infty}$ be a Bernoulli sequence on (Ω, Σ, μ), that is, a sequence of independent random variables so that $\mu(\epsilon_j = 1) = \mu(\epsilon_j = -1) = \frac{1}{2}$ for all $j \geq 1$. By [10, Prop. 2.8] it suffices to prove that the sets

$$A = \left\{ (x_j)_{j=1}^{\infty} \subset X : \left(\sum_{j=1}^{n} \epsilon_j x_j \right)_{n=1}^{\infty} \text{ is bounded in } L_p(X) \right\},$$

and

$$B = \left\{ (x_j)_{j=1}^{\infty} \subset X : \sum_{j=1}^{\infty} \epsilon_j T x_j \text{ is convergent in } L_p(Y) \right\},$$

coincide (notice that by [10, Thm 3.1], there is no difference in the choice of $0 \leq p < \infty$).

So, suppose that there exists $(x_j)_{j=1}^{\infty}$ in A and not in B. Since in particular, $\sum_{j=1}^{\infty} \epsilon_j T x_j$ is not convergent in $L_1(Y)$, there exist $\delta > 0$ and a subsequence such that

$$\int_{\Omega} \Big\| \sum_{n_k \leq j < n_{k+1}} \epsilon_j T x_j \Big\|_Y d\mu \geq \delta,$$

for $k \in \mathbb{N}$. Now, let

$$X_k = \sum_{n_k \leq j < n_{k+1}} \epsilon_j x_j, \quad \text{and} \quad Y_k = \sum_{n_k \leq j < n_{k+1}} \epsilon_j T x_j,$$

for $k \in \mathbb{N}$. Clearly $(X_k(\omega))_{k=1}^{\infty}$ belongs to A μ-a.e. However, [10, Thm. 3.1] yields that $\mu(Y_k \nrightarrow 0) > 0$.

Therefore, by scaling, we can consider $(z_j)_{j=1}^{\infty}$ in A such that $\|T z_j\| = 1$, for $j \in \mathbb{N}$. Now, by [10, Thm. 2.6], we have

$$|a_j| = \left(\int_{\Omega} \|a_j \epsilon_j(\omega) T z_j\|_Y^p d\mu \right)^{\frac{1}{p}} \leq \left(\int_{\Omega} \Big\| \sum_{j=1}^{n} a_j \epsilon_j(\omega) T z_j \Big\|_Y^p d\mu \right)^{\frac{1}{p}},$$

for $1 \leq j \leq n$ and scalars $(a_j)_{j=1}^{n}$. While [10, Lemma 4.1] yields

$$\left(\int_{\Omega} \Big\| \sum_{j=1}^{n} a_j \epsilon_j(\omega) z_j \Big\|_Y^p d\mu \right)^{\frac{1}{p}} \leq \max_{1 \leq j \leq n} |a_j| \left(\int_{\Omega} \Big\| \sum_{j=1}^{n} \epsilon_j(\omega) z_j \Big\|_Y^p d\mu \right)^{\frac{1}{p}} \leq \max_{1 \leq j \leq n} |a_j| K,$$

where

$$K = \sup_n \left(\left(\int_{\Omega} \Big\| \sum_{j=1}^{n} \epsilon_j(\omega) z_j \Big\|_Y^p d\mu \right)^{\frac{1}{p}} \right) < \infty,$$

since $(z_j)_{j=1}^{\infty} \in A$.

Hence, if we consider $T_{L_p} : L_p(X) \to L_p(Y)$ defined as usual, then we have

$$\max_{1\leq j\leq n} |a_j| \leq \Big\| T_{L_p}\Big(\sum_{j=1}^n a_j\epsilon_j z_j\Big)\Big\|_{L_p(Y)} \leq \|T\| \Big\|\sum_{j=1}^n a_j\epsilon_j z_j\Big\|_{L_p(X)} \leq \|T\|K \max_{1\leq j\leq n} |a_j|.$$

This shows that the operator T_{L_p} is an isomorphism on the subspace generated by $(\epsilon_j z_j)_{j=1}^\infty$ which is isomorphic to c_0. Therefore, by Theorem 3.1, $T : X \to Y$ is also an isomorphism on a subspace isomorphic to c_0. This finishes the proof. □

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