SUBSPACE STRUCTURE OF LORENTZ $L_{p,q}$ SPACES AND STRICTLY SINGULAR OPERATORS

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Abstract. We study subspaces of Lorentz $L_{p,q}$ spaces and provide an easy-to-check characterization of strictly singular operators defined on these spaces. As an application we obtain stability under duality for the class of strictly singular operators on $L_{p,q}$ spaces, extending a theorem of L. Weis for operators on L_p spaces.

1. Introduction

This note is devoted to the geometric properties of Lorentz spaces, mainly to the isomorphic structure of their subspaces and strictly singular operators defined between them. The Lorentz spaces $L_{p,q}$ were introduced in [16] and [17], and its importance is present in several areas of analysis such as harmonic analysis, interpolation theory... (see the surveys [4], [9]). The family of Lorentz $L_{p,q}$ spaces is a generalization of the class of classical L_p spaces, which are not fine enough to differentiate certain properties.

For instance, recall the classical Hausdorff-Young inequality which asserts that the Fourier transform $f \mapsto \hat{f}$ is bounded as an operator from $L_p(\mathbb{R}^n)$ to $L_{p'}(\mathbb{R}^n)$, for $1 < p < 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Real interpolation methods show that for $1 < p < 2$, the Fourier transform is in fact bounded from $L_p(\mathbb{R})$ to $L_{p',p}(\mathbb{R})$, which is a considerable refinement since $\|f\|_{L_{p'}} \leq \|f\|_{L_{p',p}}$ (see next section).

In order to study the subspaces of Lorentz spaces, we will make use of several techniques available from Banach lattice theory. A key result here is Kadeč-Pełczyński's dichotomy, which was originally proved for L_p spaces in [12], and generalized to more general Banach lattices in [7]. This result characterizes subspaces of Banach lattices that strongly embed in L_1 in terms of disjoint sequences. Our main result in Section 3, can be considered as a strengthened version of Kadeč-Pełczyński's Theorem for $L_{p,q}$ spaces with $p \leq q < 2$ (see Theorem 3.4).

The isomorphic structure of infinite-dimensional subspaces of a Banach space is intimately related to the class of strictly singular operators on the space. Recall that an operator is strictly singular if and only if it is never an isomorphism when restricted to any infinite dimensional subspace. This class forms a closed two-sided operator ideal that contains the ideal of compact operators, and was first introduced by T. Kato in connection with the perturbation theory of Fredholm operators

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[13]. Properties of strictly singular operators on L_p spaces have been studied in [18], [22] and more recently in [8] and [11].

In this paper we study some characterizations of strictly singular operators on $L_{p,q}$ spaces in terms of invertibility on subspaces isomorphic to ℓ_q and ℓ_2. This extends the main theorem from [22] for operators on L_p spaces, and provides a useful tool for studying duality within the class of strictly singular operators (see Theorems 4.1 and 4.3). A similar characterization for strictly singular operators on general Banach lattices has been recently obtained in [6].

The paper is organized as follows. In the second section, we introduce notation and recall the main definitions and properties of Lorentz spaces. Afterwards, Section 3 is devoted to the study of subspaces of Lorentz spaces. Here, we recall the known facts on subspaces of Lorentz spaces isomorphic to ℓ_s, and provide some facts concerning isomorphic embeddings of ℓ_q in subspaces of $L_{p,q}$ in terms of local properties of these subspaces. Namely, we will see that a subspace $X \subset L_{p,q}$ which contains uniformly isomorphic copies of ℓ_p^n or $\ell_{p,q}^n$ must also contain an almost disjoint sequence spanning ℓ_q.

In Section 4 we center our study on strictly singular operators on Lorentz spaces. Here, we provide a characterization of strictly singular operators on $L_{p,q}$ spaces in terms of ℓ_2-singular and ℓ_q-singular operators. In addition, using this characterization we study duality properties of strictly singular operators on $L_{p,q}$. In particular, we show that if p and q satisfy certain relation, an operator $T : L_{p,q} \to L_{p,q}$ is strictly singular if and only if so is T^*. An example is provided to show that these conditions on p and q are necessary (Example 4.5).

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2. Preliminaries

First of all, let us recall that for $0 < p < \infty$ and $1 \leq q < \infty$ the Lorentz space $L_{p,q}(I)$, where I is an interval of the form $(0, a)$ with $0 < a \leq \infty$ endowed with the Lebesgue measure λ, is the set of measurable functions on I such that

$$\|f\|_{p,q} = \Big(\int_0^\infty f^*(t)^q d(t^{\frac{q}{p}}) \Big)^{\frac{1}{q}} = \Big(\frac{q}{p} \int_0^\infty f^*(t)^q t^{\frac{q}{p}-1} dt \Big)^{\frac{1}{q}} < \infty$$

where here f^* denotes the decreasing rearrangement of a function f, that is

$$f^*(s) = \inf\{t \geq 0 : \lambda(\{x \in I : |f(x)| > t\}) \leq s\}.$$

By Minkowski's inequality, for $1 \leq q < p$ the expression defined by $\|f\|_{p,q}$ is a norm, while for $1 < p < q$, $\|f\|_{p,q}$ is only a quasi-norm which turns out to be equivalent to the following expression

$$\|f\|_{(p,q)} = \Big(\int_0^\infty f^{**}(t)^q d(t^{\frac{q}{p}}) \Big)^{\frac{1}{q}},$$

where $f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$. By Hardy's inequality this expression is in fact a norm (cf. [4]). Hence, for $1 < p < \infty$ and $1 \leq q < \infty$, after identifying functions which are equal almost everywhere, the space $L_{p,q}$ becomes a Banach space. However, for $0 < p \leq 1$, except the case $L_{1,1}$ which is isometric with L_1, the spaces $L_{p,q}$ are only (non-locally convex) quasi-Banach spaces.

For $1 < p < \infty$, the space $L_{p,\infty}(I)$ is analogously defined as the set of measurable functions on I such that

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty.$$

The spaces $L_{p,\infty}$ are also called weak L_p spaces. Finally, notice that for $p = \infty$ and any $1 \leq q \leq \infty$ the space $L_{\infty,q}$ coincides with L_∞.

There is also a sequence space version of Lorentz spaces. Namely, for $1 < p < \infty$ and $1 \leq q < \infty$, the space $\ell_{p,q}$ consists of all sequences (x_n) of scalars endowed with the norm

$$\|(x_n)\|_{p,q} = \Big(\sum_{k=1}^\infty (x_k^*)^q (k^{\frac{q}{p}} - (k-1)^{\frac{q}{p}} \Big)^{\frac{1}{q}},$$

where (x_k^*) denotes the decreasing rearrangement of the sequence $(|x_n|)$. Similarly, the space $\ell_{p,\infty}$ is the space of scalar sequences equipped with the norm

$$\|(x_n)\|_{p,\infty} = \sup_n n^{\frac{1}{p}} x_n^*.$$

Clearly, $\ell_{p,q}$ coincides with $L_{p,q}(\mathbb{N})$ where $\mathbb{N}$ is endowed with the counting measure.

Notice that the spaces $L_{p,q}(I)$ with the point-wise ordering (defined almost everywhere) are Banach lattices. In fact, they are rearrangement invariant spaces since given any two functions f, g with the same distribution, their decreasing rearrangements satisfy $f^* = g^*$, hence their norms coincide.

Unlike the case of L_p spaces, which coincide isometrically with $L_{p,p}$, in Lorentz spaces the measure space determines in a sense the structure of the space. For example, in $L_{p,q}(0,\infty)$, the characteristic functions $\chi_{[n,n+1]}$ for $n \in \mathbb{N}$ span a subspace isomorphic to $\ell_{p,q}$. However, for $1 < p, q < \infty$ and $p \neq q$, the space $L_{p,q}(0,1)$ does not contain a subspace isomorphic $\ell_{p,q}$, so in particular $L_{p,q}(0,1)$ and $L_{p,q}(0,\infty)$ are not isomorphic. However, in [14] it was proved that for $1 < p < \infty$, the spaces $\ell_{p,\infty}$, $L_{p,\infty}(0,1)$ and $L_{p,\infty}(0,\infty)$ are isomorphic Banach spaces.

The inclusion relations between the spaces $L_{p,q}$ are well-known (cf. [15, pp. 142-143]). For $1 \leq p < \infty$ and $1 \leq q_1 < q_2 \leq \infty$, we have $L_{p,q_1}(I) \hookrightarrow L_{p,q_2}(I)$, with $\|f\|_{p,q_2} \leq \|f\|_{p,q_1}$. Moreover, for $r<p<s$ and every q we have

$$L_{s,\infty}(I)\cap L_{r,\infty}(I)\hookrightarrow L_{p,q}(I)\hookrightarrow L_{s,1}(I)+L_{r,1}(I),$$

which in the case of finite measure reduces to $L_{s,\infty}(0,1)\hookrightarrow L_{p,q}(0,1)\hookrightarrow L_{r,1}(0,1)$.

One of the main reasons that make Lorentz $L_{p,q}$ spaces so important is the fact that they appear as real interpolates of L_p spaces. Namely, for $0<p_1<p_2\leq\infty$, $0<\theta<1$, and $1<q\leq\infty$, the space $L_{p,q}(I)$ coincides with $[L_{p_1}(I),L_{p_2}(I)]_{\theta,q}$ up to equivalence of norms, where $\frac{1}{p}=\frac{(1-\theta)}{p_1}+\frac{\theta}{p_2}$ (cf. [9]).

Recall that a Banach space X has type p (respectively, cotype q) provided there exists a constant C so that for every sequence $x_1,\ldots,x_n$ in X,

$$\Big(\int_0^1\Big\|\sum_{i=1}^n r_i(t)x_i\Big\|^p dt\Big)^{\frac{1}{p}}\leq C\Big(\sum_{i=1}^n\|x_i\|^p\Big)^{\frac{1}{p}},$$

$$\Bigg(\text{resp. }\Big(\sum_{i=1}^n\|x_i\|^q\Big)^{\frac{1}{q}}\leq C\Big(\int_0^1\Big\|\sum_{i=1}^n r_i(t)x_i\Big\|^q dt\Big)^{\frac{1}{q}}\Bigg),$$

where (r_i) denote the Rademacher functions on $[0,1]$. Also recall that a Banach lattice X is said to be p-convex (respectively, q-concave) if there is a constant M such that for every finite sequence $x_1,\ldots,x_n$ in X,

$$\Big\|\Big(\sum_{i=1}^n|x_i|^p\Big)^{\frac{1}{p}}\Big\|\leq M\Big(\sum_{i=1}^n\|x_i\|^p\Big)^{\frac{1}{p}}\qquad\Bigg(\text{resp. }\Big(\sum_{i=1}^n\|x_i\|^q\Big)^{\frac{1}{q}}\leq M\Big\|\Big(\sum_{i=1}^n|x_i|^q\Big)^{\frac{1}{q}}\Big\|\Bigg).$$

Similarly, we say that X satisfies an upper p-estimate (respectively, lower q-estimate) for disjoint vectors if there is a constant $M<\infty$ such that for every choice of pairwise disjoint elements $x_1,\ldots,x_n$ in X, we have

$$\Big\|\sum_{i=1}^n x_i\Big\|\leq M\Big(\sum_{i=1}^n\|x_i\|^p\Big)^{\frac{1}{p}}\qquad\Bigg(\text{resp. }\Big\|\sum_{i=1}^n x_i\Big\|\geq M^{-1}\Big(\sum_{i=1}^n\|x_i\|^q\Big)^{\frac{1}{q}}\Bigg).$$

We refer to [15] for a deep study of these notions and their relations.

In [3], type and convexity of Lorentz spaces were studied. Precisely, it was proved that for $1\leq q\leq p<\infty$ the space $L_{p,q}$ is q-convex, satisfies a lower p-estimate, has type $\min(2,q)$, if $p\neq 2$ has cotype $\max(2,p)$ and $L_{2,q}$ has cotype $2+\varepsilon$ for every $\varepsilon>0$. Meanwhile, for $1<p<q<\infty$, $L_{p,q}$ is q-concave, satisfies an upper p-estimate, has cotype $\max(2,q)$, if $p\neq 2$ has type $\min(2,p)$ and $L_{2,q}$ has type $2-\varepsilon$ for every $\varepsilon>0$. Moreover, an example of G. Pisier [15, Example 1.f.19] shows that $L_{p,q}$ for $1\leq q<p$ is not p-concave, so in particular $L_{2,q}$ is not of cotype 2.

Recall that a Banach lattice X is order continuous whenever every order bounded increasing sequence is convergent. Every order continuous Banach lattice X with a weak unit (i.e. with an element $e\geq 0$ for which $e\wedge x=0$ implies $x=0$) can be represented as a Banach lattice of functions on some probability space [15, Theorem 1.b.14]. Also recall that Kadeč-Pełczyński's dichotomy states that a subspace of an order continuous Banach lattice is either isomorphic to a subspace of some L_1 space or it contains a normalized sequence equivalent to a pairwise disjoint sequence (cf. [7], [12]).

We refer the reader to [15] for any unexplained terminology regarding Banach lattices.

3. Subspaces of $L_{p,q}$ spaces

The results presented in this section will be proved for the spaces $L_{p,q}(0,\infty)$ which will be simply denoted by $L_{p,q}$ and are the most general case, since these include the properties of $L_{p,q}(0,1)$ and $\ell_{p,q}$. Recall the following criterium for a sequence in $L_{p,q}$ to have a subsequence equivalent to ℓ_q (cf. [1], [4]).

Theorem 3.1. *Let (f_n) be a normalized sequence in $L_{p,q}$ ($1 < p < \infty$, $1 \leq q < \infty$) such that $f_n^* \to 0$ pointwise. Then there exists a subsequence (f_{n_k}) equivalent to the unit vector basis of ℓ_q.*

With this at hand and some more work the following property of disjoint sequences can be obtained (cf. [1], [4]).

Theorem 3.2. *Let $1 < p < \infty$, $1 \leq q < \infty$. If (f_n) is a normalized disjoint sequence in $L_{p,q}$, then its span $[f_n]$ contains a subspace isomorphic to ℓ_q. Moreover, if the sequence (f_n) is supported on a set of finite measure, then some subsequence (f_{n_k}) is already equivalent to the unit vector basis of ℓ_q.*

Next result is a stronger version of Kadec-Pelczynski's dichotomy for subspaces of $L_{p,q}$ proved in [1]. Recall that a subspace X of $L_{p,q}(0,1)$ is called strongly embedded if the norms of $L_1(0,1)$ and $L_{p,q}(0,1)$ are equivalent on X.

Theorem 3.3. *Let $1 < p < \infty$, $1 \leq q < \infty$, and let X be a subspace of $L_{p,q}$. Then either X is isomorphic to a strongly embedded subspace of $L_{p,q}(0,1)$ or X contains a complemented copy of ℓ_q.*

Recall also that for $1 < p < \infty$, $1 \leq q < \infty$, if $p \neq 2$ and $p \neq q$, then ℓ_p is not isomorphic to a subspace of $L_{p,q}$ (cf. [4, Theorem 7]). However, ℓ_p^n embed uniformly in $L_{p,q}$ (even in finite measure). Recall that ℓ_p^n are said to embed uniformly in a Banach space X if for every $n \in \mathbb{N}$ there exists an operator $T_n : \ell_p^n \to X$ such that $\sup_n \|T_n\|\|T_n^{-1}\| < \infty$. The following result shows which kind of subspaces of $L_{p,q}$ contain ℓ_p^n uniformly.

Theorem 3.4. *Let X be a subspace of $L_{p,q}(0,1)$ ($1 < p \leq q < 2$) which contains ℓ_p^n's uniformly. Then the norm of $L_{p,q}$ and that of L_1 are not equivalent on X.*

This result is inspired by [5, Section 2] where it was proved that in every subspace of L_p isomorphic to ℓ_p, the norm of L_p and that of L_1 are not equivalent (see also [21, Thm. 13]). We need a lemma first.

Lemma 3.5. *Let E be a Banach lattice of measurable functions over a probability space (Ω,Σ,μ), with type p, $1\le p\le 2$, and cotype q, $2\le q<\infty$. Let $(f_{ij})_{j=1,\,1\le i\le j}^{\infty}$ in E be a double indexed sequence of normalized elements, such that for all $j=1,2,\dots$ and scalars $c_1,\dots,c_j$,*

$$\Big(\sum_{i=1}^{j}|c_i|^p\Big)^{\frac{1}{p}}\le C\Big\|\sum_{i=1}^{j}c_if_{ij}\Big\|.$$

If M_p denotes the type p constant of E, then for all δ with $0<\delta<\frac{1}{CM_p}$, and for all $K>0$, the cardinal of the set

$$A_j=\{i\le j:\|f_{ij}\chi_{\{\omega\in\Omega:|f_{ij}(\omega)|>K\}}\|>\delta\}$$

is not uniformly bounded as $j\to\infty$.

Proof. Suppose the contrary. We can assume that there exist $N\in\mathbb{N}$, $K>0$ and $\delta<\frac{1}{CM_p}$ such that the cardinal of A_j is smaller than N for all j. Let $B_j=\{1,\dots,j\}\backslash A_j$. Hence, for every $i\in B_j$ we have

$$\|f_{ij}\chi_{\{\omega\in\Omega:|f_{ij}(\omega)|>K\}}\|\le\delta.$$

Now, let j be fixed, and let $S(i,j,K)=\{\omega\in\Omega:|f_{ij}(\omega)|>K\}$. Since E has type p with constant M_p, it follows that

$$\begin{aligned}\int_0^1\Big\|\sum_{i\in B_j}r_i(t)f_{ij}\chi_{S(i,j,K)}\Big\|dt &\le M_p\Big(\sum_{i\in B_j}\|f_{ij}\chi_{S(i,j,K)}\|^p\Big)^{\frac{1}{p}}\\ &\le M_p\delta N_j^{\frac{1}{p}},\end{aligned}\tag{1}$$

where N_j denotes the cardinal of B_j, which by hypothesis satisfies $N_j\ge j-N$.

Moreover, for every $t\in[0,1]$, we have $N_j^{\frac{1}{p}}\le C\|\sum_{i\in B_j}r_i(t)f_{ij}\|$. Hence, integrating we obtain

$$\int_0^1\Big\|\sum_{i\in B_j}r_i(t)f_{ij}\Big\|dt\ge\frac{N_j^{\frac{1}{p}}}{C}.\tag{2}$$

By the triangle inequality, putting together (1) and (2), we have

$$\begin{aligned}\int_0^1\Big\|\sum_{i\in B_j}r_i(t)f_{ij}\chi_{\Omega\backslash S(i,j,K)}\Big\|dt &\ge \int_0^1\Big\|\sum_{i\in B_j}r_i(t)f_{ij}\Big\|dt-\int_0^1\Big\|\sum_{i\in B_j}r_i(t)f_{ij}\chi_{S(i,j,K)}\Big\|dt\\ &\ge N_j^{\frac{1}{p}}\Big(\frac{1}{C}-\delta M_p\Big).\end{aligned}\tag{3}$$

While on the other side, we have

$$\begin{aligned}\int_0^1 \Big\|\sum_{i\in B_j} r_i(t) f_{ij}\chi_{\Omega\setminus S(i,j,K)}\Big\| dt &\leq M\Big\|\Big(\sum_{i\in B_j}|f_{ij}\chi_{\Omega\setminus S(i,j,K)}|^2\Big)^{\frac{1}{2}}\Big\| \\ &\leq M\Big\|\Big(\sum_{i\in B_j}K^2\Big)^{\frac{1}{2}}\Big\| \qquad (4)\\ &= MKN_j^{\frac{1}{2}},\end{aligned}$$

where M is a constant given by [15, Thm. 1.d.6.(i)]. Now, putting together (3) and (4) we obtain that

$$N_j^{\frac{1}{p}-\frac{1}{2}} \leq \frac{MK}{(\frac{1}{C}-\delta M_p)},$$

and since $j-N\leq N_j$, this is obviously false for j large enough. □

Now we can give the proof of Theorem 3.4.

of Theorem 3.4. Let X be a subspace of $L_{p,q}$ which contains ℓ_p^n's uniformly, and let $\varepsilon>0$. By [5, Lemma 2.2], we may chose (x_{ij}) in X with $\|x_{ij}\|_{L_{p,q}}=1$ for $i=1,\ldots,j$ and all j, such that

$$\Big(\sum_{i=1}^j |c_i|^p\Big)^{\frac{1}{p}} \leq (1+\varepsilon)\Big\|\sum_{i=1}^j c_i x_{ij}\Big\|_{L_{p,q}},$$

for any scalars $c_1,\ldots,c_j$. By Lemma 3.5, given $K>0$, there exists j such that

$$\|x_{ij}\chi_{\{t:|x_{ij}(t)|>K\}}\|_{L_{p,q}} \geq 1-\varepsilon,$$

for some $i\leq j$.

Then, for r fixed with $1<r<p$, we have

$$\begin{aligned}\int_{\{t:|x_{ij}(t)|>K\}}|x_{ij}|ds &\leq \frac{1}{K^{r-1}}\int_{\{t:|x_{ij}(t)|>K\}}|x_{ij}|^r ds \\ &\leq \frac{1}{K^{r-1}}\|x_{ij}\|_{L_r}^r \qquad (5)\\ &\leq \frac{1}{K^{r-1}}\|x_{ij}\|_{L_{p,q}}^r \\ &= \frac{1}{K^{r-1}}.\end{aligned}$$

On the other hand, let us denote $f = x_{ij}\chi_{\{t:|x_{ij}(t)|>K\}}$ and $g = x_{ij}\chi_{\{t:|x_{ij}(t)|\leq K\}}$. For K large enough, it follows that

$$\begin{aligned}
\|x_{ij}\|_{L_{p,q}} &= \int_0^1 ((f+g)^*(s))^q s^{\frac{q}{p}-1} ds \\
&= \int_0^{\mu(\{|x_i(t)|>K\})} f^*(s)^q s^{\frac{q}{p}-1} ds + \int_{\mu(\{|x_i(t)|>K\})}^1 g^*(s-\mu(\{|x_i(t)|>K\}))^q s^{\frac{q}{p}-1} ds \\
&= \|f\|^q_{L_{p,q}} + \int_0^{\mu(\{|x_i(t)|\leq K\})} g^*(u)^q (u+\mu(\{|x_i(t)|>K\})^{\frac{q}{p}-1} du \\
&\geq \|f\|^q_{L_{p,q}} + \|g\|^q_{L_{p,q}}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|x_{ij}\chi_{\{t:|x_{ij}(t)|\leq K\}}\|_{L_{p,q}} &\leq \left(\|x_{ij}\|^q_{L_{p,q}} - \|x_{ij}\chi_{\{t:|x_{ij}(t)|>K\}}\|^q_{L_{p,q}}\right)^{\frac{1}{q}} \\
&\leq \left(1-(1-\varepsilon)^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, it follows that

$$\begin{aligned}
\int_{\{t:|x_{ij}(t)|\leq K\}} |x_{ij}| ds &= \left\|x_{ij}\chi_{\{t:|x_{ij}(t)|\leq K\}}\right\|_{L_1} \\
&\leq \left\|x_{ij}\chi_{\{t:|x_{ij}(t)|\leq K\}}\right\|_{L_{p,q}} \\
&\leq \left(1-(1-\varepsilon)^q\right)^{\frac{1}{q}}.
\end{aligned} \tag{6}$$

Considering (5) and (6), we obtain

$$\|x_{ij}\|_{L_1} \leq \frac{1}{K^{r-1}} + \left(1-(1-\varepsilon)^q\right)^{\frac{1}{q}}.$$

Since K is arbitrarily large and ε arbitrarily small, and $\|x_{ij}\|_{L_{p,q}} = 1$, the norms of $L_{p,q}$ and L_1 cannot be equivalent on X. □

Notice that as a direct consequence of Kadec-Pelczynski's dichotomy, and the previous theorem, it holds that every subspace $X \subset L_{p,q}$ which contains ℓ_p^n uniformly, must also contain a disjoint normalized sequence. In particular, X must also contain a subspace isomorphic to ℓ_q. Moreover, recall that $\ell_{p,q}^n$ are also uniformly embedded in $L_{p,q}$: consider the span of characteristic functions of n disjoint sets with the same measure. Hence, since $\ell_{p,q}^n$ contain uniformly complemented isomorphic copies of ℓ_p^k for $k \sim n^\alpha$ (for each $0 < \alpha < 1$) [2], we also get that every subspace $X \subset L_{p,q}$ which contains $\ell_{p,q}^n$ uniformly, must contain a subspace isomorphic to ℓ_q spanned by a disjoint sequence.

To finish this section, recall that apart from ℓ_q, the space ℓ_2 can be isomorphically embedded into $L_{p,q}$ via the Rademacher functions (cf. [15, Theorem 2.b.4]). Similarly, for $1 < p < 2$ and every $s \in (p, 2]$, we can consider a subspace of $L_{p,q}$ isomorphic to ℓ_s which is spanned by independent s-stable random variables. In fact, these are the only cases in which a subspace of $L_{p,q}$

can be isomorphic to some ℓ_s, and what is more interesting, according to the following result, every subspace of $L_{p,q}$ contains one of these spaces (cf. [4, Theorem 11]).

Theorem 3.6. *Suppose $1 < p < \infty$, $1 \leq q < \infty$, with $p \neq q$, and let X be a closed subspace of $L_{p,q}$.*

a) *If $p \geq 2$, then X contains an isomorphic copy of ℓ_s for some $s \in \{2, q\}$.*
b) *If $p < 2$, then X contains an isomorphic copy of ℓ_s for some $s \in \{q\} \cup (p, 2]$.*

4. Strictly singular operators on Lorentz spaces

In this section we study some properties of strictly singular operators on a Lorentz space. Recall that an operator between Banach spaces $T : X \to Y$ is strictly singular if for every infinite dimensional subspace $Z \subseteq X$ and every $\varepsilon > 0$ there exists $z \in Z$ such that

$$\|Tz\|_Y \leq \varepsilon \|z\|_X.$$

Our aim here is to provide some characterizations of strictly singular operators in terms of invertibility in certain distinguished subspaces. To this end, given an infinite-dimensional Banach space M, we will say that an operator $T : X \to Y$ is M-singular if T is never an isomorphism when restricted to any subspace of X isomorphic to M. Clearly, an operator T is strictly singular if and only if it is M-singular for every Banach space M. However, we intend to give small families of spaces (in fact finite families) $M_1, \ldots, M_n$ such that an operator on a Lorentz space is strictly singular if and only if it is M_i-singular for $i = 1, \ldots, n$. Of course, the smaller this family is, the easier it should be to check whether an operator is strictly singular.

Recall that for operators on L_p spaces, this was accomplished by L. Weis in [22] where it was proved that an operator $T : L_p \to L_p$ is strictly singular if and only if it is ℓ_p-singular and ℓ_2-singular. This characterization has been generalized recently to more general Banach lattices in [6]. Also notice that ℓ_p-singular operators have proved to be useful for studying several properties of operators on L_p spaces (see [11]). Let us see now what the situation is for operators on $L_{p,q}$ spaces.

Theorem 4.1. *Let $T : L_{p,q} \to L_{p,q}$, with $1 < p < \infty$, $1 \leq q < \infty$. The following statements are equivalent:*

(1) *T is strictly singular.*
(2) *T is ℓ_q-singular and ℓ_2-singular.*

Moreover, if $1 < p < 2$ and $q \notin (p, 2)$, or $2 \leq p < \infty$, then these are also equivalent to

(3) *There is no subspace $M \subset L_{p,q}$, isomorphic to ℓ_q or ℓ_2, with $T(M)$ complemented in $L_{p,q}$, such that $T|_M$ is an isomorphism.*

Before the proof, we need a well-known Lemma, whose proof is implicit in [22], but we include it here for completeness.

Lemma 4.2. *Let $1 \leq r \leq 2$ and (f_n) be a seminormalized basic sequence in $L_r(\mu)$, whose closed linear span is a strongly embedded subspace of $L_r(\mu)$. Then for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that*

$$\mu(A) < \delta(\varepsilon) \Rightarrow \sup_n \Big(\int_A |f_n|^r d\mu \Big)^{\frac{1}{r}} < \varepsilon.$$

Proof. Assume the contrary. Then there exist a subsequence (f_{n_k}), a sequence of measurable sets (A_k) with $\mu(A_k) < \frac{1}{2^k}$, and some $\alpha > 0$ such that

$$\int_{A_k} |f_{n_k}|^r d\mu > \alpha,$$

for all $k \in \mathbb{N}$.

Let us consider the sets $B_k = \bigcup_{j=k}^{\infty} A_j$. Since

$$\int_{B_k - B_l} |f_{n_k}|^r d\mu \underset{l\to\infty}{\longrightarrow} \int_{B_k} |f_{n_k}|^r d\mu > \alpha,$$

then there is a further subsequence (k_i) satisfying

$$\int_{B_{k_i} - B_{k_{i+1}}} |f_{n_{k_i}}|^r d\mu > \frac{\alpha}{2}.$$

Since $(B_{k_i} - B_{k_{i+1}})$ are pairwise disjoint, it follows from [10, Lemma 2], that $(f_{n_{k_i}})$ is equivalent to the unit vector basis of ℓ_r. By [5, Theorem 2.2], ℓ_r is not strongly embedded in $L_r(\mu)$. Hence, we have reached a contradiction and the proof is finished. □

of Theorem 4.1. It is clear that (1) ⇒ (2) ⇒ (3). Let us see first that (3) ⇒ (1) whenever $1 < p < 2$ and $q \notin (p, 2)$, or $2 \leq p < \infty$. To this end, suppose that T is not *SS*, then there exists an infinite-dimensional subspace $X \subset L_{p,q}$ such that $T|_X$ invertible.

First, in the case $2 < p < \infty$, by Theorem 3.2 and the fact that $L_{p,q}(0,1) \subset L_2(0,1)$, it follows that $T(X)$ contains a subspace isomorphic to ℓ_q or ℓ_2 which is complemented in $L_{p,q}$, and we are done.

Now, for the case $1 < p \leq 2$, by Theorem 3.3 it follows that $T(X)$ either contains a subspace isomorphic to ℓ_q and complemented in $L_{p,q}$ or $T(X)$ is strongly embedded in $L_{p,q}(0,1)$. If $T(X)$ contains ℓ_q complemented we are done, so suppose that $T(X)$ is strongly embedded in $L_{p,q}(0,1)$.

We claim that this forces X not to contain a subspace isomorphic to ℓ_q. Indeed, depending on q, we distinguish four cases:(*i*) $q = 2$; (*ii*) $q = p$; (*iii*) $1 \leq q < p$; and (*iv*) $q > 2$.

In case (*i*), since $L_{p,2}$ is 2-concave and has an unconditional basis, every subspace of $L_{p,2}$ isomorphic to ℓ_2 has a subspace complemented in $L_{p,2}$ (see [19, Theorem 3.1 and Remark 4]). Hence, in this case, if X contained a subspace isomorphic to ℓ_2, then $T(X)$ would contain a subspace isomorphic to ℓ_2 and complemented, which contradicts statement (3).

Case (*ii*) follows from the fact that ℓ_p is not strongly embedded in $L_p(0,1)$ (see [5, Theorem 2.2]), which is isometric to $L_{p,p}(0,1)$.

In case (iii), consider r with $q<r<p$. Notice that $L_r(0,1)$ does not contain a subspace isomorphic to ℓ_q. Hence, if X contained a subspace isomorphic to ℓ_q, then the same would hold for $T(X)$, which is strongly embedded in $L_{p,q}(0,1)$, and in particular also strongly embedded in $L_r(0,1)$. This is clearly impossible.

Finally, in case (iv) consider r with $1<r<p$. Now ℓ_q does not embed in $L_r(0,1)$, hence if X contained a subspace isomorphic to ℓ_q, then so would $T(X)$ which is strongly embedded in $L_{p,q}(0,1)\subset L_r(0,1)$. Again a contradiction.

Therefore, in any of these cases, X does not contain a subspace isomorphic to ℓ_q, and by Theorem 3.3, we can assume that X is strongly embedded in $L_{p,q}(0,1)$, as it holds for $T(X)$.

Now, let (f_n) be a normalized weakly null unconditional basic sequence in X with $\|T(f_n)\|_{L_{p,q}} > C$, for some $C>0$. Given $1<r<p$, we have $L_{p,q}(0,1)\subset L_r(0,1)$, so $[f_n]$ is also strongly embedded in $L_r(0,1)$. By Lemma 4.2, given $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that

$$\mu(A)<\delta(\varepsilon)\Rightarrow \sup_n\Big(\int_A|f_n|^r d\mu\Big)^{\frac{1}{r}}<\varepsilon.$$

Since (f_n) is bounded in L_r, for every $\varepsilon>0$, there exists $M_\varepsilon>0$ such that $\mu(\{|f_n|>M_\varepsilon\})<\delta(\varepsilon)$.

For each $n\in\mathbb{N}$, let us consider $g_n=f_n\chi_{\{|f_n|>M_\varepsilon\}}$. Clearly, $\|g_n\|_{L_r}\leq\varepsilon$. Thus, extracting a subsequence we can assume that g_n converges weakly to some $g\in L_r(0,1)$, with $\|g\|_{L_r}\leq\varepsilon$. Choose a measurable set B and $N<\infty$, such that $\mu(B^c)<\delta(\varepsilon)$ and $|g(t)|\leq N$ for $t\in B$, and define

$$h_n=(f_n-g_n-g)\chi_B.$$

If we fix ε small enough, the sequence (h_n) satisfies the following properties:

(1) h_n is seminormalized and weakly null in L_r.
(2) $|h_n(t)|\leq M$ almost everywhere for some $M<\infty$.
(3) $\|T(h_n)\|_{L_r}>C'$ for some constant $C'>0$.

These imply that (h_n) has a subsequence (h_{n_k}) which is an unconditional basic sequence in $L_2(0,1)$. Therefore, for every $m\in\mathbb{N}$ and scalars $a_1,\dots,a_m$, we have:

$$\Big\|\sum_{i=1}^m a_iT(h_{n_i})\Big\|_{L_{p,q}}\leq\|T\|\Big\|\sum_{i=1}^m a_ih_{n_i}\Big\|_{L_{p,q}}\leq\|T\|\Big\|\sum_{i=1}^m a_ih_{n_i}\Big\|_{L_2}\leq\|T\|C_1\Big(\sum_{i=1}^m|a_i|^2\Big)^{\frac12},$$

for a certain constant C_1.

On the other hand, extracting a further subsequence we can assume that $(T(h_{n_k}))$ is also an unconditional basic sequence in $L_r(0,1)$. Hence, it follows that

$$\begin{aligned}
\Big\|\sum_{i=1}^{r} a_i T(h_{n_i})\Big\|_{L_{p,q}} &\geq \Big\|\sum_{i=1}^{r} a_i T(h_{n_i})\Big\|_{L_r} \\
&\geq K\int_0^1 \Big\|\sum_{i=1}^{r} a_i r_i(u) T(h_{n_i})\Big\|_{L_r} du \\
&\geq KD\Big\|\Big(\sum_{i=1}^{r} |a_i T(h_{n_i})|^2\Big)^{\frac{1}{2}}\Big\|_{L_r} \\
&\geq KDL\Big(\sum_{i=1}^{r} \|a_i T(h_{n_i})\|_{L_r}^2\Big)^{\frac{1}{2}} \\
&\geq KDLC'\Big(\sum_{i=1}^{r} |a_i|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

where K is the unconditional constant of $(T(h_{n_k}))$, D is the constant appearing in [15, Theorem 1.d.6], L is the 2-concavity constant of L_r and C' is the constant satisfying $\|T(h_n)\|_{L_r} > C'$.

Hence, let M be the closed linear span of (h_{n_k}) in $L_{p,q}$, which is isomorphic to ℓ_2, and where T is invertible. Now in the case $q = 2$, [19, Thm. 3.1 and Remark 4] imply that $T(M)$ contains a subspace complemented in $L_{p,2}$ and isomorphic to ℓ_2, which contradicts statement (3). While in the case $q \neq 2$, then both M and $T(M)$ are strongly embedded in $L_{p,q}$. By [19, Thm. 3.1] $T(M)$ contains a subspace, still isomorphic to ℓ_2 which is complemented in L_r. Since $T(M)$ is strongly embedded in $L_{p,q}$ and $L_{p,q} \subset L_r$, it follows that there is a subspace of $T(M)$ complemented in $L_{p,q}$, in contradiction with (3).

Thus, we have shown that (3) $\Rightarrow$ (1) under the assumption that $1 < p < 2$ and $q \notin (p, 2)$, or $2 \leq p < \infty$. Hence, to finish the proof, it is enough to so that (2) $\Rightarrow$ (1) when $1 < p < 2$ and $p < q < 2$. In this case, [3] implies that $L_{p,q}$ satisfies a lower 2-estimate, so we are in a position to use [6, Theorem A]. Therefore, if $T : L_{p,q} \to L_{p,q}$ were not strictly singular, then we could find a subspace X isomorphic to ℓ_2 or generated by a pairwise disjoint sequence, in which T would be invertible. By Theorem 3.2, this would imply that T is not ℓ_2-singular or ℓ_q-singular, so the proof is finished.

□

As an application of Theorem 4.1, we can prove a stability result for the adjoints of strictly singular operators. Notice, that unlike compact operators, strictly singular are not stable under duality in general (cf. [20], [23]): take for instance any quotient mapping $T : \ell_1 \to \ell_p$ for $1 < p < \infty$; this operator is strictly singular but T^* is even an isomorphic embedding. However, for an operator $T : L_p \to L_p$ it is true that T is strictly singular if and only if, its adjoint T^* is strictly singular. This fact was first proved for $p > 2$ by V. Milman in [18], and sometime later the proof was completed for $p < 2$ by L. Weis in [22]. We present here the extension of this result for operators on $L_{p,q}$ spaces.

Theorem 4.3. *Let $1 < p, q < \infty$ and $T : L_{p,q} \to L_{p,q}$, and consider the following statements:*

(1) *T is strictly singular,*
(2) *T^* is strictly singular.*

If $2 \leq p < \infty$, or $1 < p < 2$ and $q \notin (p, 2)$, then the implication (2) $\Rightarrow$ (1) holds. Similarly, if $1 < p \leq 2$, or $2 < p < \infty$ and $q \notin (2, p)$, then (1) $\Rightarrow$ (2) holds.

Proof. Since for $1 < p, q < \infty$ the spaces $L_{p,q}$ are reflexive and $T^{**} = T$, by duality it is enough to prove the first assertion. Hence, let p and q satisfy $2 \leq p < \infty$ or $1 < p < 2$ and $q \notin (p, 2)$, and suppose $T : L_{p,q} \to L_{p,q}$ is not strictly singular. By Theorem 4.1, there exists a subspace $M \subset L_{p,q}$ such that the restriction $T|_M$ is an isomorphism with M and $T(M)$ both complemented in $L_{p,q}$ and isomorphic to ℓ_q or ℓ_2.

Let us see how this implies that T^* cannot be strictly singular (compare to [23, Theorem 2.2]). First, recall that given a subspace M of a Banach space X, the polar $M^\perp$ denotes the subspace of X^* consisting of all functionals that annihilate M. Since $T|_M$ is an isomorphism onto $T(M)$, which is complemented in $L_{p,q}$, let $P : L_{p,q} \to T(M)$ denote this projection and consider the operator $R : L_{p,q} \to L_{p,q}$ given by

$$\begin{array}{ccc} L_{p,q} & \xrightarrow{\quad R \quad} & L_{p,q} \\ {\scriptstyle P}\downarrow & & \uparrow \\ T(M) & \xrightarrow{\ (T|_M)^{-1}\ } & M \end{array}$$

Now, if Y denotes the orthogonal complement of $T(M)$ in $L_{p,q}$ so that $L_{p,q} = T(M) \oplus Y$, then we clearly have that TR coincides with the identity on $T(M)$ and is identically zero on Y. Let us see that T^* must be invertible on $Y^\perp$ which is isomorphic to $T(M)^*$, and in particular infinite-dimensional, so that T^* is not strictly singular.

Indeed, given $f \in L_{p,q}$, let us write $f = f_1 + f_2$ with $f_1 \in T(M)$ and $f_2 \in Y$. Now, for $\varphi \in Y^\perp$ and every $f \in L_{p,q}$ we have

$$\langle R^*T^*(\varphi), f\rangle = \langle \varphi, TR(f)\rangle = \langle \varphi, TR(f_1 + f_2)\rangle = \langle \varphi, f_1\rangle = \langle \varphi, f\rangle.$$

Thus, R^*T^* coincides with the identity on $Y^\perp$, and so for any $\varphi \in Y^\perp$ we have

$$\|T^*\varphi\| = \frac{\|R^*\|}{\|R^*\|}\|T^*\varphi\| \geq \frac{1}{\|R^*\|}\|R^*T^*\varphi\| = \frac{1}{\|R^*\|}\|\varphi\|.$$

Hence, T^* is not strictly singular as we wanted to prove. □

In particular, if $1 < p < 2$ and $q \notin (p, 2)$, or $2 < p < \infty$ and $q \notin (2, p)$, or $p = 2$ and $1 < q < \infty$, then an operator $T : L_{p,q} \to L_{p,q}$ is strictly singular if and only if so is T^*.

Recall that the order continuous part of $L_{p,\infty}(0,1)$ is defined to be the closure of the simple functions in $L_{p,\infty}(0,1)$ and is denoted by $L^o_{p,\infty}$. This is a separable Banach lattice whose dual $(L^o_{p,\infty})^*$ can be identified in a canonical way with $L_{p',1}(0,1)$ (where $\frac{1}{p}+\frac{1}{p'}=1$). It can be shown that the implication (1) $\Rightarrow$ (2) of the previous theorem also holds in this case.

Proposition 4.4. *Let $1<p<\infty$. If $T: L^o_{p,\infty}(0,1)\to L^o_{p,\infty}(0,1)$ is strictly singular, then so is $T^*: L_{p',1}(0,1)\to L_{p',1}(0,1)$.*

Proof. Indeed, if T^* is not strictly singular, then, by Theorem 4.1, there exists a subspace $M\subset L_{p',1}$ isomorphic to ℓ_2 or ℓ_1 such that the restriction $T^*|_X$ is an isomorphism and $T^*(M)$ is complemented. In fact, we have that T^* is invertible on a subspace M such that either

(i) $M\simeq T^*(M)\simeq \ell_2$ with M and $T^*(M)$ strongly embedded, or
(ii) $M\simeq T^*(M)\simeq \ell_1$ with $M=[f_n]$ and $T^*(f_n)$ disjoint.

Any of these cases yields a contradiction with the fact that T is strictly singular. Indeed, if (i) holds, then this implies that T^{**} is an isomorphism on a complemented subspace Z isomorphic to ℓ_2 which is identified with the dual of the subspace $T(M)\subset L_{p',1}$, and the projection is the adjoint of the projection onto $T(M)$, $P: L_{p',1}\to L_{p',1}$. However, since $T(M)$ is strongly embedded we can factor P through the formal inclusion $L_{p',1}\hookrightarrow L_r$, for some $1<r<p'$. This implies that Z is in fact complemented in $L^o_{p,\infty}$, hence T is not strictly singular.

Now, if case (ii) holds, then as in the proof of [7, Thm. 5.1] we can find functionals F_n on $L_{p',1}$ with $\langle F_n, Tf_n\rangle = 1$ and $\langle F_n, Tf_m\rangle = 0$ for $n\neq m$. These functionals are defined by

$$\langle F_n, f\rangle = \frac{\int f(\tau(t))\chi_{[\varepsilon_n,|A_n|]}\mathtt{sgn}T^*f_n(\tau(t))t^{\frac{1}{p'}-1}dt}{\int_{\varepsilon_n}^{|A_n|}|T^*f_n(\tau(t))|t^{\frac{1}{p'}-1}dt},$$

where for each $n\in\mathbb{N}$ the set A_n denotes the support of the function T^*f_n, $\tau_n:[0,|A_n|]\to A_n$ are measure preserving functions such that $\int_0^1|T^*f_n(\tau_n(t))|t^{\frac{1}{p'}-1}dt = \|T^*f_n\|$, and $\varepsilon_n>0$ are sufficiently small (see [7, Thm. 5.1]). It follows that the functionals F_n are in fact elements of the order continuous part $L^o_{p,\infty}$ and are equivalent to the unit vector basis of c_0. Moreover, $\|TF_n\|\geq \langle F_n, T^*f_n\rangle\frac{1}{\|f_n\|}>\alpha$ for every $n\in\mathbb{N}$, and some $\alpha>0$. Hence, passing to a further subsequence, for certain constants $c, C>0$ and all scalars $a_1,\dots,a_n$, we have

$$c\max_{1\leq i\leq n}|a_i|\leq\Big\|\sum_{i=1}^n a_iTF_i\Big\|\leq\|T\|\Big\|\sum_{i=1}^n a_iF_i\Big\|\leq C\max_{1\leq i\leq n}|a_i|.$$

This implies that T is an isomorphism on a subspace isomorphic to c_0, in contradiction with the fact that T is strictly singular. □

Notice that the L_p-space version of Theorems 4.1 and 4.3 appears in [22] as a joint result. However, in the setting of $L_{p,q}$ spaces we need to state them separately in order to distinguish which implications hold depending on the parameters p and q. In fact, notice that Theorem 4.3 does not hold if the conditions on p and q are not satisfied as the following shows.

Example 4.5. *Let $1 < p < q < 2$. There exists an operator $T : L_{p,q} \to L_{p,q}$ such that T^* is strictly singular, but T is not.*

Proof. Indeed, since $1 < p < q < 2$ we can consider a sequence of independent q-stable random variables (g_n) in $L_{p,q}$. Moreover, let (f_n) be a normalized sequence of disjoint elements in $L_{p,q}$ whose span is isomorphic to ℓ_q and complemented in $L_{p,q}$. Let $P : L_{p,q} \to [f_n]$ denote this projection.

Notice, that the subspace $[g_n]$ of $L_{p,q}$ is strongly embedded in L_p. In particular, $[g_n]$ is a closed subspace of $L_{p,r}$ isomorphic to ℓ_q, for any fixed r with $p < r < q$.

Let us consider the following operator

$$\begin{array}{ccccc} L_{p,q} & & \xrightarrow{\quad T \quad} & & L_{p,q} \\ {\scriptstyle P}\downarrow & & & & \uparrow{\scriptstyle I_r} \\ [f_n] & \xrightarrow{R} & [g_n] & \xrightarrow{S} & L_{p,r} \end{array}$$

where R is an isomorphism mapping each f_n to g_n, S is the isomorphic embedding of $[g_n]$ in $L_{p,r}$, and I_r denotes the canonical inclusion from $L_{p,r}$ to $L_{p,q}$.

Clearly, T is an isomorphism on a subspace isomorphic to ℓ_q, thus it is not strictly singular. However, the adjoint operator $T^* : L_{p',q'} \to L_{p',q'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$ is strictly singular. Indeed, notice first that T^* factors through $L_{p',r'}$, hence T^* cannot be an isomorphism on any subspace isomorphic to $\ell_{q'}$, because $L_{p',r'}$ does not contain any subspace isomorphic to $\ell_{q'}$. On the other hand, T^* factors through $[g_n]^* \simeq \ell_{q'}$, hence T^* cannot be an isomorphism on any subspace isomorphic to ℓ_2. Therefore, by Theorem 4.1, T^* is strictly singular as claimed. □

Observe that the operator T given in the above example also shows that implication (3) $\Rightarrow$ (1) of Theorem 4.1 does not hold if the conditions on the parameters p and q are not satisfied.

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