INTERPOLATION OF BANACH LATTICES AND FACTORIZATION OF p-CONVEX AND q-CONCAVE OPERATORS

YVES RAYNAUD AND PEDRO TRADACETE

Abstract. We extend a result of Šestakov to compare the complex interpolation method $[X_0, X_1]_\theta$ with Calderón-Lozanovskii's construction $X_0^{1-\theta}X_1^\theta$, in the context of abstract Banach lattices. This allows us to prove that an operator between Banach lattices $T : E \to F$ which is p-convex and q-concave, factors, for any $\theta \in (0,1)$, as $T = T_2T_1$, where T_2 is $(\frac{p}{\theta+(1-\theta)p})$-convex and T_1 is $(\frac{q}{1-\theta})$-concave.

1. Introduction

In [10], J. L. Krivine showed that the composition T_2T_1 of a p-convex operator $T_1 : X \to E$ and a p-concave operator $T_2 : E \to Y$, where X, Y are Banach spaces and E is a Banach lattice, factors always through a space $L_p(\mu)$. Motivated by this fact, in this note we study factorization properties of p-convex and q-concave operators. More precisely, we consider the following question: if an operator between Banach lattices $T : E \to F$ is both p-convex and q-concave, does it necessarily factor as $T = T_2T_1$ where T_2 is p-convex and T_1 q-concave? Note that such a product is always p-convex and q-concave, hence we are interested in a converse statement.

In general, the answer to this question is negative (see Examples 4 and 5). However, we show that for every $\theta \in (0,1)$, the operator T can be written as $T = T_2T_1$ where T_2 is $(\frac{p}{\theta+(1-\theta)p})$-convex and T_1 is $(\frac{q}{1-\theta})$-concave (see Theorem 15). To prove this fact, we exhibit first a canonical way in which a p-convex (respectively q-concave) operator factors through a p-convex (resp. q-concave) Banach lattice. Afterwards, we present some interpolation results regarding the complex interpolation method and the Calderón-Lozanovskii construction for Banach lattices. In particular, we prove a comparison theorem between these two constructions that had been apparently known in the literature only in the case of Banach lattices of measurable functions. Thus, we extend this comparison theorem due to Šestakov (see [19]) to the more general setting of compatible pairs of Banach lattices which need not be function spaces (that is, ideals in the space of measurable functions on some measure space). This will constitute a key ingredient in our proof of the main factorization result.

The problem of factoring an operator through p-convex and q-concave operators had also been considered, although in a quite different manner, by S. Reisner in [17]; in particular, Theorem 1 was essentially proved in [17, Sec. 2, Lemma 6]. Moreover, this author showed that for fixed p, q, the class of operators between Banach spaces $T : E \to F$ such that the composition with the canonical inclusion $j_F : F \to F^{**}$ factors as $j_FT = UV$ with V p-convex and U q-concave, forms an operator ideal. However, his approach to an analogous statement of Theorem 1 for p-convex operators is not satisfactory for our interests, because in [17] this is only considered as a dual fact to that of factoring q-concave operators, and, as we will show in Section 3, these

Date: June 8, 2009.

2000 *Mathematics Subject Classification.* Primary: 47B60. Secondary: 46B42, 46B70.

Key words and phrases. Interpolation of Banach lattices, method of Calderón-Lozanovskii, factorization, p-convex operator, q-concave operator.

The second author was partially supported by grants MEC MTM2005-00082, AP-2004-4841 and Santander/Complutense PR34/07-15837.

factorizations do not behave in an entirely dual way. Moreover, from Theorem 1 we can only get that for a p-convex operator $T : E \to F$, the bi-adjoint $T^{**} : E^{**} \to F^{**}$ factors through a p-convex Banach lattice, which suffices for the purposes in [17], but are not enough to prove our main result on factorization (Theorem 15). We mention that our proofs of Theorems 1 and 3 have been inspired in fact by the work of P. Meyer-Nieberg in [14] on factorization of cone p-summing and p-majorizing operators (see also [15, 2.8]). Then we realized that some of the main ideas of our work were already present in the paper [17].

The organization of the paper goes as follows. Section 2 contains the proofs of the basic factorizations for p-convex (resp. q-concave) operators. It is also shown that these constructions can be equivalently obtained by means of maximality properties of factorization diagrams. The next section, Section 3, is devoted to the study of the duality relation between the factorization spaces for p-convex and q-concave operators. Next, Section 4 is mainly devoted to the proof of the extension of Šestakov's result to compatible pairs of Banach lattices. Then, in Section 5 we prove the main theorem on factorization of operators which are both p-convex and q-concave. Here the extension of Šestakov's result is used for interpolating operators which are not necessarily positive (at the difference of the situation in [17]) between Banach lattices which are perhaps not representable as ideal function spaces. In this section we show also how some examples can be used to see that in general the factorization cannot be improved much further. Finally, in Section 6 we show the connection between the constructions of the first section and the factorization theorem of Krivine.

We refer the reader to [11], [15] and [18] for any unexplained terminology on Banach lattice theory, and to [4] and [9] for those of interpolation theory.

Aknowledgement: We thank Prof. F. L. Hernández for bringing the reference [17] to our attention after that a first draft of the present paper was completed.

2. Two basic constructions

Let E be a Banach lattice and X a Banach space. Recall that an operator $T : E \to X$ is q-concave for $1 \leq q \leq \infty$, if there exists a constant $M < \infty$ so that

$$\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} \leq M \Big\|\Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}}\Big\|, \qquad \text{if } 1 \leq q < \infty,$$

or

$$\max_{1\leq i\leq n} \|Tx_i\| \leq M \Big\|\bigvee_{i=1}^n |x_i|\Big\|, \qquad \text{if } q = \infty,$$

for every choice of vectors $(x_i)_{i=1}^n$ in E (cf. [11, 1.d]). The smallest possible value of M is denoted by $M_{(q)}(T)$.

Similarly, an operator $T : X \to E$ is p-convex for $1 \leq p \leq \infty$, if there exists a constant $M < \infty$ such that

$$\Big\|\Big(\sum_{i=1}^n |Tx_i|^p\Big)^{\frac{1}{p}}\Big\| \leq M \Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}}, \qquad \text{if } 1 \leq p < \infty,$$

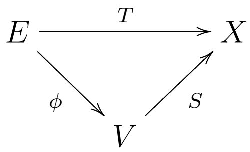
or

$$\Big\|\bigvee_{i=1}^n |Tx_i|\Big\| \leq M \max_{1\leq i\leq n} \|x_i\|, \qquad \text{if } p = \infty,$$

for every choice of vectors $(x_i)_{i=1}^n$ in X. The smallest possible value of M is denoted by $M^{(p)}(T)$. Recall that a Banach lattice is q-concave (resp. p-convex) whenever the identity operator is q-concave (resp. p-convex).

The following result was essentially proved in [17, Sec. 2, Lemma 6]. However, we include a similar proof for completeness, since we will be using the explicit construction throughout.

Theorem 1. *Let E be a Banach lattice, X a Banach space and $1 \leq q \leq \infty$. An operator $T : E \to X$ is q-concave if and only if there exist a q-concave Banach lattice V, a positive operator $\phi : E \to V$ (in fact, a lattice homomorphism with dense image), and another operator $S : V \to X$ such that $T = S\phi$.*



Proof. Let us suppose $q < \infty$. The proof for the case $q = \infty$ is trivial because every Banach lattice is ∞-concave. However, the precise construction carried out here for $q < \infty$ has its analogue for $q = \infty$.

For the "if" part, let $(x_i)_{i=1}^n$ in E. Since V is q-concave and ϕ is positive, by [11, Prop. 1.d.9] we have

$$\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} \leq \|S\| M_{(q)}(I_V)\|\phi\| \Big\| \Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}} \Big\|,$$

which yields that T is q-concave.

Now, for the other implication, given $x \in E$, let us consider

$$\rho(x) = \sup\Big\{ \Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} : \Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}} \leq |x| \Big\}.$$

If $M_{(q)}(T)$ denotes the q-concavity constant of T, then for $(x_i)_{i=1}^n$ in E, we have

$$\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{\frac{1}{q}} \leq M_{(q)}(T) \Big\| \Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}} \Big\|.$$

In particular, for all $x \in E$

$$\|Tx\| \leq \rho(x) \leq M_{(q)}(T)\|x\|.$$

Moreover, ρ is a lattice semi-norm on E. Indeed, for any $x \in E$ and $\lambda \geq 0$ it is clear that $\rho(\lambda x) = \lambda \rho(x)$. In order to prove the triangle inequality, let $x, y \in E$ and $z = |x| + |y|$, and denote $I_z \subset E$ the ideal generated by z in E, which is identified with a space $C(K)$ in which z corresponds to the function identically one [18, II.7]. Now, for every $\varepsilon > 0$ let $z_1, \ldots, z_n \in E$ such that $\big(\sum_{i=1}^n |z_i|^q\big)^{\frac{1}{q}} \leq |z|$ and

$$\rho(z) \leq \big(\sum_{i=1}^n \|Tz_i\|^q\big)^{\frac{1}{q}} + \varepsilon.$$

Since $x, y \in I_z$, they correspond to functions $f, g \in C(K)$ such that $|f(t)| + |g(t)| = 1$ for every $t \in K$. Similarly, z_i corresponds to $h_i \in C(K)$ with $\big(\sum_{i=1}^n |h_i(t)|^q\big)^{\frac{1}{q}} \leq 1$ for every $t \in K$. Hence we can consider

$$\begin{cases} f_i(t) = h_i(t) f(t), \\ g_i(t) = h_i(t) g(t), \end{cases}$$

which belong to $C(K)$ and satisfy $\big(\sum_{i=1}^n |f_i(t)|^q\big)^{\frac{1}{q}} \leq |f(t)|$ and $\big(\sum_{i=1}^n |g_i(t)|^q\big)^{\frac{1}{q}} \leq |g(t)|$. This means that we can consider $(x_i)_{i=1}^n$ and $(y_i)_{i=1}^n$ such that $\big(\sum_{i=1}^n |x_i|^q\big)^{\frac{1}{q}} \leq |x|$ and $\big(\sum_{i=1}^n |y_i|^q\big)^{\frac{1}{q}} \leq |y|$ in E, with $x_i + y_i = z_i$. Thus, it follows that

$$\rho(x+y) \leq \rho(x) + \rho(y) + \varepsilon,$$

and since this holds for every $\varepsilon > 0$, the triangle inequality is proved.

Now, if $|y| \leq |x|$, then for any $(x_i)_{i=1}^n$ such that $\left(\sum_{i=1}^{n}|x_i|^q\right)^{\frac{1}{q}} \leq |y|$, it holds that $\left(\sum_{i=1}^{n}|x_i|^q\right)^{\frac{1}{q}} \leq |x|$, hence for any such $\{x_i\}_{i=1}^n$, $\left(\sum_{i=1}^{n}\|Tx_i\|^q\right)^{\frac{1}{q}} \leq \rho(x)$. This implies that $\rho(y) \leq \rho(x)$.

Let V denote the Banach lattice obtained by completing $E/\rho^{-1}(0)$ with the norm induced by ρ. Let ϕ denote the quotient map from E to $E/\rho^{-1}(0)$, seen as a map to V. Now, for $x \in E$ let us define $S(\phi(x)) = T(x)$. Since $\|Tx\| \leq \rho(x)$, S is well defined and extends to a bounded operator $S : V \to X$, such that $T = S\phi$.

Now, let $(x_i)_{i=1}^n$ in E. For every $\varepsilon > 0$ and for every $i = 1, \ldots, n$ there exist $\{y_j^i\}_{j=1}^{k_i}$ in E such that $\left(\sum_{j=1}^{k_i}|y_j^i|^q\right)^{\frac{1}{q}} \leq |x_i|$ and

$$\rho(x_i)^q = \sup\left\{\sum_{j=1}^{k}\|Ty_j\|^q : \left(\sum_{j=1}^{k}|y_j|^q\right)^{\frac{1}{q}} \leq |x_i|\right\} \leq \sum_{j=1}^{k_i}\|Ty_j^i\|^q + \frac{\varepsilon^q}{n},$$

for every $i = 1, \ldots, n$. Therefore, we have

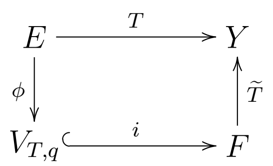
$$\left(\sum_{i=1}^{n}\rho(x_i)^q\right)^{\frac{1}{q}} \leq \rho\left(\left(\sum_{i=1}^{n}|x_i|^q\right)^{\frac{1}{q}}\right) + \varepsilon,$$

and since this holds for every $\varepsilon > 0$, the normed lattice $E/\rho^{-1}(0)$ is q-concave; hence, the same holds for its completion V. □

Since the lattice V constructed in the proof depends on the operator $T : E \to X$ and q, we will denote it by $V_{T,q}$ whenever needed. Similarly we will denote ρ_T for the expression defining the norm of $V_{T,q}$.

Remark 1. Note that $V_{T,q}$ has q-concavity constant one. In particular if E is q-concave and $T = \mathrm{id}_E$ is the identity, then $V_{T,q}$ is the usual lattice renorming of E with q-concavity constant one.

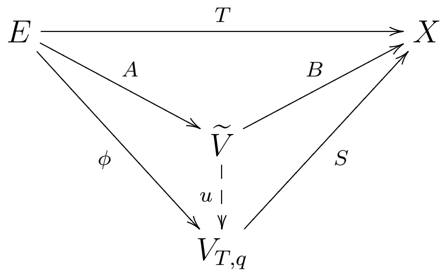
Remark 2. In [8], it was proved that an order weakly compact operator $T : E \to Y$ (i.e. $T[-x, x]$ is relatively weakly compact for every $x \in E_+$) always factors through an order continuous Banach lattice F. The Banach lattice F is constructed by means of the expression $\|x\|_F = \sup\{\|Ty\| : |y| \leq |x|\}$, for $x \in E$, which yields a Banach lattice in the usual way. Notice that if $T : E \to Y$ is q-concave, which implies being order weakly compact, then $\|x\|_F \leq \rho_T(x)$, hence we can consider a natural map $V_{T,q} \xrightarrow{i} F$ such that we can factor T as follows:



Moreover, F coincides with $V_{T,\infty}$, so in a sense the previous Theorem is an extension of [8, Thm. I.2].

The factorization given in Theorem 1 is in a certain sense maximal, as the following Proposition shows.

Proposition 2. *Let $T : E \to X$ be a q-concave operator. Suppose that T factors through a q-concave Banach lattice $\widetilde{V}$ with factors $A : E \to \widetilde{V}$ and $B : \widetilde{V} \to X$, such that A is a lattice homomorphism whose image is dense in $\widetilde{V}$, and $T = B \circ A$. Then there is a lattice* *homomorphism* $u : \widetilde{V} \to V_{T,q}$ *such that* $\phi = u \circ A$ *and* $S \circ u = B$.



Proof. Let us define for $x \in E$, $u(A(x)) = \phi(x)$. Notice that, since A is a lattice homomorphism, for $\{x_i\}_{i=1}^n$ in E, such that $\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}} \leq |x|$, we have

$$\begin{aligned}\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{\frac{1}{q}} &= \left(\sum_{i=1}^n \|BAx_i\|^q\right)^{\frac{1}{q}} \leq \|B\| \left(\sum_{i=1}^n \|Ax_i\|^q\right)^{\frac{1}{q}} \leq \|B\| M_{(q)}(I_{\widetilde{V}}) \left\| \left(\sum_{i=1}^n |A(x_i)|^q\right)^{\frac{1}{q}} \right\| \\ &= \|B\| M_{(q)}(I_{\widetilde{V}}) \left\| A\left(\left(\sum_{i=1}^n |x_i|^q\right)^{\frac{1}{q}}\right) \right\| \leq \|B\| M_{(q)}(I_{\widetilde{V}}) \|A(x)\|.\end{aligned}$$

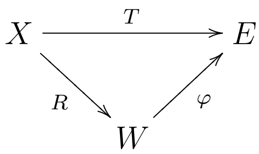
Therefore,

$$\|u(A(x))\| = \|\phi(x)\| = \rho_T(x) \leq \|B\| M_{(q)}(I_{\widetilde{V}}) \|A(x)\|.$$

Since A has dense image, the preceding inequality implies that u can be extended to a bounded operator $u : \widetilde{V} \to V_{(T,q)}$, which is clearly a lattice homomorphism and satisfies the required properties. □

There is an analogous version of Theorem 1 for p-convex operators, which could be considered, in a sense, as a predual construction to that given in Theorem 1 (see Section 3).

Theorem 3. *Let E be a Banach lattice, X a Banach space and $1 \leq p \leq \infty$. An operator $T : X \to E$ is p-convex if and only if there exist a p-convex Banach lattice W, a positive operator (an injective interval preserving lattice homomorphism) $\varphi : W \to E$ and another operator $R : X \to W$ such that $T = \varphi R$.*



Proof. Let us suppose $p < \infty$. The proof for the case $p = \infty$ is analogous, with the usual changes.

As in the proof of Theorem 1, [11, 1.d.9] yields one implication. For the non-trivial one, let $T : X \to E$ be p-convex. Let us consider the following set

$$S = \{y \in E : \, |y| \leq \left(\sum_{i=1}^k |Tx_i|^p\right)^{\frac{1}{p}}, \text{ where } \sum_{i=1}^k \|x_i\|^p \leq 1 \text{ and } k \in \mathbb{N}\}.$$

We can consider the Minkowski functional defined by its closure $\overline{S}$ in E

$$\|z\|_W = \inf\{\lambda > 0 : z \in \lambda \overline{S}\}.$$

Clearly $\overline{S}$ is solid, and since T is p-convex, it is also a bounded set of E. Let us consider the space $W = \{z \in E : \|z\|_W < \infty\}$. We claim that for any $z_1, \ldots, z_n$ in W, it holds that

$\big(\sum_{i=1}^{k}|z_i|^p\big)^{1/p}$ belongs to W and

$$\Big\|\Big(\sum_{i=1}^{k}|z_i|^p\Big)^{\frac{1}{p}}\Big\|_W \leq \Big(\sum_{i=1}^{n}\|z_i\|_W^p\Big)^{\frac{1}{p}}.$$

Indeed, given $z_1,\ldots,z_n$ in W, for every $\varepsilon>0$ and for every $i=1,\ldots,n$ there exist λ_i with $z_i\in\lambda_i\overline{S}$, such that

$$\lambda_i^p\leq\inf\Big\{\mu^p:\, z_i\in\mu\overline{S}\Big\}+\frac{\varepsilon^p}{n},$$

for each $i=1,\ldots,n$.

This means that for every $i=1,\ldots,n$, and for every $\delta>0$ there exists y_i^δ in E with $\|z_i-y_i^\delta\|_E\leq\delta$, and

$$|y_i^\delta|\leq\Big(\sum_{j=1}^{m_{i,\delta}}|Tx_j^{i,\delta}|^p\Big)^{\frac{1}{p}},$$

where $\{x_j^{i,\delta}\}_{j=1}^{m_{i,\delta}}$ satisfy

$$\Big(\sum_{j=1}^{m_{i,\delta}}\|x_j^{i,\delta}\|^p\Big)^{\frac{1}{p}}\leq\lambda_i,$$

for each $i=1,\ldots,n$, and each $\delta>0$.

Now, for each $\delta>0$ let

$$w_\delta=\Big(\sum_{i=1}^{n}|y_i^\delta|^p\Big)^{\frac{1}{p}}.$$

Notice that

$$\Big\|\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}-w_\delta\Big\|_E\leq\Big\|\Big(\sum_{i=1}^{n}|z_i-y_i^\delta|^p\Big)^{\frac{1}{p}}\Big\|_E\leq\sum_{i=1}^{n}\|z_i-y_i^\delta\|_E\leq n\delta.$$

Moreover, note that for every $\delta>0$, w_δ belongs to $\Big(\sum_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}S$. Indeed,

$$|w_\delta|=\Big(\sum_{i=1}^{n}|y_i^\delta|^p\Big)^{\frac{1}{p}}\leq\Big(\sum_{i=1}^{n}\sum_{j=1}^{m_{i,\delta}}|Tx_j^{i,\delta}|^p\Big)^{\frac{1}{p}},$$

and

$$\Big(\sum_{i=1}^{n}\sum_{j=1}^{m_{i,\delta}}\|x_j^{i,\delta}\|^p\Big)^{\frac{1}{p}}\leq\Big(\sum_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}.$$

Hence, $\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}\in\Big(\sum_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}\overline{S}$. Therefore, it follows that

$$\begin{aligned}\Big\|\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}\Big\|_W &=\inf\{\mu>0:\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}\in\mu\overline{S}\}\leq\Big(\sum_{i=1}^{n}\lambda_i^p\Big)^{\frac{1}{p}}\\ &\leq\Big(\sum_{i=1}^{n}\Big(\inf\{\mu^p:\, z_i\in\mu\overline{S}\}+\tfrac{\varepsilon^p}{n}\Big)\Big)^{\frac{1}{p}}\leq\Big(\sum_{i=1}^{n}\|z_i\|_W^p\Big)^{\frac{1}{p}}+\varepsilon.\end{aligned}$$

Since this holds for every $\varepsilon>0$, we finally have

$$\Big\|\Big(\sum_{i=1}^{n}|z_i|^p\Big)^{\frac{1}{p}}\Big\|_W\leq\Big(\sum_{i=1}^{n}\|z_i\|_W^p\Big)^{\frac{1}{p}}.$$

It follows that the Minkowski functional $\|.\|_W$ is a norm on W. Indeed since $\overline{S}$ is bounded, $\|x\|_W = 0$ implies $x = 0$. Moreover if $x, y \in W$ are non zero, set $u = \frac{|x|}{\|x\|_W}$, $v = \frac{|y|}{\|y\|_W}$, $\alpha = \frac{\|x\|_W}{\|x\|_W+\|y\|_W}$, $\beta = \frac{\|y\|_W}{\|x\|_W+\|y\|_W}$. Since $\|u\|_W = \|v\|_W = 1$, $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ we have

$$\begin{aligned}\|x+y\|_W &\leq \|\,|x|+|y|\,\|_W = (\|x\|_W + \|y\|_W)\|\alpha u + \beta v\|_W \leq (\|x\|_W + \|y\|_W)\|(\alpha u^p + \beta v^p)^{1/p}\|_W \\ &\leq (\|x\|_W + \|y\|_W)(\alpha\|u\|_W^p + \beta\|v\|^p)_W^{1/p} = \|x\|_W + \|y\|_W\end{aligned}$$

Therefore, $(W, \|\cdot\|_W)$ is a p-convex normed lattice. We claim that W is complete, and hence a p-convex Banach lattice. Indeed, let $(w_i)_{i=1}^\infty$ be a Cauchy sequence in W. Since for every $z \in E$ it holds that

$$\|z\|_E \leq M^{(p)}(T)\|z\|_W,$$

it follows that $(w_i)_{i=1}^\infty$ is also a Cauchy sequence in E. Let $w \in E$ be its limit. Notice that since w_i are bounded in W, there exists some $\lambda < \infty$ such that $w_i \in \lambda\overline{S}$ for every $i = 1, 2, \dots$ and since $\overline{S}$ is closed in E, we must have $w \in \lambda\overline{S}$. Thus, w belongs to W, and we will show that $(w_i)_{i=1}^\infty$ converges to w also in W. To this end, let $\varepsilon > 0$. Since $(w_i)_{i=1}^\infty$ is a Cauchy sequence, there exists N such that $w_i - w_j \in \varepsilon\overline{S}$ when $i, j \geq N$. Thus, if $i \geq N$ we can write

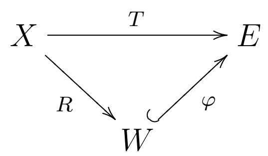
$$w - w_i = (w - w_j) + (w_j - w_i)$$

for every $j \in \mathbb{N}$, and letting $j \to \infty$ we obtain that $w - w_i \in \varepsilon\overline{S}$. This shows that $w_i \to w$ in W, and hence W is complete, as claimed.

Clearly, by the definition of W we have

$$\|Tx\|_W \leq \|x\|_X$$

for every $x \in X$. Moreover, as noticed above it also holds that $\|z\|_E \leq M^{(p)}(T)\|z\|_W$ for each $z \in E$, therefore the formal inclusion $\varphi : W \hookrightarrow E$ is clearly an injective interval preserving lattice homomorphism, and we have the following diagram



where R is defined by $Rx = Tx$ for $x \in X$. This finishes the proof. □

As with the Banach lattice constructed in Theorem 1, we will denote by $W_{T,p}$ the Banach lattice obtained in the proof of Theorem 3.

Remark 3. The operator $\varphi : W_{T,p} \to E$ constructed in the proof is an injective, interval preserving lattice homomorphism. Moreover, it satisfies that the image of the closed unit ball $\varphi(B_{W_{T,p}})$ is a closed set in E. This let us introduce the class $\mathcal{C}$ consisting of operators $T : E \to F$ between Banach lattices which are injective, interval preserving lattice homomorphisms, such that the image of the closed unit ball $T(B_E)$ is closed in F. The importance of this class will become clear next.

Remark 4. Note that if $T : X \to E$ is p-convex, then it is also p'-convex for every $1 \leq p' \leq p$. Hence, if we consider the factorization spaces $W_{T,p}$ and $W_{T,p'}$ it always holds that

$$W_{T,p'} \hookrightarrow W_{T,p},$$

with norm smaller than or equal to one.

Indeed, this follows from the following two facts. First, the set

$$S = \{y \in E : |y| \leq \left(\sum_{i=1}^k |Tx_i|^p\right)^{\frac{1}{p}}, \text{ with } \sum_{i=1}^k \|x_i\|^p \leq 1\}$$

can be equivalently described by

$$S=\{y\in E:\ |y|\leq\Big(\sum_{i=1}^{k}a_i|Tw_i|^p\Big)^{\frac{1}{p}},\text{ with }\|w_i\|\leq 1, a_i\geq 0,\sum_{i=1}^{k}a_i=1\}.$$

Furthermore, for $1\leq p'\leq p$, and $a_i\geq 0$ with $\sum\limits_{i=1}^{k}a_i=1$ it always holds that

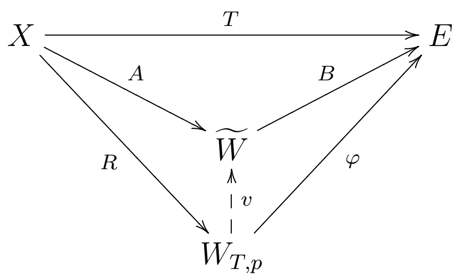
$$\Big(\sum_{i=1}^{k}a_i|Tw_i|^{p'}\Big)^{\frac{1}{p'}}\leq\Big(\sum_{i=1}^{k}a_i|Tw_i|^{p}\Big)^{\frac{1}{p}}.$$

Hence the unit ball of $W_{T,p'}$ is contained in that of $W_{T,p}$.

Remark 5. $W_{T,p}$ has p-convexity constant equal to one. If E is already p-convex and $T:E\to E$ is the identity then $W_{T,p}$ is a renorming of E with p-convexity constant one.

As for Proposition 2, the construction of Theorem 3 is in a sense minimal.

Proposition 4. *Let $T:X\to E$ be a p-convex operator, such that there exist a p-convex Banach lattice $\widetilde{W}$ and operators $A:X\to\widetilde{W}$ and $B:\widetilde{W}\to E$ with $T=BA$ and B belonging to the class $\mathcal{C}$. Then there exists an operator $v:W_{T,p}\to\widetilde{W}$ such that $vR=A$ and $Bv=\varphi$.*



Proof. Let us define v. Let $y\in W_{T,p}$ with $\|y\|_{W_{T,p}}\leq 1$. By definition, there exists a sequence $(y_n)_{n=1}^\infty$ in E such that $y_n\to\varphi(y)$ in E, and for each $n\in\mathbb{N}$,

$$|y_n|\leq\Big(\sum_{i=1}^{k_n}|Tx_i^n|^p\Big)^{\frac{1}{p}}$$

with $\sum_{i=1}^{k_n}\|x_i^n\|_X^p\leq 1$. Notice that since B is a lattice homomorphism

$$\Big(\sum_{i=1}^{k_n}|Tx_i^n|^p\Big)^{\frac{1}{p}}=\Big(\sum_{i=1}^{k_n}|BAx_i^n|^p\Big)^{\frac{1}{p}}=B\Big(\Big(\sum_{i=1}^{k_n}|Ax_i^n|^p\Big)^{\frac{1}{p}}\Big),$$

where $\Big(\sum_{i=1}^{k_n}|Ax_i^n|^p\Big)^{\frac{1}{p}}$ belongs to $\widetilde{W}$. Hence, since B is interval preserving there exists $w_n\in\widetilde{W}$ with $|w_n|\leq\Big(\sum_{i=1}^{k_n}|Ax_i^n|^p\Big)^{\frac{1}{p}}$ such that $B(w_n)=y_n$. Notice that since $\widetilde{W}$ is p-convex and $\sum_{i=1}^{k_n}\|x_i\|_X^p\leq 1$, for every n we have

$$\|w_n\|_{\widetilde{W}}\leq\Big\|\Big(\sum_{i=1}^{k_n}|Ax_i^n|^p\Big)^{\frac{1}{p}}\Big\|_{\widetilde{W}}\leq M^{(p)}(I_{\widetilde{W}})\Big(\sum_{i=1}^{k_n}\|Ax_i^n\|^p\Big)^{\frac{1}{p}}\leq M^{(p)}(I_{\widetilde{W}})\|A\|.$$

Now, since the image of the unit ball of $\widetilde{W}$ under B is closed, and $B(w_n)=y_n$ converge to $\varphi(y)\in E$, there exists $w\in\widetilde{W}$ with $\|w\|_{\widetilde{W}}\leq M^{(p)}(I_{\widetilde{W}})\|A\|$ such that $B(w)=\varphi(y)$. Moreover, this element is unique because B is injective. Let us define $v(y)=w$.

It is clear, because of the injectivity of B, that $v:W_{T,p}\to\widetilde{W}$ is linear. Moreover, by the previous argument v is a bounded operator of norm less than or equal to $M^{(p)}(I_{\widetilde{W}})\|A\|$. It is clear by construction that $B\circ v=\varphi$. Moreover, since $B\circ A=T=\varphi\circ R=B\circ v\circ R$ and B is injective, we also get that $A=v\circ R$ as desired. This finishes the proof. $\square$

Remark 6. Notice that the factorizations given in Theorems 1 and 3 also make sense in the more general context of quasi-Banach lattices and for p-convex or q-concave operators with $p,q\in(0,\infty)$ (not necessarily $p,q\geq 1$). It can be seen that in these cases, the factorization spaces are quasi-Banach lattices which need not be locally convex, except in the case when $p\geq 1$.

3. Duality relations

In this section we show the precise relation between the Banach lattices constructed in the proofs of Theorems 1 and 3. Namely we will prove the following

Theorem 5. *Let $T:X\to E$ be p-convex. By Theorem 3, T can be factored through $W_{T,p}$; moreover, since $T^*:E^*\to X^*$ is q-concave for $\frac{1}{p}+\frac{1}{q}=1$ (see* [11, Prop. 1.d.4]*), T^* can also be factored through $V_{T^*,q}$. It holds that:*

1. *$V_{T^*,q}$ is lattice isometric to a sublattice of $(W_{T,p})^*$,*
2. *$W_{T,p}$ is lattice isometric to a sublattice of $(V_{T^*,q})^*$.*

Moreover, under this identifications $V_{T^*,q}$ is always an ideal in $(W_{T,p})^*$, and if E is order continuous $W_{T,p}$ is an ideal of $(V_{T^*,q})^*$.

We need some preliminary lemmas first.

Lemma 6. *Let E be a Banach lattice, $x,y\in E_+$, $x\wedge y=0$, and $z^*\in E^*_+$. There exist u^*,v^* in E^*_+ such that $z^*=u^*+v^*$, $u^*\wedge v^*=0$ and*

$$\begin{cases}\langle z^*,x\rangle=\langle u^*,x\rangle\\ \langle z^*,y\rangle=\langle v^*,y\rangle\end{cases}$$

Proof. By [15, Lemma 1.4.3], there exist $z^*(x)$ and $z^*(y)$ in E^*_+ such that

$$\begin{cases}\langle z^*(x),u\rangle=\langle z^*,u\rangle & \text{for all } u\in E_x\\ \langle z^*(x),u\rangle=0 & \text{for all } u\in\{x\}^\perp\end{cases}$$

$$\begin{cases}\langle z^*(y),u\rangle=\langle z^*,u\rangle & \text{for all } u\in E_y\\ \langle z^*(y),u\rangle=0 & \text{for all } u\in\{y\}^\perp\end{cases}$$

where E_x denotes the principal ideal generated by x in E, and $\{x\}^\perp$ denotes the orthogonal complement of x (i.e. $\{x\}^\perp=\{u\in E: u\wedge x=0\}$).

Moreover, without loss of generality we can assume that $z^*(x),z^*(y)\leq z^*$ (simply consider $z^*(x)\wedge z^*$ and $z^*(y)\wedge z^*$), and that $z^*(x)\wedge z^*(y)=0$ (consider $z^*(x)-z^*(x)\wedge z^*(y)$ and $z^*(y)-z^*(x)\wedge z^*(y)$). Let then P be the band projection onto the band generated by $z^*(x)$ in the Dedekind complete Banach lattice E^* and Q be the complementary band projection. Then set $u^*=Pz^*$, $v^*=Qz^*$. $\square$

Lemma 7. *Let E be a Banach lattice. For any $z^*\in E^*_+$, and $x_1,\dots,x_n\in E_+$, there exist $x_1^*,\dots,x_n^*$ in E^*_+, such that $z^*=\sum_{i=1}^n x_i^*$, $x_i^*\wedge x_j^*=0$ for $i\neq j$, and*

$$\langle z^*,\bigvee_{i=1}^n x_i\rangle=\sum_{i=1}^n\langle x_i^*,x_i\rangle.$$

Proof. Given $x,y\in E_+$, Lemma 6 applied to $x-x\wedge y$ and $y-x\wedge y$ yields the result for $n=2$. An easy induction on n completes the proof. $\square$

Recall that given a set A in a Banach space X, the polar of A is the set $A^0 = \{x^* \in X^* : |\langle x^*, x\rangle| \leq 1, \forall x \in A\}$. Similarly, for a set B in X^*, the dual of a Banach space X, the prepolar of B is the set $B_0 = \{x \in X : |\langle x^*, x\rangle| \leq 1, \forall x^* \in B\}$.

Lemma 8. *Let $T : X \to E$ be p-convex, and let*

$$S := \{y \in E : |y| \leq \left(\sum_{i=1}^k |Tx_i|^p\right)^{\frac{1}{p}}, \text{ with } \sum_{i=1}^k \|x_i\|^p \leq 1\}.$$

Since $T^* : E^* \to X^*$ is q-concave (with $\frac{1}{p} + \frac{1}{q} = 1$), we can consider ρ_{T^*}, the seminorm which induces the norm on $V_{T^*,q}$ (see Theorem 1). Hence, we can also consider the convex set $U := \{y^* \in E^* : \rho_{T^*}(y^*) \leq 1\}$. Then

$$\overline{S}^0 = U,$$

where $\overline{S}$ denotes the closure of S in E.

Proof. First of all, we claim that $\overline{S} \subset U_0$.

Indeed, let $y \in E$ be such that $|y| \leq \left(\sum_{i=1}^n |Tx_i|^p\right)^{\frac{1}{p}}$ with $\sum_{i=1}^n \|x_i\|^p \leq 1$. For every $y^* \in E^*$ such that $\rho_{T^*}(y^*) \leq 1$, we have:

$$\begin{aligned} |\langle y^*, y\rangle| &\leq \langle |y^*|, \left(\sum_{i=1}^n |Tx_i|^p\right)^{\frac{1}{p}}\rangle \\ &= \langle |y^*|, \sup\left\{\sum_{i=1}^n a_i Tx_i : \sum_{i=1}^n |a_i|^q \leq 1\right\}\rangle \\ &= \sup\left\{\langle |y^*|, \bigvee_{m=1}^N \left(\sum_{i=1}^n a_i^m Tx_i\right)\rangle : \sum_{i=1}^n |a_i^m|^q \leq 1,\, m = 1, \dots, N,\, N \in \mathbb{N}\right\}. \end{aligned}$$

Where we have made use of [15, Cor. 1.3.4.ii)] in the last step.

Now, by Lemma 7, there exist $(y_m^*)_{m=1}^N$ pairwise disjoint elements of E_+^* such that $|y^*| = \sum_{m=1}^N y_m^*$ and

$$\langle |y^*|, \bigvee_{m=1}^N \left(\sum_{i=1}^n a_i^m Tx_i\right)\rangle = \sum_{m=1}^N \langle y_m^*, \sum_{i=1}^n a_i^m Tx_i\rangle.$$

Therefore, setting $z_i^* = \sum_{m=1}^N a_i^m y_m^*$, we have

$$\langle |y^*|, \bigvee_{m=1}^N \left(\sum_{i=1}^n a_i^m Tx_i\right)\rangle \leq \left(\sum_{i=1}^n \|T^* z_i^*\|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}}.$$

Note that, since $(y_m^*)_{m=1}^N$ are pairwise disjoint we have that $\left(\sum_{i=1}^n |z_i^*|^q\right)^{\frac{1}{q}} \leq \sum_{m=1}^N y_m^* = |y^*|$. Since $\rho_{T^*}(y^*) \leq 1$, this implies that $\left(\sum_{i=1}^n \|T^* z_i^*\|^q\right)^{\frac{1}{q}} \leq 1$. Therefore, for any y^* with $\rho_{T^*}(y^*) \leq 1$,

$$\begin{aligned} |\langle y^*, y\rangle| &\leq \sup\left\{\left(\sum_{i=1}^n \left\|T^*\left(\sum_{m=1}^N a_i^m y_m^*\right)\right\|^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}} : \sum_{i=1}^n |a_i^m|^q \leq 1,\, m = 1, \dots, N,\, N \in \mathbb{N}\right\} \\ &\leq 1. \end{aligned}$$

This means that $y \in U_0$. Since U_0 is closed, this proves that $\overline{S} \subseteq U_0$ as claimed.

Therefore, it follows that $(U_0)^0 \subseteq \overline{S}^0$. So in particular, $U \subseteq \overline{S}^0$.

Let us prove now the converse inclusion ($\overline{S}^0 \subseteq U$). Given $y^* \in \overline{S}^0$, we want to show that $\rho_{T^*}(y^*) \leq 1$. To this end, let $y_1^*, \ldots, y_k^*$ be elements in E^*, such that

$$\left(\sum_{i=1}^{k} |y_i^*|^q\right)^{\frac{1}{q}} \leq |y^*|.$$

Notice that since S is solid, then so is $\overline{S}^0$. In particular, $|y^*| \in \overline{S}^0$ whenever $y^* \in \overline{S}^0$.

Now, for every $\varepsilon > 0$ there exist $x_1, \ldots, x_k$ in X, such that $\sum_{i=1}^{k} \|x_i\|^p \leq 1$, and

$$\left(\sum_{i=1}^{k} \|T^* y_i^*\|^q\right)^{\frac{1}{q}} \leq |\sum_{i=1}^{k} \langle T^* y_i^*, x_i\rangle| + \varepsilon.$$

Moreover, by [11, Prop. 1.d.2] we have

$$|\sum_{i=1}^{k} \langle T^* y_i^*, x_i\rangle| = |\sum_{i=1}^{k} \langle y_i^*, Tx_i\rangle| \leq \langle \left(\sum_{i=1}^{k} |y_i^*|^q\right)^{\frac{1}{q}}, \left(\sum_{i=1}^{k} |Tx_i|^p\right)^{\frac{1}{p}} \rangle \leq 1$$

because $|y^*| \in \overline{S}^0$. Therefore, $\rho_{T^*}(y^*) \leq 1$ for every $y^* \in \overline{S}^0$. This finishes the proof.

□

Remark 7. Note that the equality $\overline{S}^0 = U$ proved above, yields in particular that U is weak*-closed. Hence, by the bipolar theorem it also holds that $\overline{S} = U_0$.

Now we can give the proof of Theorem 5.

Proof of Theorem 5. We stick to the notation of Theorems 1 and 3. Let us consider the inclusion $\varphi : W_{T,p} \hookrightarrow E$. Hence, we also have $\varphi^* : E^* \to (W_{T,p})^*$. Notice that for every $y^* \in E^*$ we have

$$\begin{aligned}
\|\varphi^*(y^*)\|_{(W_{T,p})^*} &= \sup\{\langle \varphi^*(y^*), y\rangle : \|y\|_{W_{T,p}} \leq 1\} \\
&= \sup\{\langle y^*, \varphi(y)\rangle : y \in \overline{S}\} \\
&= \inf\{\lambda > 0 : y^* \in \lambda \overline{S}^0\} \\
&= \rho_{T^*}(y^*),
\end{aligned}$$

by Lemma 8. Thus $\ker \varphi^* \supset \rho_T^{-1}(0)$, which allows us to define

$$\begin{array}{rccc}
A: & E^*/\rho_{T^*}^{-1}(0) & \longrightarrow & (W_{T,p})^* \\
 & y^* + \rho_{T^*}^{-1}(0) & \longmapsto & \varphi^*(y^*)
\end{array}$$

Moreover, A can be extended to an isometry from $V_{T^*,q}$ into $(W_{T,p})^*$.

Furthermore, since the unit ball of $W_{T,p}$ is a solid subset of E, then φ is interval preserving (i.e. $\varphi([0,x]) = [0, \varphi(x)]$ for $x \in W_{T,p}^+$). Thus, φ^* is a lattice homomorphism (cf. [1, Theorem 1.35]). Now, for $v \in V_{T^*,q}$, we can consider a sequence (y_n^*) in E^* such that $\lim_n y_n^* + \rho_{T^*}^{-1}(0) = v$ in $V_{T^*,q}$. Hence, we have

$$A(|v|) = \lim_n A(|y_n^*| + \rho_{T^*}^{-1}(0)) = \lim_n \varphi^*(|y_n^*|) = \lim_n |\varphi^*(y_n^*)| = \lim_n |A(y_n^* + \rho_{T^*}^{-1}(0))| = |A(v)|.$$

Therefore, A is a lattice homomorphism, which implies that $V_{T^*,q}$ is lattice isometric to a sublattice of $(W_{T,p})^*$.

In order to see that $V_{T^*,q}$ is in fact an ideal of $(W_{T,p})^*$, let $y \in (W_{T,p})^*$ with $0 \leq y \leq A(x)$ for some $x \in V_{T^*,q}$. Notice that $x = \lim \phi(x_n^*)$ in $V_{T^*,q}$, where (x_n^*) belong to E^*. Thus,

$$A(x) = \lim A(\phi(x_n^*)) = \lim \varphi^*(x_n).$$

If we denote $y_n = y \wedge \varphi^*(x_n)$, then we clearly have that y_n tends to y in $(W_{T,p})^*$. Moreover, since φ is a lattice homomorphism, by [1, Thm. 1.35], it follows that φ^* is interval preserving. Hence, since $0 \leq y_n \leq \varphi^*(x_n^*)$, for every $n \in \mathbb{N}$, there exists $u_n^* \in [0, x_n^*]$, such that $y_n = \varphi^*(u_n^*)$. Notice that $\varphi^*(u_n^*)$ tends to y in $(W_{T,p})^*$. In particular, we have $\rho_{T^*}(u_n^* - u_m^*) = \|\varphi^*(u_n^* - u_m^*)\| \to 0$ when $n, m \to \infty$, which yields that $\phi(u_n^*)$ tends to some u^* in $V_{T^*,q}$. By construction, we obtain that $A(u^*) = y$, which implies that A is interval preserving. This shows that $V_{T^*,q}$ is an ideal of $(W_{T,p})^*$, as claimed.

On the other hand, we can also define a mapping $B : W_{T,p} \to (V_{T^*,q})^*$. Indeed, given $y \in \overline{S}$ and $y^* \in E^*$, since $\overline{S} = U_0$, we have $\langle y^*, \varphi(y)\rangle \leq \rho_{T^*}(y^*)$. Therefore, for every $y \in W_{T,p}$ and $y^* \in E^*$ we get $\langle y^*, \varphi(y)\rangle \leq \rho_{T^*}(y^*)\|y\|_{W_{T,p}}$. Hence, there exists a unique element $B(y) \in (E^*/\rho_{T^*}^{-1}(0))^*$ such that

$$\langle y^* + \rho_{T^*}^{-1}(0), B(y)\rangle = \langle y^*, \varphi(y)\rangle$$

for every $y^* \in E^*$. Clearly, $B(y)$ is a linear functional which is continuous for the norm in $V_{T^*,q}$, thus, it can be extended to an element of $(V_{T^*,q})^*$, with $\|B(y)\|_{(V_{T^*,q})^*} \leq \|y\|_{W_{T,p}}$. Hence, $B : W_{T,p} \to (V_{T^*,q})^*$ is a linear mapping which is bounded of norm ≤ 1. Moreover, for $y \in W_{T,p}$ we have

$$\|B(y)\|_{(V_{T^*,q})^*} = \sup\{\langle v, B(y)\rangle : \|v\|_{V_{T^*,q}} \leq 1\} = \sup\{\langle y^*, \varphi(y)\rangle : \rho_{T^*}(y^*) \leq 1\}$$

which is the value of the Minkowski functional of $U_0 = \overline{S}$ at $\varphi(y)$. Hence,

$$\|B(y)\|_{(V_{T^*,q})^*} = \inf\{\lambda \geq 0 : \varphi(y) \in \lambda\overline{S}\} = \|y\|_{W_{T,p}}.$$

This means that B is an isometry.

Moreover, for $y^* \in E^*_+$ and every $y \in W_{T,p}$ we have

$$\begin{aligned}\langle y^* + \rho_{T^*}^{-1}(0), |B(y)|\rangle &= \sup\{|\langle x^* + \rho_{T^*}^{-1}(0), B(y)\rangle| : |x^*| \leq y^*\}\\ &= \sup\{|\langle x^*, \varphi(y)\rangle| : |x^*| \leq y^*\}\\ &= \langle y^*, |\varphi(y)|\rangle,\end{aligned}$$

and since φ is a lattice homomorphism we have

$$\langle y^* + \rho_{T^*}^{-1}(0), |B(y)|\rangle = \langle y^*, \varphi(|y|)\rangle = \langle y^* + \rho_{T^*}^{-1}(0), B(|y|)\rangle.$$

Since this holds for every $y^* \in E^*_+$, we have that $|B(y)| = B(|y|)$. Therefore, B is a lattice homomorphism and the claimed result follows.

To prove the last statement, let $u \in (V_{T^*,q})^*$ such that $0 \leq u \leq B(y)$ for some $y \in W_{T,p}$. We consider $\phi : E^* \to V_{T^*,q}$ the operator induced by the quotient map. Since ϕ is positive, so is $\phi^* : (V_{T^*,q})^* \to E^{**}$. It holds that

$$\phi^*(u) \leq \phi^*(B(y)) = \varphi(y).$$

Indeed, for every $y^* \in E^*$ we have

$$\langle \phi^*(B(y)), y^*\rangle = \langle B(y), \phi(y^*)\rangle = \langle \varphi(y), y^*\rangle.$$

Hence, $\phi^*(u) \in [0, \varphi(y)]$ in E^{**}. However, if E is order continuous and $\varphi(y)$ belongs to E, then we have $[0, \varphi(y)] \subset E$. Moreover, since φ is interval preserving, there exists $x \in [0, y]$ in $W_{T,p}$, such that $\phi^*(u) = \varphi(x)$. This implies that $u = B(x)$, which means that $W_{T,p}$ is an ideal in $(V_{T^*,q})^*$. □

Notice that the isometries A and B given in the proof of Theorem 5 may not be surjective, as the following examples show. Moreover, if E is not order continuous, $W_{T,p}$ may not be an ideal in $(V_{T^*,q})^*$.

Example 1. Let $T : L_\infty(0,1) \hookrightarrow L_1(0,1)$ denote the formal inclusion. Clearly, for every $1 \leq p \leq \infty$, T is p-convex. First, notice that the set

$$S = \{f \in L_1(0,1) : |f| \leq \Big(\sum_{i=1}^n |Tf_i|^p\Big)^{\frac{1}{p}},\ \sum_{i=1}^n \|f_i\|^p_{L_\infty} \leq 1\},$$

satisfies that $\overline{S} = \{f \in L_1(0,1) : \|f\|_{L_\infty} \le 1\}$. This implies that
$$W_{T,p} = L_\infty(0,1).$$
On the other hand, if we consider the adjoint operator $T^* : L_1(0,1)^* \to L_\infty(0,1)^*$, which is p'-concave (for $\frac{1}{p}+\frac{1}{p'}=1$), then for $f \in L_\infty(0,1) = L_1(0,1)^*$ we clearly have $\|T^*f\|_{L_\infty^*} = \|f\|_{L_1}$.

From here, it follows that the expression
$$\rho_{T^*,p'}(f) = \sup\left\{ \Big(\sum_{i=1}^n \|T^* f_i\|_{L_\infty^*}^{p'} \Big)^{\frac{1}{p'}} : \Big(\sum_{i=1}^n |f_i|^{p'} \Big)^{\frac{1}{p'}} \le |f| \right\},$$
trivially satisfies $\|f\|_{L_1} \le \rho_{T^*,p'}$.

While on the other hand, for $f \in L_\infty(0,1)$ and $(f_i)_{i=1}^n$ with $(\sum_{i=1}^n |f_i|^{p'})^{1/p'} \le |f|$ we have
$$\Big(\sum_{i=1}^n \|T^* f_i\|_{L_\infty^*}^{p'} \Big)^{\frac{1}{p'}} = \Big(\sum_{i=1}^n \|f_i\|_{L_1}^{p'} \Big)^{\frac{1}{p'}} \le \Big\| \Big(\sum_{i=1}^n |f_i|^{p'} \Big)^{\frac{1}{p'}} \Big\|_{L_1} \le \|f\|_{L_1}.$$
Thus, $\rho_{T^*,p'}(f) = \|f\|_{L_1}$, which implies that $V_{T^*,p'} = L_1(0,1)$. Hence, the isometry $A : V_{T^*,p'} \to (W_{T,p})^*$ given in Theorem 5 is not surjective.

Example 2. Let $T : \ell_1 \hookrightarrow \ell_\infty$ denote the formal inclusion. Clearly, T is ∞-convex. Moreover, it is easy to see that the set
$$S = \{y \in \ell_\infty : |y| \le \bigvee_{i=1}^n |y_i|, \bigvee_{i=1}^n \|y_i\|_{\ell_1} \le 1\},$$
satisfies $\overline{S} = B_{c_0}$. Hence,
$$W_{T,\infty} = c_0.$$
On the other hand, let $T^* : \ell_\infty^* \to \ell_1^*$ be the adjoint operator, which is 1-convex. It is well known that $\ell_\infty^* = \ell_1^{**}$ can be decomposed as
$$\ell_1^{**} = J(\ell_1) \oplus J(\ell_1)^\perp,$$
where $J(\ell_1)$ denotes the canonical image of ℓ_1 in its bidual, and $J(\ell_1)^\perp$ its disjoint complement.

Notice that every $y \in J(\ell_1)^\perp$, viewed as an element of ℓ_∞^*, satisfies $y|_{c_0} = 0$. Indeed, for every $n \in \mathbb{N}$, let e_n denote the sequence formed by zeros except 1 in the n^{th} entry. For $y \in J(\ell_1)^\perp$, by disjointness we have
$$\begin{aligned} 0 = \langle |y| \wedge J(e_n), e_n\rangle &= \inf\{\langle |y|, x\rangle + \langle e_n, z\rangle : x, z \in \ell_\infty^+, x+z = e_n\} \\ &= \inf\{\lambda \langle |y|, e_n\rangle + 1 - \lambda : \lambda \in [0,1]\} \\ &= \langle |y|, e_n\rangle, \end{aligned}$$
for every $n \in \mathbb{N}$, which clearly implies $y|_{c_0} = 0$. In particular, for $y \in J(\ell_1)^\perp$ we have
$$\begin{aligned} \|T^*(y)\| &= \sup\{\langle T^*(y), x\rangle : x \in \ell_1, \|x\|_{\ell_1} \le 1\} \\ &= \sup\{\langle y, Tx\rangle : x \in \ell_1, \|x\|_{\ell_1} \le 1\} \\ &= 0, \end{aligned}$$
since $Tx \in c_0 \subset \ell_\infty$ for every $x \in \ell_1$. Therefore, for $y \in J(\ell_1)^\perp$, since $J(\ell_1)^\perp$ is solid, we have
$$\rho_{T^*,1}(y) = \sup\left\{ \sum_{i=1}^n \|T^* y_i\|_{\ell_\infty} : \sum_{i=1}^n |y_i| \le |y| \right\} = 0.$$
While for $y \in J(\ell_1)$ we have
$$\rho_{T^*}(y) = \sup\left\{ \sum_{i=1}^n \|T^* y_i\|_{\ell_\infty} : \sum_{i=1}^n |y_i| \le |y| \right\} = \|y\|_{\ell_1}.$$

Hence, $V_{T^*,1} = \ell_1$, which implies that the isometry $B : W_{T,\infty} \to (V_{T^*,1})^*$ of Theorem 5 is not surjective.

Example 3. Let $T : \ell_1 \to c$ be defined by $T(x_1, x_2, \ldots, x_n, \ldots) = (x_1, x_1 + x_2, \ldots, \sum_{k=1}^n x_k, \ldots)$, where c denotes the space of convergent sequences of real numbers with the supremum norm. Clearly, T is positive and p-convex for every $1 \le p \le \infty$. Notice that the set

$$S = \{y \in c : |y| \le \Big(\sum_{i=1}^n |Ty_i|^p\Big)^{\frac{1}{p}}, \sum_{i=1}^n \|y_i\|_{\ell_1}^p \le 1\},$$

contains the constant sequence equal to one, so since S is solid, $\overline{S}$ coincides with the closed unit ball of c. Hence, $W_{T,p} = c$.

Now, we can consider the adjoint operator $T^* : c^* \to \ell_1^*$, which is clearly q-concave for every $1 \le q \le \infty$. Recall that c^* can be identified with the space $\ell_1(\mathbb{N})$ in the following way: for an element $x = (x_0, x_1, \ldots)$ in $\ell_1(\mathbb{N})$ and another element $y = (y_1, y_2, \ldots)$ in c, we set

$$\langle x, y\rangle = x_0 \lim y_n + \sum_{n=1}^\infty x_n y_n.$$

Therefore, for a positive element $x \in c^*$ we have

$$\begin{aligned} \|T^*x\|_{\ell_1^*} &= \sup\{\langle T^*x, y\rangle : \|y\|_{\ell_1} \le 1\} \\ &= \sup\{\langle x, Ty\rangle : \|y\|_{\ell_1} \le 1\} \\ &\ge \langle x, Te_1\rangle = \textstyle\sum_{n=0}^\infty x_n = \|x\|_{c^*}. \end{aligned}$$

Since $\|T\| \le 1$, it holds that $\|T^*x\|_{\ell_1^*} = \|x\|_{c^*}$ for every positive $x \in c^*$. This implies that

$$\rho_{T^*,q}(x) = \sup\left\{\Big(\sum_{i=1}^n \|T^*x_i\|_{\ell_1^*}^q\Big)^{\frac{1}{q}} : \Big(\sum_{i=1}^n |x_i|^q\Big)^{\frac{1}{q}} \le |x|\right\} = \|x\|_{c^*},$$

which yields that $V_{T^*,q} = c^*$.

Notice that, in particular, the operator $\varphi : c \hookrightarrow c$ defined in Theorem 3 coincides with the identity on c, and the operator $\phi : c^* \to c^*$ defined in Theorem 1 coincides as well with the identity on c^*. Now, by the definition of the operator $B : W_{T,p} \to (V_{T^*,q})^*$ in Theorem 5, it follows that for every $y \in c$ and $y^* \in c^*$ we have

$$\langle B(y), y^*\rangle = \langle B(y), \phi(y^*)\rangle = \langle \varphi(y), \phi(y^*)\rangle = \langle y, y^*\rangle.$$

Hence, $B = J$, where $J : c \to c^{**}$ denotes the canonical inclusion of c into its bidual. Now since c is not order continuous, it follows that $B(c)$ is not an ideal in $(V_{T^*,q})^*$, and this shows that the last statement of Theorem 5 does not hold without the assumption of order continuity on E.

4. Interpolation of Banach lattices

Throughout this section we will be using the complex interpolation method for Banach lattices, hence we need to consider complex Banach lattices. However, our final results, which are given in the next section, remain true for real Banach lattices by means of "complexifying" and considering the real part after the interpolation constructions. Notice that the results presented in the previous section work equally for both real or complex Banach lattices. We refer to [15, Section 2.2] for the notion of complex Banach lattice.

Recall that a compatible pair of Banach spaces (X_0, X_1) is a pair of Banach spaces X_0, X_1 which are continuously included in a topological vector space X. In the context of Banach lattices, we will say that two Banach lattices X_0, X_1 form a *compatible pair of Banach lattices*

(X_0, X_1) if there exists a complete Riesz space X, and inclusions $i_j : X_j \hookrightarrow X$ which are continuous interval preserving lattice homomorphisms, for $j = 0, 1$. In this way, the space

$$X_0 + X_1 = \{x \in X : x = x_0 + x_1, \text{ with } x_0 \in X_0,\ x_1 \in X_1\},$$

equipped with the norm $\|x\| = \inf\{\|x_0\|_{X_0} + \|x_1\|_{X_1} : x = x_0 + x_1\}$ is a Banach lattice which contains X_0 and X_1 as (non-closed) ideals.

Given a compatible pair of Banach lattices, (X_0, X_1), for each $\theta \in [0, 1]$ we will consider three different constructions:

(1) $X_0^{1-\theta} X_1^{\theta}$ denotes the space of elements $x \in X_0 + X_1$ such that

$$|x| \leq \lambda |x_0|^{1-\theta} |x_1|^{\theta},$$

for some $\lambda > 0$, $x_0 \in X_0$ and $x_1 \in X_1$, with $\|x_0\|_{X_0} \leq 1$, $\|x_1\|_{X_1} \leq 1$. Notice that the expressions of the form $|f|^{1-\theta}|g|^{\theta}$ can be defined in any Banach lattice by means of the functional calculus due to Krivine (see [11, pp. 40-43]). The norm in this space is given by

$$\|x\|_{X_0^{1-\theta} X_1^{\theta}} = \inf\{\lambda > 0 : |x| \leq \lambda |x_0|^{1-\theta}|x_1|^{\theta} \text{ for some } \|x_0\|_{X_0} \leq 1,\ \|x_1\|_{X_1} \leq 1\}.$$

(2) $[X_0, X_1]_\theta$ denotes the space of elements $x \in X_0 + X_1$ which can be represented as $x = f(\theta)$ for some $f \in \mathcal{F}(X_0, X_1)$. Here $\mathcal{F}(X_0, X_1)$ denotes the linear space of functions $f(z)$ defined in the strip $\Pi = \{z \in \mathbb{C} : z = x + iy,\ 0 \leq x \leq 1\}$, with values in the space $X_0 + X_1$, such that
- $f(z)$ is continuous and bounded for the norm of $X_0 + X_1$ in Π,
- $f(z)$ is analytic for the norm of $X_0 + X_1$ in the interior of Π,
- $f(it)$ assumes values in X_0 and is continuous and bounded for the norm of X_0, while $f(1 + it)$ assumes values in X_1 and is continuous and bounded for the norm of X_1.

In $\mathcal{F}(X_0, X_1)$ we can consider the norm $\|f\|_{\mathcal{F}(X_0,X_1)} = \max\{\sup_t \|f(it)\|_{X_0}, \sup_t \|f(1 + it)\|_{X_1}\}$. The norm in $[X_0, X_1]_\theta$ is given by

$$\|x\|_{[X_0,X_1]_\theta} = \inf\{\|f\|_{\mathcal{F}(X_0,X_1)} : f(\theta) = x\}.$$

(3) $[X_0, X_1]^\theta$ denotes the space of elements $x \in X_0 + X_1$ which can be represented as $x = f'(\theta)$ for some $f \in \overline{\mathcal{F}}(X_0, X_1)$. Now $\overline{\mathcal{F}}(X_0, X_1)$ denotes the linear space of functions $f(z)$ defined in the strip $\Pi = \{z \in \mathbb{C} : z = x + iy,\ 0 \leq x \leq 1\}$, with values in the space $X_0 + X_1$, such that
- $\|f(z)\|_{X_0+X_1} \leq c(1 + |z|)$ for some constant $c > 0$ and for every $z \in \Pi$,
- $f(z)$ is continuous in Π and analytic in the interior of Π for the norm of $X_0 + X_1$,
- $f(it_1) - f(it_2)$ has values in X_0 and $f(1 + it_1) - f(1 + it_2)$ in X_1 for any $-\infty < t_1 < t_2 < \infty$ and

$$\|f\|_{\overline{\mathcal{F}}(X_0,X_1)} = \max\left\{\sup_{t_1,t_2}\left\|\frac{f(it_2) - f(it_1)}{t_2 - t_1}\right\|_{X_0}, \sup_{t_1,t_2}\left\|\frac{f(1 + it_2) - f(1 + it_1)}{t_2 - t_1}\right\|_{X_1}\right\} < \infty.$$

The norm in $[X_0, X_1]^\theta$ is given by

$$\|x\|_{[X_0,X_1]^\theta} = \inf\{\|f\|_{\overline{\mathcal{F}}(X_0,X_1)} : f'(\theta) = x\}.$$

These spaces are Banach lattices provided that (X_0, X_1) is a compatible pair of Banach lattices. Moreover, $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$ are always interpolation spaces, while $X_0^{1-\theta} X_1^{\theta}$ is an intermediate space between X_0 and X_1 which is an interpolation space under certain extra assumptions. We refer to [6], [9], [12], and [13] for more information on these spaces.

Next theorem extends a result of Šestakov [19], which was originally proved only for the case of Banach lattices of measurable functions, showing how these constructions are related to each other.

Theorem 9. *Let X_0, X_1 be a compatible pair of Banach lattices. For every $\theta \in (0,1)$ it holds that*

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^{[X_0,X_1]_\theta} = \overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^\theta},$$

with equality of norms.

Before the proof of Theorem 9 we need the following.

Lemma 10. *Let $F : \Pi \to X_0 + X_1$ be a function in $\mathcal{F}(X_0, X_1)$ of the form*

$$F(z) = e^{\delta z^2} \sum_{j=1}^N x_j e^{\lambda_j z},$$

where $\delta > 0$, the λ_j are real, and $x_j \in X_0 \cap X_1$. It holds that

$$\|F(\theta)\|_{X_0^{1-\theta}X_1^\theta} \le \|F\|_{\mathcal{F}(X_0,X_1)}.$$

Proof. Let $F : \Pi \to X_0 \cap X_1$ be a function in $\mathcal{F}(X_0, X_1)$ of the form

$$F(z) = e^{\delta z^2} \sum_{j=1}^N x_j e^{\lambda_j z},$$

where $\delta > 0$, the λ_j are real, and $x_j \in X_0 \cap X_1$. Let $x = \sum_{j=1}^N |x_j|$. We can consider the principal (non closed) ideal in $X_0 \cap X_1$ generated by x, equipped with the norm that makes it isomorphic to a $C(K)$ space for some compact K (i.e. $\|y\| = \inf\{\lambda > 0 : |y| \le \lambda x\}$, cf. [18, Chapter II. §7]). We clearly have inclusions

$$C(K) \hookrightarrow X_0 \cap X_1 \hookrightarrow X_0 + X_1,$$

which are bounded lattice homomorphisms. Moreover, since $|x_j| \le x$, we have $x_j \in C(K)$, so we can consider

$$F(\omega, z) = e^{\delta z^2} \sum_{j=1}^N x_j(\omega) e^{\lambda_j z},$$

as a function of $\omega \in K$, and $z \in \Pi$. For each $z \in \Pi$, $F(\cdot, z)$ belongs to $C(K)$. Hence, applying [6, §9.4, ii)], for any $\omega \in K$ we have

$$|F(\omega,\theta)| \le \left[\frac{1}{1-\theta}\int_{-\infty}^{+\infty} |F(\omega, it)|\mu_0(\theta,t)dt\right]^{1-\theta} \left[\frac{1}{\theta}\int_{-\infty}^{+\infty} |F(\omega, 1+it)|\mu_1(\theta,t)dt\right]^{\theta},$$

where μ_0 and μ_1 are the Poisson kernels for the strip Π, given by (see [6, §9.4]):

$$\mu_0(\theta,t) = \frac{e^{-\pi t}\sin \pi\theta}{\sin^2 \pi\theta + [\cos \pi\theta - e^{-\pi t}]^2} \qquad \mu_1(\theta,t) = \frac{e^{-\pi t}\sin \pi\theta}{\sin^2 \pi\theta + [\cos \pi\theta + e^{-\pi t}]^2}.$$

Hence setting

$$g(\omega) = \frac{1}{1-\theta}\int_{-\infty}^{+\infty} |F(\omega, it)|\mu_0(\theta,t)dt, \quad \text{and} \quad h(\omega) = \frac{1}{\theta}\int_{-\infty}^{+\infty} |F(\omega, 1+it)|\mu_1(\theta,t)dt,$$

we find that g and h belong to $C(K)$. Indeed, for any ω_1, ω_2 in K, we have

$$\begin{aligned} |g(\omega_1) - g(\omega_2)| &\leq \frac{1}{1-\theta} \int_{-\infty}^{+\infty} |F(\omega_1, it) - F(\omega_2, it)| \mu_0(\theta, t) dt \\ &\leq \frac{1}{1-\theta} \int_{-\infty}^{+\infty} \left| \sum_{j=1}^{N} (x_j(\omega_1) - x_j(\omega_2)) e^{i\lambda_j t} \right| e^{-\delta t^2} \mu_0(\theta, t) dt \\ &\leq \frac{1}{1-\theta} \int_{-\infty}^{+\infty} \sum_{j=1}^{N} |x_j(\omega_1) - x_j(\omega_2)| e^{-\delta t^2} \mu_0(\theta, t) dt \\ &\leq \sum_{j=1}^{N} |x_j(\omega_1) - x_j(\omega_2)|, \end{aligned}$$

since $\int_{-\infty}^{+\infty} \mu_0(\theta, t) dt = 1-\theta$ (see [6, §29.4]). This inequality together with the fact that x_j belongs to $C(K)$ for $j = 1, \ldots, N$, proves that $g \in C(K)$. The proof for h is identical. Moreover,

$$\begin{aligned} \|g\|_{X_0} &= \left\| \frac{1}{1-\theta} \int_{-\infty}^{+\infty} |F(\omega, it)| \mu_0(\theta, t) dt \right\|_{X_0} \\ &\leq \frac{1}{1-\theta} \int_{-\infty}^{+\infty} \|F(\omega, it)\|_{X_0} \mu_0(\theta, t) dt \\ &\leq \|F\|_{\mathcal{F}(X_0,X_1)} \frac{1}{1-\theta} \int_{-\infty}^{+\infty} \mu_0(\theta, t) dt \\ &= \|F\|_{\mathcal{F}(X_0,X_1)}, \end{aligned}$$

and similarly

$$\|h\|_{X_1} \leq \|F\|_{\mathcal{F}(X_0,X_1)}.$$

Since $|F(\theta)| \leq g^{1-\theta} h^{\theta}$ (in $C(K)$, and thus in $X_0 + X_1$ since Krivine's calculous is preserved under lattice homomorphisms), we have therefore

$$\|F(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F\|_{\mathcal{F}(X_0,X_1)}.$$

And the proof is finished. □

Proof of Theorem 9. If x is an element in $\overline{X_0 \cap X_1}^{[X_0,X_1]_\theta}$, by the definition of the norm in $[X_0, X_1]_\theta$, for every $\varepsilon > 0$, we can take F in $\mathcal{F}(X_0, X_1)$, such that $F(\theta) = x$ and

$$\|F\|_{\mathcal{F}(X_0,X_1)} \leq \|x\|_{[X_0,X_1]_\theta} + \varepsilon.$$

By [9, Chapter IV, Thm. 1.1], we can consider a sequence $(F_n)_{n=1}^{\infty}$ in $\mathcal{F}(X_0, X_1)$, of elements of the form

$$e^{\delta z^2} \sum_{j=1}^{N} x_j e^{\lambda_j z},$$

where $x_j \in X_0 \cap X_1$ and $\lambda_j \in \mathbb{R}$, such that $\|F - F_n\|_{\mathcal{F}(X_0,X_1)} \to 0$. Then we have

$$\|F_n(\theta) - x\|_{[X_0,X_1]_\theta} = \|F_n(\theta) - F(\theta)\|_{[X_0,X_1]_\theta} \leq \|F_n - F\|_{\mathcal{F}(X_0,X_1)} \to 0.$$

By Lemma 10, for $n, m \in \mathbb{N}$ we have

$$\|F_n(\theta) - F_m(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F_n - F_m\|_{\mathcal{F}(X_0,X_1)} \to 0,$$

when $n, m \to \infty$, and

$$\|F_n(\theta)\|_{X_0^{1-\theta} X_1^{\theta}} \leq \|F_n\|_{\mathcal{F}(X_0,X_1)} \to \|F\|_{\mathcal{F}(X_0,X_1)} \leq \|x\|_{[X_0,X_1]_\theta} + \varepsilon.$$

Therefore, $F_n(\theta)$ also converges to a limit in $X_0^{1-\theta} X_1^{\theta}$ of norm not exceeding $\|x\|_{[X_0,X_1]_\theta} + \varepsilon$. However, since $X_0^{1-\theta} X_1^{\theta}$ and $[X_0, X_1]_\theta$ are both continuously embedded in $X_0 + X_1$, it follows that x is also the limit of $F_n(\theta)$ for the norm of $X_0^{1-\theta}X_1^\theta$. Hence, $x \in X_0^{1-\theta}X_1^\theta$ and $\|x\|_{X_0^{1-\theta}X_1^\theta} \leq \|x\|_{[X_0,X_1]_\theta} + \varepsilon$. Since this is true for all $\varepsilon > 0$, we have

$$\|x\|_{X_0^{1-\theta}X_1^\theta} \leq \|x\|_{[X_0,X_1]_\theta}.$$

We will show now that $X_0^{1-\theta}X_1^\theta \subset [X_0, X_1]^\theta$ and the inclusion mapping

$$X_0^{1-\theta}X_1^\theta \hookrightarrow [X_0, X_1]^\theta$$

is bounded with norm smaller than or equal to one. Indeed, let $x \in X_0^{1-\theta}X_1^\theta$ be such that $\|x\|_{X_0^{1-\theta}X_1^\theta} \leq 1$. Then for every $\varepsilon > 0$ we have $g \in X_0^+$, and $h \in X_1^+$ such that $\|g\|_{X_0} \leq 1$, $\|h\|_{X_1} \leq 1$, and $|x| \leq (1+\varepsilon)g^{1-\theta}h^\theta$ in $X_0 + X_1$.

Now, if $\mathcal{I}$ denotes the (non closed) order ideal generated by $g \vee h$ in $X_0 + X_1$, then $\mathcal{I}$ can be viewed as a space $C(K)$ over some compact Hausdorff space K. Since $|x| \leq (1+\varepsilon)g^{1-\theta}h^\theta$ in $X_0 + X_1$, we can consider

$$f(t) = \frac{x(t)}{g^{1-\theta}(t)h^\theta(t)},$$

which is well defined for all $t \in K$ such that $g(t)h(t) \neq 0$. This allows us to define

$$F(t,z) = \begin{cases} f(t)g(t)^{1-z}h(t)^z & \text{if } g(t)h(t) \neq 0, \\ 0 & \text{in any other case.} \end{cases}$$

Note that, since $g, h \leq g \vee h$, we have $\|g\|_{C(K)}, \|h\|_{C(K)} \leq 1$; hence, for every $z \in \Pi$,

$$\sup_{t \in K} |F(t,z)| \leq 1 + \varepsilon.$$

Clearly, for $z \in \overset{\circ}{\Pi}$ we can consider $\phi(z) \in C(K)$ defined by $\phi(z)(t) = F(t,z)$. It is routine to verify that the map

$$\phi : \overset{\circ}{\Pi} \to C(K)$$

is continuous. We claim that it is analytic. Indeed, note that for every $t \in K$ fixed, $F(t, \cdot)$ is analytic on $\overset{\circ}{\Pi}$. Hence,

$$\phi(z)(t) = F(t,z) = \frac{1}{2\pi i}\int_\gamma \frac{F(t,\xi)}{\xi - z} d\xi$$

for every $t \in K$, and for any circumference γ of center z contained in $\overset{\circ}{\Pi}$. Since this identity is valid for every $t \in K$, we get

$$\phi(z) = \frac{1}{2\pi i}\int_\gamma \frac{\phi(\xi)}{\xi - z} d\xi.$$

This means that $\phi : \overset{\circ}{\Pi} \to C(K)$ is analytic.

Now, let us define

$$F_1(t,z) = \int_{\gamma_z} F(t,\xi) d\xi,$$

for $t \in K$ and $z \in \Pi$, where γ_z is any path joining $\frac{1}{2}$ and z, with all its points except possibly z inside $\overset{\circ}{\Pi}$. Note that since F is analytic in $\overset{\circ}{\Pi}$ and $\sup_{t \in K}|F(t,z)| \leq 1 + \varepsilon$, for all $z \in \Pi$, F_1 is independent of the path γ_z, so it is well defined. Therefore, we can define $\phi_1 : \Pi \to B(K)$, where $B(K)$ denotes the bounded measurable functions on K, by

$$\phi_1(z) = F_1(\cdot, z) = \int_{\gamma_z} \phi(\xi) d\xi,$$

for $z \in \Pi$. Since $\phi : \overset{\circ}{\Pi} \to C(K)$ is analytic, so is ϕ_1 on $\overset{\circ}{\Pi}$, and clearly $\phi_1(\overset{\circ}{\Pi}) \subseteq C(K)$. Moreover,

$$\|\phi_1(z) - \phi_1(z')\|_{C(K)} \leq (1+\varepsilon)|z - z'|$$

for $z, z' \in \overset{\circ}{\Pi}$. Now, for any z in the border of Π, let $z_n \in \overset{\circ}{\Pi}$ be such that $z_n \to z$. Since $\|\phi_1(z_n) - \phi_1(z_m)\|_{C(K)} \leq (1+\varepsilon)|z_n - z_m|$, we get that $\phi_1(z_n)$ is a Cauchy sequence in $C(K)$, hence convergent to some $\psi \in C(K)$. In particular, for every $t \in K$, $\phi_1(z_n)(t) \to \psi(t)$ and since $\phi_1(z_n)(t) = \int_{\gamma_{z_n}} F(t,\xi)d\xi$ we get that $\psi(t) = \int_{\gamma_z} F(t,\xi)d\xi$. This implies that $\phi_1(\Pi) \subseteq C(K)$, and

$$\|\phi_1(z) - \phi_1(z')\|_{C(K)} \leq (1+\varepsilon)|z - z'|$$

for $z, z' \in \Pi$. Thus $\phi_1 : \Pi \to C(K)$ is continuous.

Now, for $u, v \in \mathbb{R}$, and for every $\alpha \in (0,1)$ let γ_α be the path formed by the rectilinear segments $[iu, \alpha + iu]$, $[\alpha + iu, \alpha + iv]$ and $[\alpha + iv, iv]$. Hence, for every $\alpha \in (0,1)$ and $t \in K$ such that $g(t)h(t) \neq 0$

$$\begin{aligned} |F_1(t,iu) - F_1(t,iv)| &\leq \int_{\gamma_\alpha} |F(t,\xi)|d\xi \\ &= \int_{[iu,\alpha+iu]} |F(t,\xi)|d\xi + \int_{[\alpha+iu,\alpha+iv]} |F(t,\xi)|d\xi + \int_{[\alpha+iv,iv]} |F(t,\xi)|d\xi \\ &\leq \alpha(1+\varepsilon) + (1+\varepsilon)g(t)^{1-\alpha}h(t)^{\alpha}|u-v| + \alpha(1+\varepsilon) \\ &\leq (g(t)^{1-\alpha}h(t)^{\alpha}|u-v| + 2\alpha)(1+\varepsilon). \end{aligned}$$

Thus, letting $\alpha \to 0^+$, we get

$$\frac{|F_1(t,iu) - F_1(t,iv)|}{|u-v|} \leq (1+\varepsilon)g(t)$$

for $t \in K$ with $g(t)h(t) \neq 0$. Since the same inequality holds trivially if $g(t) = 0$, we have that

$$\frac{|\phi_1(iu) - \phi_1(iv)|}{|u-v|} \leq (1+\varepsilon)g$$

in X_0. Analogously we have

$$\frac{|\phi_1(1+iu) - \phi_1(1+iv)|}{|u-v|} \leq (1+\varepsilon)h$$

in X_1. Since X_0 and X_1 are order ideals, it clearly follows that $\frac{|\phi_1(iu)-\phi_1(iv)|}{|u-v|} \in X_0$ and $\frac{|\phi_1(1+iu)-\phi_1(1+iv)|}{|u-v|} \in X_1$.

Therefore, since $\frac{d\phi_1}{dz}\Big|_{z=\theta} = \phi(\theta) = x$, we get that $x \in [X_0, X_1]^\theta$ and

$$\|x\|_{[X_0,X_1]^\theta} \leq \max\left[\sup_{u,v}\left\|\frac{|\phi_1(iu) - \phi_1(iv)|}{|u-v|}\right\|_{X_0}, \sup_{u,v}\left\|\frac{|\phi_1(1+iu) - \phi_1(1+iv)|}{|u-v|}\right\|_{X_1}\right] \leq 1+\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we have proved that $X_0^{1-\theta}X_1^\theta \hookrightarrow [X_0, X_1]^\theta$ is continuous with norm smaller than or equal to one. In particular, we have an inclusion

$$\overline{X_0 \cap X_1}^{X_0^{1-\theta}X_1^\theta} \hookrightarrow \overline{X_0 \cap X_1}^{[X_0,X_1]^\theta}$$

with norm smaller than one. Now, by [3], we have

$$\overline{X_0 \cap X_1}^{[X_0,X_1]^\theta} = \overline{X_0 \cap X_1}^{[X_0,X_1]_\theta},$$

with equality of norms. This proves the theorem. □

5. Factorization for operators which are both p-convex and q-concave

In section 2, it was proved that every p-convex (resp. q-concave) operator factors in a nice way through a p-convex (resp. q-concave) Banach lattice. However, if the operator is both p-convex and q-concave, can this factorization be improved? It is well-known that if E is a q-concave Banach lattice and F a p-convex Banach lattice, then every operator $T : E \to F$ is both p-convex and q-concave. Moreover, if an operator $T : E \to F$ between Banach lattices, has a factorization of the following form

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\psi} \\ E_1 & \xrightarrow{R} & F_1 \end{array}$$

where ϕ and ψ are positive, E_1 q-concave, and F_1 p-convex, then T is both p-convex and q-concave [10]. Hence, the following question is natural:

Can a p-convex and q-concave operator $T : E \to F$ factor always in this way? According to Theorems 1 and 3, this is true if the operator $T : E \to F$ can be written as $T = T_1 \circ T_2$, where T_1 is p-convex, and T_2 is q-concave. In fact, it turns out that the previous question is equivalent to the following one.

If $T : E \to F$ is p-convex and q-concave, do there exist operators T_1 and T_2, such that $T = T_1 \circ T_2$, where T_1 is p-convex, and T_2 is q-concave?

In general, the answer to this question is negative, as the following examples show.

Proposition 11. *Let $T : E \to F$ be an operator from an ∞-convex Banach lattice (an AM-space) E to a q-concave Banach lattice F ($q < \infty$). If T can be factored as $T = SR$, with R q-concave and S ∞-convex, then T is compact.*

Proof. If $T : E \to F$ has such a factorization, then by Theorems 1 and 3 we must have

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle\phi}\downarrow & & \uparrow{\scriptstyle\varphi} \\ V & \xrightarrow{T_1} & W \end{array}$$

where V is q-concave, W an AM-space, and ϕ, φ lattice homomorphisms.

Since ϕ and φ are positive and take values in q-concave Banach lattices, by [11, Prop. 1.d.9], they are q-concave operators. Moreover, since both operators are defined on AM-spaces, by [11, Theorem 1.d.10], ϕ and φ are q-absolutely summing.

Therefore, $T = \varphi \circ (T_1 \circ \phi)$ is a product of two q-absolutely summing operators, hence it is compact, because every q-absolutely summing operator is weakly compact and Dunford-Pettis (cf. [2, Cor. 8.2.15]). □

Example 4. The formal inclusion $T : C(0,1) \hookrightarrow L_q(0,1)$ is q-concave and ∞-convex, but it does not factor as $T = T_1 \circ T_2$, with T_1 ∞-convex, and T_2 q-concave.

Proof. Since T is positive, it is q-concave and ∞-convex by [11, Prop. 1.d.9]. However T is not compact since the closure in $L_q(0,1)$ of the unit ball of $C(0,1)$ contains the Rademacher functions. □

By duality, Proposition 11 immediately yields the following.

Corollary 12. *Let $T : E \to F$ be an operator from a p-convex Banach lattice E to a 1-concave Banach lattice (an AL-space) F. If T can be factored as $T = SR$, with R 1-concave and S p-convex, then T is compact.*

A different argument can be used to see that the formal inclusion $i : L_p(0,1) \hookrightarrow L_q(0,1)$ with $1 < q < p < \infty$ (which is clearly p-convex and q-concave) does not factor as $i = T_2T_1$ with T_1 q-concave and T_2 p-convex. First we need the following lemma:

Lemma 13. *Let $1 < q < p < \infty$. There is no disjointness preserving nonzero operator $T : L_q(0,1) \to L_p(0,1)$.*

Proof. Assume $f := Th \neq 0$ for some $h \in L_q(0,1)$ with $\|h\|_q = 1$. If $U : L_q(0,1) \to L_q(0,1)$ is a linear isometry such that $U\chi_{[0,1]} = h$, then $S := TU$ is also disjointness preserving. For each $n \in \mathbb{N}$, let us consider the partition $\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$. Notice that for each $n \in \mathbb{N}$ there must exist $k_n \leq n$ such that

$$\|S(\chi_{[\frac{k_n-1}{n},\frac{k_n}{n}]}\|_p \geq \frac{\|f\|_p}{n^{1/p}}.$$

Otherwise, by the fact that S is disjointness preserving, we would have

$$\|f\| = \|\sum_{k=1}^n S(\chi_{[\frac{k-1}{n},\frac{k}{n}]})\| = \Big(\sum_{k=1}^n \|S(\chi_{[\frac{k-1}{n},\frac{k}{n}]})\|^p\Big)^{\frac{1}{p}} < \Big(\sum_{k=1}^n \frac{\|f\|^p}{n}\Big)^{\frac{1}{p}} = \|f\|,$$

which is clearly a contradiction.

Hence, since $\|\chi_{[\frac{k-1}{n},\frac{k}{n}]}\|_q = \frac{1}{n^{1/q}}$ for every $k = 1,\dots,n$, we have

$$\|S(\chi_{[\frac{k_n-1}{n},\frac{k_n}{n}]})\|_p \geq \frac{\|f\|_p}{n^{1/p}} = \frac{\|f\|_p}{n^{1/p-1/q}}\|\chi_{[\frac{k_n-1}{n},\frac{k_n}{n}]}\|_q.$$

Therefore, since $q < p$, for n large enough we get a contradiction with the fact that S is bounded. □

Recall that given a Banach lattice E and a Banach space X, an operator $T : E \to X$ is called AM-compact if $T[-x,x]$ is relatively compact for every positive $x \in E$.

Theorem 14. *If a lattice homomorphism $T : L_p(0,1) \to L_q(0,1)$ $(q < p)$ can be factored as $T = T_2T_1$ with T_1 q-concave and T_2 p-convex, then T is AM-compact.*

Proof. Suppose that we have

$$\begin{array}{ccc} L_p(0,1) & \xrightarrow{\ T\ } & L_q(0,1) \\ & {\scriptstyle T_1}\searrow \quad \nearrow{\scriptstyle T_2} & \\ & X & \end{array}$$

with T_1 q-concave and T_2 p-convex. Hence, by Theorems 1 and 3 we have

$$\begin{array}{ccc} L_p(0,1) & \xrightarrow{\ T\ } & L_q(0,1) \\ {\scriptstyle \phi}\downarrow & & \uparrow{\scriptstyle \varphi} \\ V & \xrightarrow{\ S\ } & W \end{array}$$

where V is q-concave, W p-convex, and ϕ, φ are lattice homomorphisms. Now, by Krivine's Theorem ([11, Theorem 1.d.11]) we can factor

$$\begin{array}{ccc} L_p(0,1) & \xrightarrow{\ \phi\ } & V \\ & {\scriptstyle \phi_1}\searrow \quad \nearrow{\scriptstyle \phi_2} & \\ & L_q(\mu) & \end{array} \qquad \begin{array}{ccc} W & \xrightarrow{\ \varphi\ } & L_q(0,1) \\ & {\scriptstyle \varphi_1}\searrow \quad \nearrow{\scriptstyle \varphi_2} & \\ & L_p(\nu) & \end{array}$$

where ϕ_i and φ_i are still lattice homomorphisms. Therefore, we can consider the closure of $\phi_1(L_p(0,1))$ in $L_q(\mu)$, which is lattice isomorphic to some $L_q(\widetilde{\mu})$, and the quotient $L_p(\nu)/\ker(\varphi_2)$

which is lattice isomorphic to $L_p(\widetilde{\nu})$ for certain measures $\widetilde{\mu}$ and $\widetilde{\nu}$. Thus, we can consider the following diagrams:

$$\begin{array}{ccccc} L_p(0,1) & & \xrightarrow{\ \phi_1\ } & & L_q(\mu) \\ & \searrow_{\widetilde{\phi_1}} & & \nearrow_{i} & \\ & & L_q(\widetilde{\mu}) & & \end{array} \qquad \begin{array}{ccccc} L_p(\nu) & & \xrightarrow{\ \varphi_2\ } & & L_q(0,1) \\ & \searrow_{\pi} & & \nearrow_{\widetilde{\varphi_2}} & \\ & & L_p(\widetilde{\nu}) & & \end{array}$$

Now, let $R : L_q(\widetilde{\mu}) \to L_p(\widetilde{\nu})$ be defined by $R = \pi\varphi_1 S\phi_2 i$. It follows that R is a lattice homomorphism. Indeed, given $x \in L_q(\widetilde{\mu})$, we can consider (x_n) in $L_p(0,1)$ such that $\widetilde{\phi_1}(x_n) \to x$ in $L_q(\widetilde{\mu})$. Since T is a lattice homomorphism $T(|x_n|) = |Tx_n|$ for every n, and since $\widetilde{\varphi_2}$ is an injective lattice homomorphism we get that $R\widetilde{\phi_1}(|x_n|) = |R\widetilde{\phi_1}(x_n)|$, and by continuity and the fact that $\widetilde{\phi_1}$ is also a lattice homomorphism, we achieve $R(|x|) = |R(x)|$.

Hence, considering the quotient by $\ker(R)$ we can factor R through an injective lattice homomorphism from some L_q space to an L_p space. By considering the diffuse and atomic parts of these spaces we can decompose them as $L_q(0,1) \oplus \ell_q$ and $L_p(0,1) \oplus \ell_p$ (lattice isomorphically). Accordingly, every operator between them can be decomposed into four parts acting between each of the summands, that is $R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$ with

$$\begin{array}{ll} R_{11} : L_q(0,1) \to L_p(0,1) & R_{12} : \ell_q \to L_p(0,1) \\ R_{21} : L_q(0,1) \to \ell_p & R_{22} : \ell_q \to \ell_p. \end{array}$$

Clearly if R is a lattice homomorphism, so are R_{ij}, and since the intervals in ℓ_p and ℓ_q are compact, we have that R_{12}, R_{21} and R_{22} are AM-compact. Finally, by Lemma 13 we see that R_{11} has to be the zero operator. This finishes the proof. □

Example 5. For $1 < q < p < \infty$, the formal inclusion $i : L_p(0,1) \hookrightarrow L_q(0,1)$ cannot be factored as $i = T_2T_1$ with T_1 q-concave and T_2 p-convex.

Proof. Since i is positive, by [11, Prop. 1.d.9], i is q-concave and p-convex. Moreover, since $i : L_p(0,1) \hookrightarrow L_q(0,1)$ is a lattice homomorphism and it is not AM-compact (consider for instance the Rademacher functions), by Theorem 14, we conclude that it cannot be factored as $i = T_2T_1$ with T_1 q-concave and T_2 p-convex. □

Despite these facts, as an application of the results of section 4, we have the following factorization for operators which are both p-convex and q-concave.

Theorem 15. *Let E and F be Banach lattices, and let $T : E \to F$ be both p-convex and q-concave. For every $\theta \in (0,1)$ we can factor T in the following way*

$$\begin{array}{ccc} E & \xrightarrow{\ T\ } & F \\ {\scriptstyle \phi_\theta}\downarrow & & \uparrow{\scriptstyle \varphi_\theta} \\ E_\theta & \xrightarrow{\ R_\theta\ } & F_\theta \end{array}$$

where ϕ_θ and φ_θ are interval preserving lattice homomorphisms, E_θ is $(\frac{q}{1-\theta})$-concave, and F_θ is $(\frac{p}{\theta+(1-\theta)p})$-convex.

Before the proof, we need the some lemmas first. Recall, that given a Banach space X, and $1 \le p < \infty$, $\ell_p(X)$ denotes the space of sequences (x_n) of X such that $(\|x_n\|_X)$ belongs to ℓ_p. This is a Banach space with the norm

$$\|(x_n)\|_{\ell_p(X)} = \Big(\sum_{n=1}^{\infty} \|x_n\|_X^p\Big)^{\frac{1}{p}}.$$

In order to keep a unified notation, for $p=\infty$, $\ell_\infty(X)$ will denote the space of sequences (x_n) of X such that $(\|x_n\|_X)$ belongs to c_0, equipped with the norm

$$\|(x_n)\|_{\ell_\infty(X)} = \sup \|x_n\|_X.$$

Notice that this space is usually denoted $c_0(X)$ in the literature.

Analogously, given a Banach lattice E, and $1\le p\le\infty$, $E(\ell_p)$ denotes the completion of the space of eventually null sequences (x_n) of E under the norm

$$\|(x_n)\|_{E(\ell_p)} = \begin{cases} \sup\limits_n \Big\| \Big(\sum\limits_{i=1}^n |x_i|^p\Big)^{\frac{1}{p}} \Big\|_E & \text{if } 1\le p<\infty, \\ \sup\limits_n \Big\| \bigvee\limits_{i=1}^n |x_i| \Big\|_E & \text{if } p=\infty. \end{cases}$$

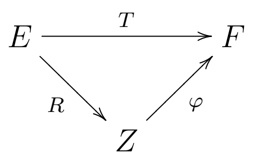
The following lemma consists of standard results. In the case of Banach lattices of measurable functions, this can be obtained from [5, Theorem 3], however, in general we cannot use the measurability tools and thus some functional calculus needs to be carried out.

Lemma 16. *Let (F,G) be a compatible pair of Banach lattices, let $r,s\in[1,+\infty]$ and $\theta\in(0,1)$. For $\frac{1}{t}=\frac{1-\theta}{r}+\frac{\theta}{s}$, we have:*

1. *$\ell_r(F)^{1-\theta}\ell_s(G)^\theta = \ell_t(F^{1-\theta}G^\theta)$, with equality of norms.*
2. *$\overline{\ell_r(F)\cap\ell_s(G)}^{\ell_r(F)^{1-\theta}\ell_s(G)^\theta} = \ell_t(\overline{F\cap G}^{F^{1-\theta}G^\theta})$.*
3. *the inclusion $F(\ell_r^n)^{1-\theta}G(\ell_s^n)^\theta \hookrightarrow F^{1-\theta}G^\theta(\ell_t^n)$ is bounded of norm ≤ 1.*
4. *$E(\ell_r)^{1-\theta}E(\ell_s)^\theta = E(\ell_t)$, with equality of norms.*

We skip the proof of the lemma and proceed with the proof of the main result.

Proof of Theorem 15. Since T is p-convex, it can be factored through a p-convex Banach lattice Z as in Theorem 3:



where $\varphi : Z\to F$ is an injective interval preserving lattice homomorphism, and $Rx=Tx$ for all $x\in E$. Therefore, (Z,F) can be considered as a compatible interpolation pair of Banach lattices, and we can interpolate $T:E\to F$ and $R:E\to Z$ by the complex method of interpolation (see [6]) with parameter θ, (thus, we complexify E and Z if they are not complex Banach lattices) and we get a Banach lattice $F_\theta=[(Z,F)]_\theta$, and an operator $T_\theta : E\to F_\theta$. Moreover, since φ is an inclusion, F_θ is also continuously included in F. Let us denote this inclusion by $\varphi_\theta : F_\theta\hookrightarrow F$.

We claim that F_θ is p_θ convex, with $\frac{1}{p_\theta}=\frac{\theta}{p}+\frac{1-\theta}{1}$, that is $p_\theta=\frac{p}{\theta+(1-\theta)p}$. Indeed, first notice that if Z is p-convex then $F^{1-\theta}Z^\theta$ is p_θ-convex. This is because for any positive operator S it holds that

$$S(|x_0|^{1-\theta}|x_1|^\theta) \le (S|x_0|)^{1-\theta}(S|x_1|)^\theta.$$

This implies that for any positive operator S acting simultaneously from X_0 into Y_0 and from X_1 into Y_1 the interpolated operator $S: X_0^{1-\theta}X_1^\theta\to Y_0^{1-\theta}Y_1^\theta$ is bounded. In our particular case, (see the discussion following [11, 1.d.3]) for every $n\in\mathbb{N}$, we have operators

$$\begin{array}{rcl} \hat I_n : \ell_1^n(F) & \longrightarrow & F(\ell_1^n) \\ (x_1,\dots,x_n) & \longmapsto & (x_1,\dots,x_n) \end{array} \qquad \begin{array}{rcl} \hat I_n : \ell_p^n(Z) & \longrightarrow & Z(\ell_p^n) \\ (x_1,\dots,x_n) & \longmapsto & (x_1,\dots,x_n) \end{array}$$

which are bounded uniformly on $n \in \mathbb{N}$. Since they are clearly positive, by the previous remark the following operators are also uniformly bounded

$$\begin{array}{rcl} I_n : \ell_1^n(F)^{1-\theta}\ell_p^n(Z)^\theta & \longrightarrow & F(\ell_1^n)^{1-\theta}Z(\ell_p^n)^\theta \\ (x_1,\ldots,x_n) & \longmapsto & (x_1,\ldots,x_n) \end{array}$$

Using (1) and (3) of Lemma 16 we get that the operators

$$\begin{array}{ccccccc} \ell_{p_\theta^n}(F^{1-\theta}G^\theta) & \longrightarrow & \ell_1^n(F)^{1-\theta}\ell_p^n(Z)^\theta & \longrightarrow & F(\ell_1^n)^{1-\theta}Z(\ell_p^n)^\theta & \longrightarrow & F^{1-\theta}G^\theta(\ell_{p_\theta}^n) \\ (x_1,\ldots,x_n) & \longmapsto & (x_1,\ldots,x_n) & \longmapsto & (x_1,\ldots,x_n) & \longmapsto & (x_1,\ldots,x_n) \end{array}$$

are also uniformly bounded on n. This means that $F^{1-\theta}G^\theta$ is p_θ-convex. Now, by Theorem 9, $F_\theta = \overline{F\cap Z}^{F^{1-\theta}Z^\theta}$, and since $F\cap Z$ is a sublattice of $F^{1-\theta}Z^\theta$, F_θ is also p_θ-convex.

Now we claim that T_θ is $(\frac{q}{1-\theta})$-concave. Indeed, since $T : E \to F$ is q-concave and $R : E \to Z$ is ∞-concave, the following maps are bounded:

$$\begin{array}{rclrcl} \check{T} : E(\ell_q) & \longrightarrow & \ell_q(F) & \check{R} : E(\ell_\infty) & \longrightarrow & \ell_\infty(Z) \\ (x_1,x_2,\ldots) & \longmapsto & (Tx_1,Tx_2,\ldots) & (x_1,x_2,\ldots) & \longmapsto & (Rx_1,Rx_2,\ldots) \end{array}$$

Therefore, the interpolated map

$$\check{T}_\theta : [(E(\ell_q),E(\ell_\infty))]_\theta \to [(\ell_q(F),\ell_\infty(Z))]_\theta$$

is also bounded (cf. [4] or [6, §4]). Note that by Theorem 9 and (4) of Lemma 16, we have

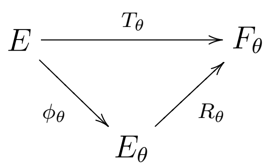
$$[(E(\ell_q),E(\ell_\infty))]_\theta = \overline{E(\ell_q)\cap E(\ell_\infty)}^{E(\ell_q)^{1-\theta}E(\ell_\infty)^\theta} = \overline{E(\ell_q)}^{E(\ell_q)^{1-\theta}E(\ell_\infty)^\theta} = E(\ell_{q_\theta}),$$

where $\frac{1}{q_\theta} = \frac{\theta}{\infty} + \frac{1-\theta}{q}$. And by Lemma 16, we have the identity

$$[(\ell_q(F),\ell_\infty(Z))]_\theta = \overline{\ell_q(F)\cap \ell_\infty(Z)}^{\ell_q(F)^{1-\theta}\ell_\infty(Z)^\theta} = \ell_{q_\theta}(\overline{F\cap Z}^{F^{1-\theta}Z^\theta}) = \ell_{q_\theta}(F_\theta),$$

with equality of norms. Therefore, the map $\check{T}_\theta : E(\ell_{q_\theta}) \to \ell_{q_\theta}(F_\theta)$ is bounded, which means that T_θ is q_θ-concave ($q_\theta = \frac{q}{1-\theta}$).

Hence, we can now apply Theorem 1 to $T_\theta : E \to F_\theta$, and we get the factorization



through the q_θ-concave Banach lattice E_θ. Therefore, T can be factorized as claimed. □

Remark 8. It is easy to see that in the case when the spaces E and F are real Banach lattices, after complexifying and making the previous argument, the obtained operators are all "complexified" operators, i.e. $T_{\mathbb{C}}(x+iy) = T(x) + iT(y)$. Hence, by considering the restriction to the real part in each space, we obtain the same factorization result for real Banach lattices.

6. Connections with Krivine's theorem

Recall the classical result proved in [10]: Given Banach spaces X, Y and a Banach lattice E, if $T_1 : X \to E$ is p-convex and $T_2 : E \to Y$ is p-concave, then T_2T_1 factors through $L_p(\mu)$ for certain measure μ. We remark that the factorization Theorems 1 and 3 allow us to reduce Krivine's theorem to the following purely lattice theoretical version:

Lemma 17. *If W, V are quasi-Banach lattices with W p-convex and V p-concave, then every lattice homomorphism $h : W \to V$ factors through some space $L_p(\mu)$, and the factors are lattice homomorphisms.*

Proof. We may assume by renorming that the p-convexity constant of W, resp. the q-concavity constant of V, are equal to one (see Rem. 1 and 5). Notice that by an elementary p-concavification/ convexification argument (see [11, pp. 53-54]), the proof of the Lemma reduces itself to the case $p = 1$ (this is because a lattice homomorphism $h : W \to V$ is bounded if and only if it is bounded between the p-convexifications $h : W^{(p)} \to V^{(p)}$). In this case, Krivine's argument becomes transparent: indeed, let us consider

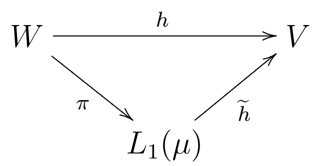
$$F_1 = \{x \in W : \|x\|_W < 1\} \text{ and } F_2 = h^{-1}(\{y \in V : y \geq 0 \text{ and } \|y\|_V \geq \|h\|\}).$$

Clearly, both sets are convex and satisfy $F_1 \cap F_2 = \emptyset$. Hence, by Hahn-Banach's Theorem we can find a functional $f \in W^*$ such that $f(x) \leq 1$ for each $x \in F_1$ and $f(x) \geq 1$ for each $x \in F_2$. Thus, f is positive and for $x \in W$ we have $\frac{1}{\|h\|}\|h(x)\|_V \leq f(|x|) \leq \|x\|_W$.

This allows us to define a seminorm on W by $x \mapsto f(|x|)$ which induces a lattice norm norm on the vector lattice $W/\{x \in W : f(|x|) = 0\}$, the completion of which (for this new norm) is, by Kakutani's theorem [11, Theorem 1.b.2], isomorphic as normed lattice to a space $L_1(\mu)$ for a certain measure μ. Moreover, if π denotes the map $W \to L_1(\mu)$ induced by the quotient map $W \to W/\{x \in W : f(|x|) = 0\}$, we have

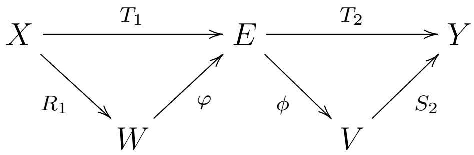
$$\frac{1}{\|h\|}\|h(x)\|_V \leq \|\pi(x)\| \leq \|x\|_W.$$

This means that we can factor



where π and $\widetilde{h}$ are lattice homomorphisms and $\widetilde{h}$ is defined so that $\widetilde{h}(\pi(x)) = h(x)$. □

Now, let $T_1 : X \to E$ be a p-convex operator and $T_2 : E \to Y$ be p-concave. Using Theorems 1 and 3 we have



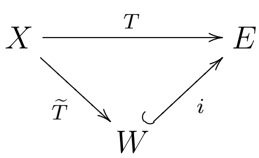
where W is p-convex, V p-concave, and φ, ϕ are lattice homomorphisms. This diagram shows clearly how Krivine's theorem can be obtained from the previous Lemma.

Remark 9. The same argument plus a standard application of Maurey's Theorem [2, Theorem 7.1.2] yields that if $T_1 : X \to E$ is p-convex and $T_2 : E \to Y$ is q-concave, with $p > q$, then T_2T_1 can be factored through the canonical inclusion $i : L_p(\mu) \to L_q(\mu)$ for a certain measure μ (this was essentially proved in [17, Sec. 2, Corollary 7] when Y is reflexive).

In a similar direction, as another application of Theorem 9, we have the following result (compare with [17, Sec. 3, Proposition 2]).

Proposition 18. *Let $T : X \to E$ be p-convex and $S : E \to Y$ q-concave. For every $\theta \in (0,1)$ we can factor ST through a Banach lattice U_θ which is p_θ-convex and q_θ-concave (with as usual $p_\theta = \frac{p}{p(1-\theta)+\theta}$ and $q_\theta = \frac{q}{1-\theta}$).*

Proof. By Theorem 3, we can factor T in the following way, where W is a p-convex Banach lattice and i a positive operator:



Moreover, since $S \circ i : W \to Y$ is q-concave, by Theorem 1 we have the lattice seminorm $\rho_{S\circ i}$ which is continuous with respect to the norm in W ($\rho_{S\circ i}(x) \leq M_q(S \circ i)\|x\|_W$), and such that $W/\rho_{S\circ i}^{-1}(0)$ with the norm that $\rho_{S\circ i}$ induces becomes a q-concave Banach lattice, such that $S \circ i$ factors through it. But, since W is p-convex and $\rho_{S\circ i}^{-1}(0)$ is a closed ideal, it follows that $W/\rho_{S\circ i}^{-1}(0)$ with its quotient norm is also p-convex.

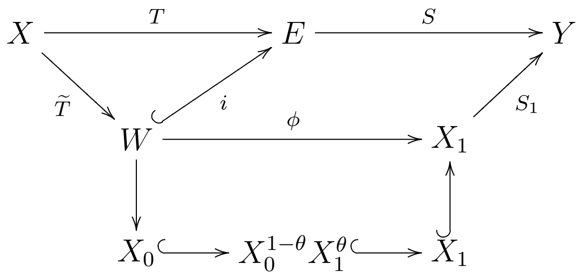
We can consider $X_0 = W/\rho_{S\circ i}^{-1}(0)$ with its quotient norm, and $X_1 = \widehat{W/\rho_{S\circ i}^{-1}(0)}$ (the completion under $\rho_{S\circ i}$) with the norm induced by $\rho_{S\circ i}$. Note that, for all y with $\rho_{S\circ i}(y) = 0$, we have that

$$\rho_{S\circ i}(x) = \rho_{S\circ i}(x+y) \leq M_{(q)}(S \circ i)\|x+y\|.$$

Thus,

$$\|x\|_{X_1} = \rho_{S\circ i}(x) \leq M_{(q)}(S \circ i) \inf\{\|x+y\| : \rho_{S\circ i}(y) = 0\} = M_{(q)}(S \circ i)\|x\|_{X_0},$$

which means that the inclusion $X_0 \hookrightarrow X_1$ is bounded of norm less than or equal to $\leq M_{(q)}$. Therefore, we can interpolate X_0 and X_1. Since X_0 is p-convex and X_1 is q-concave, by [16] we get that $U_\theta = X_0^{1-\theta}X_1^\theta$ is p_θ-convex and q_θ-concave. The following diagram illustrates the situation:



□

Remark 10. A similar result to Proposition 18 was also given in [17, Sec. 3, Proposition 2] for interval preserving lattice homomorphisms from a p-convex to a q-concave Banach lattice with essentially the same proof. The idea of using interpolation to produce this kind of factorization has been initiated both by S. Reisner in [17] and, in parallel, by T. Figiel in [7] using the real method of interpolation.

Corollary 19. *If $T : E \to E$ is p-convex and q-concave, then T^2 factors through a p_θ-convex and q_θ-concave Banach lattice. In particular, it factors through a super reflexive Banach lattice.*

References

[1] Y. A. Abramovich and C. D. Aliprantis, *An invitation to operator theory.* Graduate Studies in Mathematics, vol. 50, American Mathematical Society, Providence, RI, 2002.

[2] F. Albiac, N. J. Kalton, *Topics in Banach space theory.* Graduate Texts in Mathematics, **233**. Springer, New York, 2006.

[3] J. Bergh, *On the relation between the two complex methods of interpolation.* Indiana Univ. Math. J. **28** (1979), no. 5, 775–778.

[4] J. Bergh, J. Löfström, *Interpolation spaces. An introduction.* Grundlehren der Mathematischen Wissenschaften, No. **223**. Springer-Verlag, Berlin-New York, 1976.

[5] A. V. Bukhvalov, *Interpolation of linear operators in spaces of vector-valued functions and with mixed norm.* (Russian) Sibirsk. Mat. Zh. **28** (1987), no. 1, i, 37–51. English translation: Siberian Math. J. **28** (1987), no. 1, 24–36.

[6] A. P. Calderón, *Intermediate spaces and interpolation, the complex method.* Studia Math. **24**, (1964) 113–190.

[7] T. Figiel, *Uniformly convex norms on Banach lattices.* Studia Math. **68**, (1980) 215–247.

[8] N. Ghoussoub, W. B. Johnson, *Factoring operators through Banach lattices not containing $C(0,1)$.* Math. Z. **194**, (1987) 153-171.

[9] S. G. Kreĭn, Yu. Ī. Petunīn, E. M. Semënov, *Interpolation of linear operators.* Translations of Mathematical Monographs, 54. American Mathematical Society, 1982.

[10] J. L. Krivine, *Théorèmes de factorisation dans les espaces réticulés.* (French) Séminaire Maurey-Schwartz 1973–1974: *Espaces L^p, applications radonifiantes et géométrie des espaces de Banach,* Exp. Nos. 22 et 23. Centre de Math., École Polytech., Paris, 1974.
[11] J. Lindenstrauss, L. Tzafriri, *Classical Banach Spaces II: Function Spaces.* Berlin ; New York : Springer-Verlag, (1979).
[12] G.Ya. Lozanovskii, *Certain Banach lattices.* (Russian) Sibirsk. Mat. Ž. **10**, (1969) pp. 584–599.
[13] G.Ya. Lozanovskii, *Certain Banach lattices IV.* (Russian) Sibirsk. Mat. Ž. **14**, (1973), pp. 140–155.
[14] P. Meyer-Nieberg, *Kegel p-absolutsummierende und p-beschränkende Operatoren.* (German) Nederl. Akad. Wetensch. Indag. Math. **40** (1978), no. 4, 479–490.
[15] P. Meyer-Nieberg, *Banach Lattices.* Springer-Verlag (1991).
[16] G. Pisier, *Some applications of the complex interpolation method to Banach lattices.* J. Analyse Math. **35**, (1979), 264–281.
[17] S. Reisner, *Operators which factor through convex Banach lattices.* Canad. J. Math. **32** (1980), 1482–1500.
[18] H.H. Schaefer *Banach lattices and positive operators.* Die Grundlehren der mathematischen Wissenschaften, Band **215**. Springer-Verlag 1974.
[19] V. A. Šestakov, *Complex interpolation in Banach spaces of measurable functions.* (Russian) Vestnik Leningrad. Univ. No. 19 Mat. Meh. Astronom. Vyp. 4 (1974), 64–68, 171.

Institut de Mathématiques de Jussieu, CNRS and UPMC-Univ. Paris-06, case 186, 75005 Paris, France
E-mail address: yves.raynaud@upmc.fr

Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040, Madrid, Spain.
E-mail address: tradacete@mat.ucm.es