DOMINATION PROBLEMS FOR STRICTLY SINGULAR OPERATORS AND OTHER RELATED CLASSES

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Abstract. We survey recent results on domination properties of strictly singular operators and related operator ideals, as well as Banach-Saks operators, Narrow operators and p-summing operators.

Dedicated to the memory of Charalambos D. Aliprantis

1. Introduction

A central question in the theory of Positive Operators between Banach lattices is the *domination* problem: Let $0 \leq R \leq T : E \to F$ be positive operators between Banach lattices. Assume that T satisfies some property $(*)$.

(Q) Does the operator R inherit the property $(*)$?

It may be the case that for certain properties the answer is the best possible, i.e. (Q) has a positive answer. For example this is the case for the class of integral operators (those representable as $Tf(x) = \int K(x,.)f(.)d\mu$). But in general, the answer to (Q) can be negative and thus the problem turns into determining the weakest conditions on the Banach lattices involved in order to have a positive answer.

The domination problem for the class of *compact* operators was solved by P. Dodds and D. Fremlin in their seminal paper [14]: Let E and F be Banach lattices with E^* and F order continuous, and consider $0 \leq R \leq T : E \to F$. If T is compact, then R is also compact. In the special setting of $L^p(\mu)$-spaces this was also solved independently by L. Pitt in [42].

These compactness domination results have been applied in several different areas such as mathematical biology (see the survey [3]).

The domination for *weakly compact operators* was first considered by Y. Abramovich in [1], showing that if F does not contain a subspace isomorphic to c_0 then $0 \leq R \leq T : E \to F$ implies that R is weakly compact whenever T is. This result was later improved by A. Wickstead in [50], characterizing the domination property of weakly compact operators by the fact that E^* or F are order continuous.

For the class of *Dunford-Pettis* operators, the first domination results go back to [6]. These were later improved by N. Kalton and P. Saab [32] showing that if F is order continuous and T is Dunford-Pettis then R is also Dunford-Pettis (see [53]

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in this volume for more comments on the history of these results). Note also that converse domination results have been given in [51], [52] and [8].

A related question is the so-called *power problem* for dominated endomorphisms: Given $0 \leq R \leq T : E \longrightarrow E$ it is interesting to study whether some iteration (power) of the operator R inherits the property (*) of the operator T, under *no assumptions* on the Banach lattice E.

This approach was first developed by C. D. Aliprantis and O. Burkinshaw in [4] and [5] for compact and weakly compact operators obtaining

- If T is compact then R^3 is also compact.
- If T is weakly compact then R^2 is also weakly compact.

These results are also shown to be optimal. Moreover, the power problem for the class of Dunford-Pettis operators was also studied by N. Kalton and P. Saab in ([32]): If T is Dunford-Pettis, then R^2 is also Dunford-Pettis.

The aim of this paper is to survey several recent results on domination for other important operator classes. Thus Section 3 is devoted to *strictly singular* operators with some remarks on the related classes of *strictly co-singular* and *super-strictly singular* operators. In Section 4, the domination of *Banach-Saks* operators is considered. Section 5 deals with *Narrow* operators and Section 6 with the class of *p-summing* operators. Last Section collects several open problems on this topic. Finally, we would like to mention that there are other domination results for operator classes which have not been considered here (for example, for Asplund and Radon-Nikodym operators in [33]).

We refer the reader to the monographs [2], [7], [35], [37] and [54] for unexplained terminology from Banach lattices and positive operators theory.

2. Preliminaries

In this section we recall some notions and tools required throughout this paper. First, recall the *Kadeč-Pełczyński disjointification* method for order continuous Banach lattices (cf. [35]): Let X be any subspace of an order continuous Banach lattice E. Then, either

(1) X contains an almost disjoint normalized sequence, that is, there exist a normalized sequence $(x_n)_n \subset X$ and a disjoint sequence $(z_n)_n \subset E$ such that $\|z_n - x_n\| \to 0$, or,
(2) X is isomorphic to a closed subspace of $L_1(\Omega, \Sigma, \mu)$.

Notice that if X is separable, then it can be included in some ideal H of E with a weak order unit. Therefore, this ideal has a representation as a Köthe function space over a finite measure space (Ω, Σ, μ) ([35, Thm. 1.b.14]), and in this case the previous dichotomy says that either X contains an almost disjoint sequence or the natural inclusion $J : H \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is an isomorphism when restricted to X.

If E is an order continuous Banach function space on a finite measure space (Ω, Σ, μ), a bounded subset $A \subset E$ is *equi-integrable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f\chi_B\|_E < \varepsilon$ for every $B \in \Sigma$ with $\mu(B) < \delta$ and every $f \in A$.

An order continuous Banach lattice E satisfies the *subsequence splitting property* (cf.[49]) if for every bounded sequence $(f_n)_n$ in E there is a subsequence $(n_k)_k$ and sequences $(g_k)_k$, $(h_k)_k$ in E with $|g_k| \wedge |h_k| = 0$ and

$$f_{n_k} = g_k + h_k$$

such that $(g_k)_k$ is equi-integrable and $|h_k| \wedge |h_l| = 0$ if $k \neq l$.

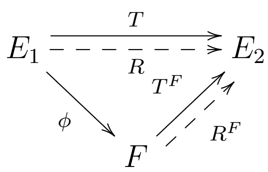
Every p-concave Banach lattice ($p < \infty$) has the subsequence splitting property (cf. [49]). Recall that E is p-concave if there exists a constant $M < \infty$ such that for every choice of elements $(x_i)_{i=1}^n$ we have

$$\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p} \leq M \left\| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \right\|.$$

Given a Banach lattice E and a Banach space Y, an operator $T : E \to Y$ is *order weakly compact* if $T[-x, x]$ is relatively weakly compact for every $x \in E_+$. Order weakly compact operators can be characterized as those operators not preserving a positive disjoint order-bounded isomorphic copy of c_0 (cf. [37, Cor.3.4.5]). Also, if X is a Banach space and F a Banach lattice, an operator $T : X \to F$ does not preserve an isomorphic copy of ℓ_1 which is complemented in F if and only if its adjoint T^* is order weakly compact (cf. [37, Thm.3.4.14]).

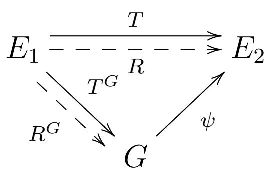
The following factorization results for positive order weakly compact operators are useful tools ([24]):

Theorem 2.1. *Let E_1, E_2 be Banach lattices and consider operators $0 \leq R \leq T : E_1 \to E_2$. There exist a Banach lattice F, a lattice homomorphism $\phi : E_1 \to F$ and operators $0 \leq R^F \leq T^F$ such that $T = T^F \phi$ and $R = R^F \phi$:*



Moreover, $T : E_1 \to E_2$ is order weakly compact if and only if F is order continuous.

Theorem 2.2. *Let E_1, E_2 be Banach lattices and consider operators $0 \leq R \leq T : E_1 \to E_2$. There exist a Banach lattice G, a lattice homomorphism $\psi : G \to E_2$ and operators $0 \leq R^G \leq T^G$ such that $T = \psi T^G$ and $R = \psi R^G$:*



Moreover $T^* : E_2^* \to E_1^*$ is order weakly compact if and only if G^* is order continuous.

Considering the domination problem for an operator ideal (in the sense of Pietsch), some comments are in order. Suppose we have two positive operators $0 \leq R \leq T : E \to F$, when does R belong to the closed ideal generated by T? The investigation of this question in the general framework of operator ideals is of importance in studying specific domination problems.

Freudenthal's theorem addresses this issue. Briefly recall that if $T : G \to F$ is a positive operator between two vector lattices G and F with F Dedekind-complete, then a positive operator $S : G \to F$ is said to be a *component* of T if $S \wedge (T - S) = 0$ holds in $\mathcal{L}^r(G, F)$. The operators of the form QTP, where Q is a band projection on F and P is a band projection on G, are called *elementary components* of T. A *simple* *component* of T is a component of the form $\bigvee_{i=1}^{n} Q_iTP_i$. The collection of all simple components (resp. components) of T will be denoted by $\mathcal{S}_T$ (resp. $\mathcal{C}_T$). A regular operator S from G to F is said to be a *T-step operator* if there exist pairwise disjoint components $T_1, \dots, T_n$ of T with $T_1 + \cdots + T_n = T$, and real numbers $\alpha_1, \dots, \alpha_n$ satisfying

$$S = \sum_{i=1}^{n} \alpha_i T_i.$$

With this notation Freudenthal's result ([37, Section 1.2]) is recalled next

Theorem 2.3 (Freudenthal)**.** *Let $R, T : G \to F$ be positive operators between two vector lattices, with F Dedekind-complete. If $0 \leq R \leq T$ holds, then there exists a sequence $(S_n)_n$ of T-step operators satisfying*

$$0 \leq R - S_n \leq n^{-1}T \quad \textit{for each } n, \quad \textit{and} \quad 0 \leq S_n \uparrow R.$$

Notice that condition $0 \leq R - S_n \leq n^{-1}T$ implies that $R - S_n$ tends to zero with respect to the operator norm by the equality $\|R - S_n\| = \sup\{\|(R - S_n)x\| : x \in ball(G)_+\}$. In particular this means that as long as the T-step operators belong to the (norm-closed) class to which T belongs we can conclude that R itself belongs to the same class. Therefore, we are interested in the case when order approximation by operators in the algebraic ideal of T yields norm approximation, and we say that an operator T has *order continuous norm* whenever every sequence of positive operators with $|T| \geq T_n \downarrow 0$ in $\mathcal{L}^r(E, F)$ satisfies $\|T_n\| \downarrow 0$. Consider the set

$$I_T = \{S \in \mathcal{L}^r(E, F) : \text{there exists } n \in \mathbb{N} \text{ such that } |S| \leq n|T|\},$$

and denote by $Ring(T)$ the closure in $\mathcal{L}(E, F)$ of the set of operators of the form $\sum_{i=1}^{n} R_iTS_i$ with $S_i \in \mathcal{L}(E)$, $R_i \in \mathcal{L}(F)$. The following theorem sumarizes the previous lines (see [7, Thm. 5.70]).

Theorem 2.4. *Let E be a Banach lattice which is either σ-Dedekind complete or has a quasi-interior point, and let F be a Dedekind complete Banach lattice. If T has order continuous norm, then*

$$I_T \subseteq Ring(T).$$

P. Dodds and D. Fremlin [14, Thm. 5.1] provided a very useful characterization of the order continuity of the norm of an operator. Recall that an operator between Banach lattices $T : E \to F$ is *M-weakly compact* if $\|Tx_n\| \to 0$ for every norm bounded disjoint sequence $(x_n)_n$ in E. An operator $T : E \to F$ is *L-weakly compact* if every disjoint sequence in the solid hull of $T(B_E)$ tends to zero in F.

Theorem 2.5. *Let E and F be Banach lattices with F Dedekind complete. An operator $T : E \to F$ has order continuous norm if and only if it is both L-weakly compact and M-weakly compact.*

These techniques for approximation of bounded operators have been further generalized by N. Kalton and P. Saab. The main result in this direction is the following (see [32]).

Theorem 2.6. *Let E and F be Banach lattices with quasi-interior points. Let $T : E \to F$ be a positive operator, and let $A \subset E$ and $B \subset F^*$ be solid bounded sets such that for every positive disjoint sequences $(a_n)_n$ in A, and $(b_n)_n$ in B the following conditions hold:*

(1) $Ta_n \to 0$ *in the weak topology,*
(2) $T^*b_n \to 0$ *in the weak***-topology,*
(3) $\langle Ta_n, b_n\rangle \to 0$.

Suppose that $S, R : E \to F$ *satisfy* $|S| \le |R| \le T$ *in* $\mathcal{L}(E, F^{**})$. *Then, for every* $\varepsilon > 0$ *there exist multipliers* $M_1, \dots, M_n \in \mathcal{L}(E)$, *and* $L_1, \dots, L_n \in \mathcal{L}(F)$ *such that*

$$S_0 = \sum_{i=1}^{n} L_i R M_i$$

satisfies

$$|\langle Sa - S_0 a, b\rangle| \le \varepsilon, \text{ for every } a \in A, \text{ and } b \in B.$$

This result was used in [32] to provide a domination theorem for Dunford-Pettis operators.

3. Domination by strictly singular operators

In this section we consider the problem of domination by strictly singular operators and some related classes.

Recall that an operator $T : X \to Y$ between Banach spaces is said to be *strictly singular* (or *Kato*) if for every infinite dimensional (closed) subspace M of X, the restriction $T|_M$ is not an isomorphism into Y, i.e. there is no infinite dimensional subspace M of X with a constant $c > 0$ such that for every $x \in M$

$$c\|x\| \le \|Tx\|$$

This class forms a closed operator ideal, which properly contains the ideal of compact operators. For example the inclusion operators $L^\infty[0,1] \hookrightarrow L^p[0,1]$, $1 \le p < \infty$ are strictly singular (but no compact) (cf. [47] Thm.5.2). Strictly singular operators are very useful in Fredholm and perturbation theory (cf. [2], [34]). It is well-known that an operator $T : X \to Y$ between Banach spaces is strictly singular if and only if for every infinite dimensional subspace M of X there exists another infinite dimensional subspace $N \subset M$ such that the restriction $T|_N$ is compact. In general the fact that T (resp. the adjoint T^*) is strictly singular does not imply that T^* (resp. T) is strictly singular.

Let us give some examples of strictly singular inclusions between *rearrangement invariant* (r.i) function spaces E (see [35] for definitions). In the *finite* measure case the canonical inclusions $L^\infty[0,1] \hookrightarrow E[0,1]$ are always strictly singular for any r.i. space $E \ne L^\infty$ ([38]). On the other side inclusion $E[0,1] \hookrightarrow L^1[0,1]$ is strictly singular if and only if $E[0,1]$ does not contain the order continuous Orlicz space $L_0^{\exp x^2}[0,1]$ ([26]).

In the *infinite* measure case the inclusions $L^1 \cap L^\infty \hookrightarrow E \hookrightarrow L^1 + L^\infty$ hold. The strict singularity of the left inclusions $L^1 \cap L^\infty \hookrightarrow E$ is characterized in [28] in terms of the associated fundamental function: $\lim\limits_{t\to 0} \phi_E(t) = \lim\limits_{t\to\infty} \frac{\phi_E(t)}{t} = 0$. The behavior of the right inclusions $E \hookrightarrow L^1 + L^\infty$ is more complicated, since the corresponding criteria involve other conditions than the mere behavior of the associated fundamental functions (see [29]).

In the context of Banach lattices, a useful variant of strict singularity is the following ([27]): given a Banach lattice E and a Banach space Y, an operator

$T : E \to Y$ is called *disjointly strictly singular* if it is not invertible on any subspace of E generated by a disjoint sequence.

Clearly, every strictly singular operator is also disjointly strictly singular, but the converse is not true. For example the inclusions $L^q[0,1] \hookrightarrow L^p[0,1]$, $1 \le p < q < \infty$, are disjointly strictly singular but not strictly singular. This follows from Khintchine's inequality for the Rademacher functions (r_n):

$$\Big(\int_0^1 |\sum_{n=1}^{\infty} a_n r_n|^p \, d\mu\Big)^{1/p} \sim \big(\sum_{n=1}^{\infty} |a_n|^2\big)^{1/2}$$

In some special cases, both concepts coincide: for instance for spaces with a Schauder basis of disjoint vectors or for $C(K)$-spaces. The class of all disjointly strictly singular operators is stable by addition and by composition on the right but in general it is not an operator ideal.

We present now recent domination results for positive strictly singular operators between Banach lattices. First let us show that this is not true in general:

Example 3.1. *There exist operators* $0 \le R \le T : \ell^1 \to L^\infty[0,1]$ *such that* T *is strictly singular but* R *is not.*

Indeed, take $\widetilde{R} : \ell^1 \to L^\infty[0,1]$ the isometry defined by $\widetilde{R}(e_n) = r_n$. Consider also the positive operators $R_1, R_2 : \ell^1 \to L^\infty[0,1]$ defined by $R_1(e_n) = r_n^+$ and $R_2(e_n) = r_n^-$ respectively, where r_n^+ and r_n^- are the positive and negative part of r_n. Clearly $\widetilde{R} = R_1 - R_2$. Moreover $0 \le R_1, R_2 \le T$, where T is the rank-one operator:

$$T(x) = \Big(\sum_{n=1}^{\infty} x_n\Big)\chi_{[0,1]}.$$

The operator T is clearly strictly singular, but neither the operator R_1 nor R_2 are strictly singular. Now the equalities $T = R_1 + R_2$ and $\widetilde{R} = R_1 - R_2$ proof the claim.

Using the above, and the existence of lattices isomorphic copies of ℓ^∞ in non order continuous Dedekind complete Banach lattices, a general negative result can be given:

Proposition 3.2. *Let* E *and* F *two Banach lattices with* F *Dedekind-complete; assume that neither* E^* *nor* F *are order continuous. Then there exist two positive operators* $0 \le R \le T : E \to F$ *such that* T *is strictly singular but* R *is not.*

An important step in the obtention of our domination theorem is the following domination result for the class of disjointly strictly singular operators ([16]):

Theorem 3.3. *Let* E *and* F *be Banach lattices such that* F *is order continuous. If* T *is disjointly strictly singular and* $0 \le R \le T : E \to F$ *then* R *is also disjointly strictly singular.*

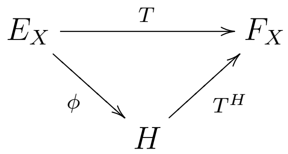
Another important step is to consider the case when the range space is an $L^1(\mu)$-space, or more generally, a space with the positive Schur property. Recall that a Banach lattice E has the *positive Schur property* if every positive weakly null sequence is convergent. Other examples of Banach lattices with the positive Schur property are the Orlicz spaces $L^{x \log^p(1+x)}[0,1]$, $p > 0$, and the Lorentz spaces $L^{p,1}[0,1]$, $1 < p < \infty$. Observe that the positive Schur property implies that E does not contain an isomorphic copy of c_0 (in particular E is order continuous).

In this context the following result was given in [18]; we include here an improved version of the proof.

Proposition 3.4. *Let E and F be Banach lattices such that F has the positive Schur property. If $0 \leq R \leq T : E \to F$ and T is strictly singular, then R is also strictly singular.*

Proof. Suppose that R is not strictly singular, then, there exists an infinite-dimensional subspace X (which can be assumed separable) in E such that $R|_X$ is an isomorphism. Since F does not contain an isomorphic copy of c_0, neither does $R(X)$. Moreover, if $R(X)$ contained an isomorphic copy of ℓ_1, then R would be an isomorphism on the span of a disjoint sequence equivalent to the canonical basis of ℓ_1 (see [37, Sect.3.4],also [12]); however T is disjointly strictly singular and F order continuous, so by Theorem 3.3, this yields a contradiction. Therefore, by [35, Thm.1.c.5], $R(X)$, and hence X, must be reflexive.

Now, we consider the ideal E_X generated by X in E, and F_X the ideal generated by $T(X)$ in F. Let A denote the solid hull of B_X the unit ball of X, and B the unit ball of F_X^*. We claim that $T|_{E_X} : E_X \to F_X$, A and B satisfy the conditions of Theorem 2.6. Indeed, by [24, Thm I.2(c)] we have the following factorization



where ϕ is a lattice homomorphism and H is a Banach lattice which does not contain an isomorphic copy of c_0. Since B_X is a weakly compact set, $\phi(B_X)$ is also weakly compact, and since H does not contain c_0, by [7, Thm 4.39], the solid hull $so(\phi(B_X))$ is also weakly compact. Since ϕ is a lattice homomorphism, $\phi(A) = \phi(so(B_X)) \subset so(\phi(B_X))$, and therefore $\phi(A)$ is also weakly compact.

Now, let $(a_n)_n$ be a positive disjoint sequence in A. We claim that $\|T(a_n)\| \to 0$ in F. Suppose, this is not the case, hence passing to a further subsequence we can assume that $\|T(a_n)\| \geq \alpha > 0$. By the previous argument $(\phi(a_n))_n$ must have a weakly convergent subsequence. We will show that this sequence is in fact weakly null. In order to see this, we make use of the representation [35, Thm 1.b.14], and consider the closed ideal generated by $e = \sum_{n=1}^{\infty} 2^{-n}\phi(a_n)$ in H as a space of measurable functions included in $L_1(\Omega, \Sigma, \mu)$. Since $(a_n)_n$ are pairwise disjoint, and ϕ is a lattice homomorphism, $(\phi(a_n))_n$ are also pairwise disjoint, so in particular the sequence $(\phi(a_n))_n$, which is weakly convergent in H, must satisfy $\|\phi(a_n)\|_1 \to 0$ when $n \to \infty$. This implies that the weak limit of $(\phi(a_n))_n$ in H has to be zero, as desired. In particular, $T(a_n) \to 0$ weakly, and since T is positive, by the positive Schur property of F, we get that $T(a_n) \to 0$ in the norm of F. This is a contradiction with the assumption that $\|T(a_n)\| \geq \alpha > 0$.

Therefore, we have proved that for every positive disjoint sequence in A, we have $\|T(a_n)\| \to 0$. This proves the first and third conditions of Theorem 2.6. To prove the second condition, notice that since F is order continuous, by [37, Cor. 2.4.3] every disjoint sequence $(b_n)_n$ in B (the unit ball of F_X^*) is weak-$*$ convergent to zero. Hence, we also have that $T^* b_n \to 0$ in the weak*-topology, as desired.

Thus , by Theorem 2.6, for every $\varepsilon > 0$ there exists operators $M_1, \ldots, M_n \in \mathcal{L}(E_X)$, and $L_1, \ldots, L_n \in \mathcal{L}(F_X)$ such that

$$R_\varepsilon = \sum_{i=1}^{n} L_i T M_i$$

satisfies $|\langle Ra - R_\varepsilon a, b\rangle| \leq \varepsilon$, for every $a \in A$, and $b \in B$. In particular, this implies that $\|R|_X - R_\varepsilon|_X\| < \varepsilon$. Since the class of strictly singular operators is a closed operator ideal, and T is strictly singular we have that R_ε is strictly singular for every $\varepsilon > 0$, and so is $R|_X$. This is a contradiction with the assumption that $R|_X$ was invertible, and the proof is finished. □

A general domination result for strictly singular operators, given recently by the authors in [18], is stated next; it improves a weaker version in [17] by removing the extra hypothesis of order continuity on E^* there imposed.

Theorem 3.5. *Let E be a Banach lattice with the subsequence splitting property and F an order continuous Banach lattice. If $0 \leq R \leq T : E \to F$ with T strictly singular, then R is also strictly singular.*

Notice that if the Banach lattice E has finite cotype, then an alternative proof of this result can be given using a recent characterization of strictly singular operators on Banach lattices ([19]):

Theorem 3.6. *Let E and F be Banach lattices such that E has finite cotype and F be order continuous. A regular operator $T : E \to F$ is strictly singular if and only if it is both disjointly strictly singular and AM-compact.*

The statement follows now from a domination result for AM-compact operators and from Theorem 3.3 .

We pass to study the *power* problem for strictly singular endomorphisms. That this problem is not trivial is shown with the following

Example 3.7. *There exist operators $0 \leq R \leq T : L^2[0,1] \oplus \ell^\infty \to L^2[0,1] \oplus \ell^\infty$ such that T is strictly singular but R is not.*

Indeed, consider the rank-one operator $Q : L^1[0,1] \to \ell^\infty$ defined by

$$Q(f) = \Big(\int_0^1 f, \int_0^1 f, \ldots\Big)$$

Take also an isometry $S : L^1[0,1] \to \ell^\infty$ given by $S(f) = (h_n'(f))_n$, where $(h_n)_n$ is a dense sequence in the unit ball of $L^1[0,1]$, and $(h_n')_n$ is a sequence of norm one functionals such that $h_n'(h_n) = \|h_n\|$ for all n. If $J : L^2[0,1] \hookrightarrow L^1[0,1]$ denotes the canonical inclusion, then the operator $SJ : L^2[0,1] \to \ell^\infty$ is not strictly singular.

Since ℓ^∞ is Dedekind complete we have that $|SJ|$, $(SJ)^+$ and $(SJ)^-$ are also continuous operators between $L^2[0,1]$ and ℓ_∞. It is easy to see that $|SJ| \leq QJ$. Since SJ is not strictly singular, we must have that either $(SJ)^+$ or $(SJ)^-$ is not strictly singular, so let us assume, that $(SJ)^+$ is not strictly singular. Now consider the matrices of operators:

$$R = \begin{pmatrix} 0 & 0 \\ (SJ)^+ & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ QJ & 0 \end{pmatrix},$$

which clearly define operators on $L^2[0,1] \oplus \ell_\infty$ with the required properties.

We pass now on to the positive results for the power problem. For the class of disjointly strictly singular operators we have the following ([16]):

Theorem 3.8. *Let E be a Banach lattice and operators $0 \le R \le T : E \to E$. If T is disjointly strictly singular then R^2 is also disjointly strictly singular.*

In general, for the class of strictly singular operators we have the following

Theorem 3.9. *Let E be a Banach lattice and operators $0 \le R \le T : E \to E$. If T is strictly singular, then R^4 is also strictly singular.*

This is deduced from a more general statement proved by the authors in [18] using factorization methods:

Theorem 3.10. *Let*

$$E_1 \underset{R_1}{\overset{T_1}{\rightrightarrows}} E_2 \underset{R_2}{\overset{T_2}{\rightrightarrows}} E_3 \underset{R_3}{\overset{T_3}{\rightrightarrows}} E_4 \underset{R_4}{\overset{T_4}{\rightrightarrows}} E_5$$

be operators between Banach lattices, such that $0 \le R_i \le T_i$ for $i = 1,2,3,4$.

If T_1, T_3 are strictly singular, and T_2, T_4 are order weakly compact, then $R_4R_3R_2R_1$ is also strictly singular.

As a direct consequence we have Theorem 3.9. Indeed, since T is strictly singular, it cannot preserve an isomorphic copy of c_0, so, in particular, it is order weakly compact. Therefore, it suffices to apply Theorem 3.6 to $E_i = E$, $R_i = R$ and $T_i = T$ for all i.

Corollary 3.11. *Let $0 \le R \le T : E \to F$, and $0 \le S \le V : F \to G$. If F and G are order continuous Banach lattices, and T and V are strictly singular operators, then the composition SR is strictly singular.*

In particular, if $0 \le R \le T : E \to E$ with T strictly singular and E order continuous, then R^2 is strictly singular

Indeed, since F is order continuous, the identity $I_F : F \to F$ is order weakly compact. Consider $E_1 = E$, $E_2 = F$, $E_3 = F$, $E_4 = G$ and $E_5 = G$; and the operators $T_1 = T$, $T_2 = I_F$, $T_3 = V$ and $T_4 = I_G$. Then, by Theorem 3.10 we obtain that $I_G S I_F R = SR$ is strictly singular.

Note that in the above example the Banach lattice $L^2[0,1] \oplus \ell^\infty$ is not order continuous and the square R^2 is the zero operator.

We provide now some applications to related operator ideals. Given two Banach spaces X and Y, a bounded operator $T : X \to Y$ is *super strictly singular* (or finitely strictly singular) if there does not exist a number $c > 0$ and a sequence of subspaces $(E_n)_n$ of X, with dim $E_n = n$, such that

$$\|Tx\| \ge c\,\|x\| \qquad \text{for all } x \in \bigcup_n E_n$$

In other words, T is super strictly singular if the *Bernstein* numbers $b_n(T) \searrow 0$, as $n \to \infty$, where

$$b_n(T) = \sup \inf_{x \in S(E_n)} \|Tx\|,$$

with the supremum taken among all n-dimensional subspaces E_n of X and $S(E_n)$ denotes the unit sphere of E_n. Clearly super-strict singularity implies strict singularity, and compact operators are always super strictly singular. The inclusions $L^\infty[0,1] \hookrightarrow E[0,1]$ for r.i. spaces $E \neq L^\infty$ are (non-compact) super strictly singular operators. We refer to Plichko ([43]) for properties of this operator class.

There is also a characterization of super strict singularity in terms of ultraproducts. Namely, an operator $T : E \to F$ is super strictly singular if and only if its ultrapower $T_{\mathcal{U}} : E_{\mathcal{U}} \mapsto F_{\mathcal{U}}$ is strictly singular for every (free) ultrafilter $\mathcal{U}$. Using this fact together with the previous results, a domination result for super strictly singular operators can be deduced:

Proposition 3.12. *Let $0 \leq R \leq T : E \to F$ be two positive operators from a Banach lattice E to a Banach lattice F. If E and F have finite cotype and T is super strictly singular then R is also super strictly singular.*

The conditions on non-trivial concavity play here an important role, as the following example shows ([20]):

Let $1 \leq p < q < \infty$, $E = (\oplus_{n=1}^\infty \ell_1^n)_p$ and $F = (\oplus_{n=1}^\infty \ell_\infty^{2^n})_q$. Consider for every n the operator $T_n : \ell_1^n \to \ell_\infty^{2^n}$ which sends an arbitrary finite sequence $(a_k)_1^n$ to the sequence $(\sum a_k)(1,1,\ldots,1)$ of $\ell_\infty^{2^n}$. Consider the isometry $R_n : \ell_1^n \to \ell_\infty^{2^n}$ represented by the $(2^n \times n)$ matrix with $\{1,-1\}$-entries defined as

$$R_n \equiv (x_{k,l}) = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 1 & \ldots & 1 & -1 \\ 1 & 1 & \ldots & -1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ -1 & -1 & \ldots & -1 & -1 \end{pmatrix}$$

(we set $x_{k,l} = \epsilon_k(l)$, where ϵ_k, for $k = 1, ..., 2^n$ is an enumeration of $\{-1,1\}^n$).

Consider now the operators $T = \oplus T_n$ and $R = \oplus R_n$ from E into F. The operator T is positive and factorizes through the inclusion $i : \ell_p \hookrightarrow \ell_q$. Indeed, the operator $\varphi : E \to \ell_p$ defined as $\varphi(x_n)_n = (\sigma(x_n))_n$, where $\sigma(x_n) = \sum_{i=1}^n x_{n,i}$, is well defined and bounded. Consider next the bounded operator $\psi : \ell_q \to F$ defined as $\psi(a_n)_n = \oplus a_n 1_{2^n}$, where 1_{2^n} is the unit of $\ell_\infty^{2^n}$. Notice that $T = \psi\ i\ \varphi$. Since i is super strictly singular (cf. [43]), the operator T itself is super strictly singular. On the other hand the operator R is not super strictly singular since R_n is an isometry for every n. Standard facts show that R is regular and that the inequalities $0 \leq R^-, R^+ \leq |R| \leq T$ hold true. From this we obtain that R^+ and R^- cannot be simultaneously super strictly singular since R is not (Notice that F is reflexive and E has the subsequence splitting property).

In the case of endomorphisms, using the fact that $(R_{\mathcal{U}})^n = (R^n)_{\mathcal{U}}$ for every $n \in \mathbb{N}$, and Theorem 3.9 we have the following

Proposition 3.13. *Let $0 \leq R \leq T : E \to E$ be positive operators on a Banach lattice. If T is super strictly singular then R^4 is also super strictly singular.*

Let us briefly mention some applications to the class of strictly co-singular operator: Recall that an operator $T : X \to Y$ between two Banach spaces is said to be *strictly co-singular* (or Pelczynski [41]) if for every (closed) subspace M of Y of infinite codimension the composition operator πT is not surjective , where π

denotes the quotient map from Y onto Y/M. For example the canonical inclusion $L^2[0,1] \hookrightarrow L^1[0,1]$ is strictly co-singular .

Strictly co-singular operators are also a closed operator ideal which is partially related by duality to the strictly singular operator class (recall that in general strict singularity is not stable under duality): If the adjoint operator $T^* : Y^* \to X^*$ is strictly singular (resp. co-singular) then $T : X \to Y$ is strictly co-singular (resp. singular). However the converse statements are not true in general. Using this fact and above results we can deduce the following:

Proposition 3.14. *Let E and F be two Banch lattices with E^* order continuous and F be reflexive with F^* satisfying the subsequence splitting property. If $0 \le R \le T : E \mapsto F$ and T is strictly co-singular then R is also strictly co-singular.*

Finally notice that a "super" version of strictly co-singular operators have been also considered in [43], for which domination results can be also deduced (see [20].

4. Domination by Banach-Saks operators

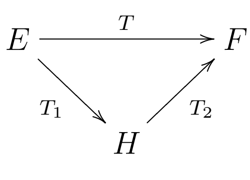
We consider now the class of Banach-Saks operators in the present context of domination. Recall that an operator $T : X \to Y$, between two Banach spaces X and Y, is *Banach-Saks* if every bounded sequence $(x_n)_n$ in X has a subsequence such that $(Tx_{n_k})_k$ is Cesàro convergent, that is, the sequence of arithmetic means $(\frac{1}{N}\sum_{k=1}^N T(x_{n_k}))_N$ is convergent in the norm of Y.

A Banach space is said to have the *Banach-Saks property* if the identity operator is Banach-Saks. Banach-Saks spaces are known to be reflexive but not conversely ([9]). Hence Banach-Saks operators are weakly compact; in fact they factorize through Banach-Saks spaces as Beauzamy has shown ([10]).

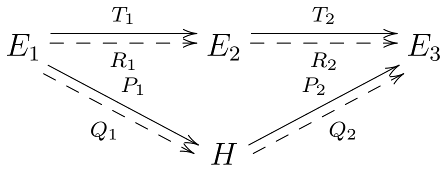
We will exploit the fact that order continuous Banach lattices with a weak unit are continuously included into L^1-spaces over probability spaces which in turn enjoy a weak version of the Banach-Saks property. More precisely, recall that a Banach space X has the *weak Banach-Saks property* (or is *weakly Banach-Saks*) if every weakly null sequence in X has a Cesàro convergent subsequence. Szlenk ([48]) has shown that $L^1(\Omega, \mu)$ is weakly Banach-Saks where (Ω, Σ, μ) is a probability space.

This result by Szlenk is used in the proof of the following factorization theorem together with the simple observation that the sequence of Cesàro sums of an equi-integrable sequence is also equi-integrable, and in combination with the Lions-Peetre interpolation construction (see [23] for details).

Theorem 4.1. *Let E and F be Banach lattices and $T : E \to F$ a positive Banach-Saks operator. If F is order continuous, then there exist a Banach lattice H with the Banach-Saks property, and operators $T_1 : E \to H$, $T_2 : H \to F$, such that the following factorization diagram holds:*



Observe that given $0 \le R_1 \le T_1 : E_1 \to E_2$ and $0 \le R_2 \le T_2 : E_2 \to E_3$, with T_1 Banach-Saks and T_2 order weakly compact, the proof of Theorem 4.1 can in fact be adapted to obtain the factorization



where H is a Banach lattice with the Banach-Saks property, $0 \leq Q_1 \leq P_1$ and $0 \leq Q_2 \leq P_2$. From here a domination result for Banach-Saks operators is easily obtained. Notice, however, that it is possible to derive an alternative domination result without making use of interpolation. This is the content of the following theorem which improves previous results in [22]. Its proof is again based on previous remarks together with Theorem 2.1

Theorem 4.2. *Let E_1, E_2 and E_3 be Banach lattices and $0 \leq R_i \leq T_i : E_i \to E_{i+1}$ be positive operators for $i = 1, 2$. If T_1 is a Banach-Saks operator and T_2 is order weakly compact, then the composition R_2R_1 is a Banach-Saks operator.*

Since Banach-Saks operators are weakly compact - hence order weakly compact - we obtain that $n = 2$ is the optimal answer for the power problem in this class.

Corollary 4.3. *Let E be a Banach lattice and $0 \leq R \leq T : E \to E$ be positive operators. If T is Banach-Saks, then R^2 is also Banach-Saks.*

Also from Theorem 4.2 and the fact that order intervals of order continuous Banach lattices are weakly compact [35, p. 28] follows

Corollary 4.4. *Let E and F be Banach lattices, such that F is order continuous. If $0 \leq R \leq T : E \to F$, with T Banach-Saks, then R is also a Banach-Saks operator.*

The operators in Example 3.1 show that the order continuity of F in Corollary 4.4 is not superfluous. In fact those operators can be used to show that Corollary 4.3 is actually sharp; indeed consider as usual the operators $0 \leq \widetilde{R} \leq \widetilde{T} : \ell_1 \oplus L^\infty \to \ell_1 \oplus L^\infty$ defined by

$$\widetilde{R} = \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \qquad \widetilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix}.$$

Clearly $\widetilde{T}$ is Banach-Saks, but $\widetilde{R}$ is not. Notice that $\widetilde{R}^2 = 0$.

5. Domination by narrow operators

In this section we consider the problem of domination by narrow operators. Given a Köthe function space $E(\mu)$ on an atomless probability space (Ω, Σ, μ) and a Banach space Y, we recall that a linear operator $T : E(\mu) \to Y$ is *narrow* if for each measurable set $A \in \Sigma$ and every $\varepsilon > 0$ there is a partition $\{A_1, A_2\}$ of the set A, with $\mu(A_1) = \mu(A_2)$ such that $\|T(\chi_{A_1} - \chi_{A_2})\| < \varepsilon$.

Plichko and Popov provided in [44] the first systematic study of this class of operators which will be denoted by $\mathcal{N}(E(\mu), Y)$; there they prove that if $E(\mu)$ is a rearrangement invariant function space different from $L^\infty(\mu)$ and Y is a Banach space, then every compact operator $T : E(\mu) \to Y$ is narrow.

Other references to narrow operators prior to the monographic treatment in [44] can be found in ([25], [11], [45], [46]) in the context of "norm-sign-preserving operators"on $L^1[0, 1]$. We refer also to ([15], [30]) for the relation between complementably singular operators and narrow operators ([15], [30]). For instance in [30] it is proved that every operator $T : L^p(\mu) \to L^p(\nu)$, $1 \leq p < 2$ which is L^p-complementably singular must be narrow.

In [44] the following questions were posed regarding strictly singular and narrow operators: Let $1 < p < \infty$, $p \neq 2$ and Y a Banach space. Is every strictly singular operator $T : L^p(\Omega, \Sigma, \mu) \to Y$ narrow? Is every l_2-singular operator $T : L^p(\Omega, \Sigma, \mu) \to Y$ narrow? In [21] a positive answer to these questions for regular operators was given. In particular every bounded operator $T : L^1[0,1] \to F$, where F is a Banach lattice containing no isomorphic copy of c_0, that is l_2-singular was shown to be narrow.

Notice that, as in previous sections, we cannot expect a general domination result without extra hypothesis on the lattices as shown in ([21, Ex. 3.3]) where a positive rank one operator in $\mathcal{N}(L^1[0,1], l_\infty)$ dominates a non-narrow operator. In fact the example can be easily adapted to show that there is no domination result in the class $\mathcal{N}(E(\mu), l_\infty)$.

Notice that the "ring approximation"in Theorem 2.4 initially fails when applied to the class of narrow operators due to its surprising lack of structure. Most remarkably narrow operators fail to have a vector space structure (cf. [44, §8 Ex.2])) and they are not stable by composition with bounded operators. This evident drawback demands some extra work.

First, based on standard results, the following remark is established

Proposition 5.1. *Let Y and Z be two Banach spaces and $T : E(\mu) \to Y$ a narrow operator.*

a) *If $S \in \mathcal{L}(Y, Z)$, then $ST \in \mathcal{N}(E(\mu), Z)$.*
b) *If P is a band projection on $E(\mu)$, then $TP \in \mathcal{N}(E(\mu), Y)$.*

In other words, if F is a Dedekind-complete normed vector lattice and $T : E(\mu) \to F$ is a positive narrow operator, then every elementary component of T is narrow.

In order to extend this result to simple components we need to use a known technical lemma (see [21] for a proof).

Lemma 5.2. *Let G and F be vector lattices, F Dedekind-complete, and T a positive operator from G to F. For every $S \in \mathcal{S}_T$ there exists a finite set of mutually disjoint elementary components of T, $Q_1TP_1, \cdots, Q_nTP_n$, such that $S = \sum_{i=1}^{n} Q_iTP_i$. Moreover the projections $P_1, \ldots P_n$ can be taken pairwise disjoint.*

A further step must be taken in order to extend this result to arbitrary components of T. We do it first for the case that the operator $T : E(\mu) \to F$ is M-weakly compact. In this case the operator norm is order continuous on the order-interval $[0, T]$ if F is order continuous (Thm. 2.5). Since components of T were shown by De Pagter ([39]) to be approximated in order by simple components of T, we obtain in this context that they can be approximated in norm by simple components of T, which were already shown to be narrow; thus the components of T turn out to be narrow as narrow operators form a closed subset of the space of bounded operators. Notice in addition that every operator T defined on a Banach lattice with order continuous dual norm and taking values in $L^1(\mu)$ must be M-weakly compact ([21, Prop. 3.11]), hence from the above every component of T must be narrow when considered as taking values in $L^1(\mu)$. An appeal to ([21, Prop. 2.3.]) shows that every component of T is narrow.

Once we have proved that under order continuity in F and $E(\mu)^*$ every component of T is narrow we want to show that every T-step operator is narrow. Again the proof cannot be obtained by simple addition in the absence of vector space structure.

Proposition 5.3. *Let $E(\mu)$ and $F(\nu)$ be Köthe function spaces on the probability spaces (Ω, Σ, μ) and (Ω', Σ', ν) respectively. If $E(\mu)^*$ and $F(\nu)$ are order continuous and $T : E(\mu) \to F(\nu)$ is a positive narrow operator, then every positive T-step operator is narrow.*

Once this result is established we have Freudenthal's result at our disposal to prove the following

Proposition 5.4. *Let $E(\mu)$ be a Köthe function space over an atomless probability space (Ω, Σ, μ) such that $E(\mu)^*$ is order continuous, and let F be an order continuous Banach lattice. For every positive narrow operator $T : E(\mu) \to F$ the inclusion $[0, T] \subset \mathcal{N}(E(\mu), F)$ holds.*

Notice that the restriction $T|_{L^\infty(\mu)} : L^\infty(\mu) \to F$ is narrow if T is. Since $L^\infty(\mu)^*$ is order continuous the operator $S|_{L^\infty(\mu)} : L^\infty(\mu) \to F$ is narrow by the previous proposition, but again this is tantamount to $S : E \to F$ being narrow.

Thus the following general domination result is obtained

Theorem 5.5. *Let (Ω, Σ, μ) be an atomless probability space and $E(\mu)$ a Köthe function space on (Ω, Σ, μ). Let F be an order continuous Banach lattice and $0 \leq R \leq T : E \to F$ be two positive operators. If T is narrow then R is narrow.*

We conclude this section with a word of caution regarding terminology as different concepts have been depicted under the common name narrow (see f.i. [31]). Also it was only recently that the notion of narrow operator was extended in [36] to the abstract Banach lattice setting with a definition that clearly preserves the meaning of the initial one. In that paper the authors improve the domination result in the sense that under the same order continuity assumptions as those given here they not only obtain domination but also the band structure of narrow operators in the lattice of all regular operators between two Banach lattices.

6. Domination by p-summing operators

This section is devoted to presenting some recent results on domination of p-summing operators based on [40]. Notice that the ideal of p-summing operators, as many others, is not closed under the operator norm. This fact makes a difference with the classes considered so far, where approximation arguments were available (see Theorem 2.6). In a certain sense, this might suggest that no domination result can be expected for non-closed operator ideals, however, this is far from true as we shall see.

Recall that a sequence $(x_n)_n$ in a Banach space X is weakly p-summable if

$$\sup_{x^* \in B_{X^*}} \Big(\sum_n |\langle x^*, x_n \rangle|^p \Big)^{\frac{1}{p}}$$

is bounded, or equivalently if $x_n = T(e_n)$ for some $T : \ell_{p'} \to X$ (where (e_n) is the unit vector basis of $\ell_{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$). Similarly, a sequence $(x_n)_n$ is strongly summable when $\big(\sum_n \|x_n\|^p\big)^{\frac{1}{p}}$ converges. An operator $T : X \to Y$, is called *p-summing* if there is a constant $C < \infty$ such that for every $(x_n)_n$ and every $m \in \mathbb{N}$

$$\big(\sum_{n=1}^{m} \|Tx_n\|^p\big)^{\frac{1}{p}} \leq C \sup_{x^*\in B_{X^*}} \big(\sum_{n=1}^{m} |\langle x^*, x_n\rangle|^p\big)^{\frac{1}{p}}.$$

We set $\pi_p(T)$ to be the smallest of these constants, and it follows that $\pi_p(\cdot)$ defines a norm for the ideal of p-summing operators. Clearly, an operator $T : X \to Y$ is p-summing if it maps weakly p-summable sequences into strongly p-summable. We refer to [13] for the theory of p-summing operators.

Note, that by the classical theorem of Dvoretzky-Rogers (cf. [13]) the identity operator on any infinite dimensional Banach space is never a p-summing operator. In particular, p-summing operators are always strictly singular; since in addition every finite-rank operator is p-summing, we can use the operators given in Example 3.1 to provide a counterexample for a general domination result of p-summing operators.

An important reason for the failure of a general domination result for p-summing operators stems from the fact that weak summability is not a lattice property. Namely, the sequence $(|x_n|)_n$ need not be weakly p-summable although $(x_n)_n$ was. This is just because an operator $T : \ell_{p'} \to X$ need not have a bounded modulus (cf. [7]).

Notice that since the ideal of p-summing operators is not closed for the operator norm, but for the $\pi_p(\cdot)$ norm, we will consider a "quantified" version of the domination problem. That is, given Banach lattices E and F, we look for a constant $C > 0$ such that if $0 \leq R \leq T : E \to F$, then $\pi_p(R) \leq C\pi_p(T)$.

In order to motivate the first positive results, we consider operators on a Hilbert space. In finite dimension, an operator $T : \ell_2^n \to \ell_2^n$ can be considered as an $n \times n$ matrix $(a_{ij})_{i,j=1}^n$. Clearly, ℓ_2^n with the coordinate-wise ordering becomes a Banach lattice, where two operators $T = (a_{ij})$ and $R = (b_{ij})$ satisfy $0 \leq R \leq T$ whenever $0 \leq b_{ij} \leq a_{ij}$ for every $i, j = 1, \dots, n$. Notice that for operators on Hilbert space, the classes of 2-summing operators and Hilbert-Schmidt operators coincide. Moreover

$$\pi_2(T) = \Big(\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\Big)^{\frac{1}{2}},$$

Hence, it is clear that $\pi_2(R) \leq \pi_2(T)$ whenever $0 \leq R \leq T : \ell_2^n \to \ell_2^n$. Note that the same proof works for operators on the infinite dimensional ℓ_2 (see also Proposition 6.2).

Other simple cases in which a domination theorem holds are the following.

Proposition 6.1. *Let F be a Banach lattice and $0 \leq R \leq T : C(K) \to F$ be positive operators. If T is p-summing for some $1 \leq p < \infty$, then R is p-summing.*

Proof. By [35, Theorem 1.d.10], every positive operator from the space $C(K)$ is p-summing if and only if it is p-concave. The result follows from the fact that

$$\Big(\sum_{k=1}^{n} \|R(x_k)\|^p\Big)^{\frac{1}{p}} \leq \Big(\sum_{k=1}^{n} \|T(|x_k|)\|^p\Big)^{\frac{1}{p}} \leq M_{(p)}(T)\Big\|\Big(\sum_{k=1}^{n} |x_k|^p\Big)^{\frac{1}{p}}\Big\|.$$

□

The following provides a domination result for absolutely summing operators into Hilbert spaces, and also plays a key role in the proof of Theorem 6.3.

Proposition 6.2. *Let $0 \le R \le T : E \to L_2(\mu)$. If T is absolutely summing, then so is R. Moreover, there exists a constant $C > 0$ independent of R and T such that $\pi_1(R) \le C\pi_1(T)$.*

In [40] a domination theorem for p-summing operators ($1 \le p < \infty$) is obtained as far as the Banach lattices involved have cotype 2. This is the content of the following

Theorem 6.3. *Let $1 \le p < \infty$. Let E and F be Banach lattices with cotype 2 and $0 \le R \le T : E \to F$. There exists a universal constant C_p (depending only on p and the cotype constants of E and F) such that*

$$\pi_p(R) \le C_p \pi_p(T).$$

In particular, if T is p-summing, then R is also p-summing.

Notice that if the Banach lattice F has cotype greater than 2, then such a result cannot hold as the following example shows

Example 6.4. *Given $1 \le p < \infty$ and $1 \le q \le \infty$, for any $n \in \mathbb{N}$ there exist positive operators $0 \le R_n \le T : L_p(0,1) \to L_q(0,1)$ such that*

$$\pi_2(T) = 1 \quad \textit{and} \quad \pi_2(R_n) \ge C n^{\frac{1}{2} - \frac{1}{q}},$$

for certain constant C independent of n.

In particular, if $q > 2$ there is no domination theorem for 2-summing operators between L_p and L_q.

Proof. Let $T : L_p \to L_q$ be given by $T(f) = \int f d\mu \chi_{(0,1)}$, and for $n \in \mathbb{N}$ let $R_n : L_p \to L_q$ be defined by

$$R_n(f) = \sum_{k=1}^{n} \int f r_k^+ d\mu \, \chi_{A_k},$$

where r_k^+ denotes the positive part of the k-th Rademacher function, and $A_k = (\frac{k-1}{n}, \frac{k}{n})$ for $k = 1, \ldots, n$.

Clearly, T is a rank one operator, so we have $\pi_2(T) = \|T\| = 1$. Now, to compute $\pi_2(R_n)$, notice that the sequence $(r_k)_{k=1}^n$ is weakly 2-summable with

$$\sup\{\sum_{k=1}^{n} \langle x^*, r_k\rangle^2 : x^* \in B_{L_p}\} \le B_p,$$

where B_p is the constant appearing in Khintchine's inequality (hence independent of n, cf. [13]). Therefore, we have

$$\begin{aligned} \pi_2(R_n) &\ge \tfrac{1}{B_p}\Big(\textstyle\sum_{k=1}^n \|R_n(r_k)\|_q^2 \Big)^{\frac{1}{2}} \\ &= \tfrac{1}{B_p}\Big(\textstyle\sum_{k=1}^n \| \sum_{j=1}^n \int r_k r_j^+ d\mu \chi_{A_j} \|_q^2 \Big)^{\frac{1}{2}} \\ &= \tfrac{1}{B_p}\Big(\textstyle\sum_{k=1}^n \| \frac{1}{2}\chi_{A_k} \|_q^2 \Big)^{\frac{1}{2}} \\ &= \tfrac{1}{2B_p} n^{\frac{1}{2}-\frac{1}{q}}. \end{aligned}$$

In particular, when $2 < q$ there cannot be a constant K such that whenever $0 \le R \le T : L_p \to L_q$, we had $\pi_2(R) \le K\pi_2(T)$. □

Using this kind of construction, a characterization of Banach lattices satisfying a lower 2-estimate can be provided in terms of domination properties of 2-summing operators (see [40]).

Theorem 6.5. *A Banach lattice F satisfies a lower 2-estimate if and only if for every 2-concave Banach lattice E, there exists a constant $K < \infty$ such that $\pi_2(R) \leq K\pi_2(T)$ whenever $0 \leq R \leq T : E \to F$.*

7. Final remarks

In this section we present some remarks and open problems related to the domination properties of positive operators.

Let us begin with the domination properties of strictly singular operators. The following questions have been left unanswered:

Problem 1. *Given an order continuous Banach lattice E, and $0 \leq R \leq T : E \to E$ with T strictly singular, must R be strictly singular too? Corollary 3.11 asserts that R^2 is strictly singular, but this might not be optimal.*

Problem 2. *Given a Banach lattice E and $0 \leq R \leq T : E \to E$, if T is strictly singular, then R^4 is strictly singular. Can we expect R^3 or even R^2 to be strictly singular?*

Notice that both problems are almost equivalent, since by Theorem 2.1 every strictly singular operator factors through an order continuous Banach lattice. Connected with this problems are those for the related classes of strictly co-singular and super strictly singular operators:

Problem 3. *Let E be a Banach lattice and consider operators $0 \leq R \leq T : E \to E$, with T strictly co-singular. Which is the smallest $n \in \mathbb{N}$ such that R^n is also strictly co-singular?*

Problem 4. *Given a Banach lattice E and operators $0 \leq R \leq T : E \to E$, with T super strictly singular, Proposition 3.13 asserts that R^4 is super strictly singular. As in Problem 2, can R^3 or even R^2 be expected to be super strictly singular?*

In the context of Banach-Saks operators, if F is an order continuous Banach lattice and $0 \leq R \leq T : E \to F$ with T Banach-Saks, then Corollary 4.4 asserts that R is also Banach-Saks. We leave the following open:

Problem 5. *Can we replace in Corollary 4.4 the hypothesis of order continuity on F with order continuity on E^* to reach the same conclusion?*

Notice that taking duals will not do here because being Banach-Saks is not a property stable under duality (think of the identity operator on the Baernstein space which is known to have a Banach-Saks dual). We carry on with the power problem for Narrow operators:

Problem 6. *Is every narrow operator order weakly compact? A positive answer would settle the power problem for $n = 2$ (by means of the usual factorization technique).*

For p-summing operators in connection with Theorem 6.3, the following problem was left open:

Problem 7. *Let E be a Banach lattice of cotype greater than 2 and F be a Banach lattice with cotype 2. Let $0 \leq R \leq T : E \to F$. Is there a universal constant $C > 0$ such that $\pi_2(R) \leq C\pi_2(T)$?*

Notice that for $p > 2$, if $0 \leq R \leq T : L_p \to L_2$, then by Proposition 6.2 there is a domination result for absolutely summing operators, hence we cannot expect a counterexample in the form of 6.4.

The following more general question is of interest:

Problem 8. *Provide an operator ideal $\mathfrak{I}$ and operators $T \in \mathfrak{I}$ and $0 \leq R \leq T$ such that $R^n \notin \mathfrak{I}$ for any $n \in \mathbb{N}$.*

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