DISJOINTLY HOMOGENEOUS BANACH LATTICES AND COMPACT PRODUCTS OF OPERATORS

JULIO FLORES, PEDRO TRADACETE, AND VLADIMIR G. TROITSKY

Abstract. The notion of disjointly homogeneous Banach lattice is introduced. In these spaces every two disjoint sequences share equivalent subsequences. It is proved that on this class of Banach lattices the product of a regular AM-compact and a regular disjointly strictly singular operators is always a compact operator.

1. Introduction

This note is a continuation of a previous work by the authors [FTT08] where it was proved that, on a wide class of Banach lattices (which includes those with finite cotype), the product of a regular AM-compact operator and a regular disjointly strictly singular operator is strictly singular and has invariant subspaces. In particular, if T is regular, AM-compact, and disjointly strictly singular, then the square T^2 is strictly singular. Here we show that in a certain class of Banach lattices better compactness properties can be obtained.

To this end, the notion of disjointly homogeneous Banach lattice is introduced. Namely, a Banach lattice E is called *disjointly homogeneous* if for two arbitrary disjoint sequences in E there exist subsequences which are equivalent. This forms a class of Banach lattices that includes for instance the spaces $L_p(\mu)$ $(1 \leq p \leq \infty)$, Lorentz spaces $L_{p,q}(\mu)$ and some others.

For this class of Banach lattices, the following holds.

Theorem. *Let E be a disjointly homogeneous Banach lattice. If $T : E \to E$ is regular, disjointly strictly singular, and AM-compact, then T^2 is compact.*

In particular, as a consequence of Lomonosov's Theorem we get that under these hypotheses such operators have hyperinvariant subspaces.

We will routinely use the following well known facts. Suppose that E is an order continuous Banach lattice with a weak order unit. Then E can be represented as a

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Köthe Function space over some probability measure space (Ω,Σ,μ) with continuous inclusions:

$$L_\infty(\mu)\hookrightarrow E\hookrightarrow L_1(\mu).$$

Moreover, the dual E^* can be identified with the space of all μ-measurable functions g such that $\sup\bigl\{\int_\Omega fgd\mu\ :\ \|f\|_E\leqslant 1\bigr\}<\infty$, and the value taken by the functional corresponding to g at $f\in E$ is $\int_\Omega fgd\mu$. See [LT79, Thm. 1.b.14] for details.

Recall that, given $\varepsilon>0$, the ***Kadec-Pełczyński*** set $M(\varepsilon)$ is defined as follows:

$$M(\varepsilon)=\bigl\{x\in E\ :\ \mu\bigl(\sigma(x,\varepsilon)\bigr)\geqslant\varepsilon\bigr\}$$

where $\sigma(x,\varepsilon)=\bigl\{t\in\Omega\ :\ |x(t)|\geqslant\varepsilon\|x\|_E\bigr\}$. It is known ([LT79, Proposition 1.c.8]) that $\|x\|_1\geqslant\varepsilon^2\|x\|_E$ for all $x\in M(\varepsilon)$; hence the norms $\|\cdot\|_E$ and $\|\cdot\|_1$ are equivalent on every subspace of E contained in $M(\varepsilon)$ for some $\varepsilon>0$. On the other hand, if a normalized sequence (x_n) in E is not contained in any $M(\varepsilon)$, then there is a subsequence (x_{n_k}) and a disjoint (unconditional basic) sequence (y_k) in E equivalent to (x_{n_k}) with $\|x_{n_k}-y_k\|_E\to 0$.

Recall that an operator $T:E\to E$ is positive if it maps positive elements to positive elements. Moreover, an operator is regular if it is a difference of two positive operators. By [Wei82, Theorem 2.2], every regular operator $T\colon E\to E$ can be extended to a bounded operator $\widetilde{T}\colon L_1(\mu)\to L_1(\mu)$. It was shown in [FTT08, Theorem 2.2] that $T\colon E\to E$ is AM-compact if and only if $\widetilde{T}\colon L_1(\mu)\to L_1(\mu)$ is Dunford-Pettis.

Recall that a Banach lattice is weakly sequentially complete if and only if it does not contain a subspace which is isomorphic to c_0, if and only if it does not contain a sublattice which is lattice isomorphic to c_0. Such a Banach lattice is called a ***KB-space***. Every KB-space is order continuous; a dual Banach lattice is a KB-space if and only if it is order continuous. See [AB85] for more details.

2. Disjointly homogeneous Banach lattices

A Banach lattice E is said to be ***disjointly homogeneous*** if for every seminormalized sequences (x_n) and (y_m) with $|x_i|\wedge|x_j|=0$ and $|y_i|\wedge|y_j|=0$ for $i\neq j$, there exist equivalent subsequences, that is, there exist a constant $C>0$ and subsequences (n_k), (m_k) such that

$$C^{-1}\Bigl\|\sum_{k=1}^N a_kx_{n_k}\Bigr\|\leq\Bigl\|\sum_{k=1}^N a_ky_{m_k}\Bigr\|\leq C\Bigl\|\sum_{k=1}^N a_kx_{n_k}\Bigr\|,$$

for every scalars $(a_k)_{k=1}^N$.

Observe that a Banach lattice E is disjointly homogeneous if for any pair of disjoint positive normalized sequences (x_n) and (y_n), there exist subsequences which are equivalent.

Also note that the definition of a disjointly homogeneous Banach lattice depends on the lattice structure, that is, it is not preserved under isomorphisms in general. For instance, for any $1 < p < \infty$, $p \neq 2$, the function space $L_p[0,1]$ is isomorphic as a Banach space to the atomic Banach lattice H_p given by the unconditional Haar basis (see, e.g., [LT77, p. 19]), and this lattice has disjoint sequences equivalent to ℓ_2 and ℓ_p; thus, with the atomic structure H_p is not disjointly homogeneous.

Examples of disjointly homogeneous spaces include the spaces $L_p(\mu)$ for $1 \leq p \leq \infty$ and every measure μ, because every normalized disjoint sequence in $L_p(\mu)$ is equivalent to the unit vector basis of ℓ_p. Moreover, in [FJT75] and [CD88] it was shown that every disjoint normalized sequence in the Lorentz function spaces $\Lambda_{W,q}(\mu)$, or $L_{p,q}$ contains a subsequence equivalent to the unit vector basis of ℓ_q (for $q < \infty$).

Motivated by these examples, we say that a Banach lattice is p-disjointly homogeneous if every normalized disjoint sequence has a subsequence equivalent to the unit vector basis of ℓ_p (c_0 in the case $p = \infty$). Clearly, the spaces $\ell_p(X_n)$ where X_n is a sequence of finite dimensional Banach lattices, are p-disjointly homogeneous. So are the Baernstein spaces B_p introduced by C. Seifert (see [CS89, p. 7]).

One could ask whether every disjointly homogeneous Banach lattice has to be p-disjointly homogeneous for some $p \in [1, \infty]$. The following example shows that this is not the case.

Example. Let T be Tsirelson's space (see [Tsir74]). We claim that T with the lattice structure given by its unconditional basis (t_n) is disjointly homogeneous, and clearly does not contain any disjoint sequence equivalent to the unit vector basis of ℓ_p or c_0.

Proof. If $x \in T$ with $x = \sum_{i=1}^{\infty} \alpha_i t_i$, then we denote $\operatorname{supp} x = \{i \in \mathbb{N} : \alpha_i \neq 0\}$. For $x, y \in T$ we write $\operatorname{supp} x < \operatorname{supp} y$ if $i < j$ whenever $i \in \operatorname{supp} x$ and $j \in \operatorname{supp} y$. Given two normalized disjoint sequences in T, (x_n) and (y_n), we will show that they have equivalent subsequences.

By truncating each x_n, we may assume by Proposition 1.a.9 of [LT77] that each x_n has finite support. By passing to a subsequence, we may further assume that $\operatorname{supp} x_n < \operatorname{supp} x_{n+1}$ for all n. Similarly, we may assume that $\operatorname{supp} y_n < \operatorname{supp} y_{n+1}$ for all n. Now it is easy to construct subsequences (x_{n_k}) and (y_{n_k}) so that

$$\operatorname{supp} x_{n_1} < \operatorname{supp} y_{n_1} < \operatorname{supp} x_{n_2} < \operatorname{supp} y_{n_2} \ldots$$

It follows from [CS89, Proposition II.4] that (x_{n_k}) and (y_{n_k}) are equivalent. □

Proposition 2.1. *Suppose that E is a disjointly homogeneous Banach lattice. Then either E or E^* (or both) is a KB-space. Precisely we have that*

(i) *E is not a KB-space if and only if E is ∞-disjointly homogeneous.*
(ii) *E^* is not a KB-space if and only if E is 1-disjointly homogeneous.*

Proof. The equivalence in (i) follows immediately from the definition of a KB-space. [AB85, Theorem 14.21] asserts that E^* is not a KB-space iff E contains a lattice copy of ℓ_1, this yields the equivalence in (ii). Finally, since no subsequence of the unit vector basis of c_0 is equivalent to the unit vector basis of ℓ_1 and vice versa, the two pairs of conditions are incompatible, hence at least one of the two spaces has to be a KB-space. □

A natural question in this setting is wether disjointly homogeneous spaces are stable under duality. In this direction we have the following result.

Theorem 2.2. *If E is an ∞-disjointly homogeneous Banach lattice, then E^* is a 1-disjointly homogeneous Banach lattice.*

Proof. Every disjoint sequence in E has a subsequence equivalent to the unit vector basis of c_0. In particular, E^* is order continuous. Let (x_n^*) be a normalized disjoint positive sequence in E^*. Consider a sequence (x_n) of elements in E_+ of norm one, such that $x_n^*(x_n) = 1$. By [MN91, Proposition 2.3.1], for any $\varepsilon > 0$ there exist a subsequence (k_n) and a disjoint sequence $(v_n) \subset E_+$ such that $v_n \leqslant x_{k_n}$ and $x_{k_n}^*(v_n) \geq 1 - \varepsilon$. By hypothesis, there exist a constant $C > 0$ and a subsequence of (v_n) which we still denote (v_n) such that

$$C^{-1} \sup_{n=1,\dots,m} |b_n| \leqslant \Big\| \sum_{n=1}^{m} b_n v_n \Big\| \leqslant C \sup_{n=1,\dots,m} |b_n|.$$

Therefore, for any sequence of scalars $(a_n)_{n=1}^m$ we have:

$$\begin{aligned} \Big\| \sum_{n=1}^{m} a_n x_{k_n}^* \Big\| &= \Big\| \sum_{n=1}^{m} |a_n| x_{k_n}^* \Big\| = \sup \Big\{ \Big(\sum_{n=1}^{m} |a_n| x_{k_n}^* \Big)(y) \ : \ y \in E, \ \|y\| \leqslant 1 \Big\} \\ &\geqslant \Big(\sum_{n=1}^{m} |a_n| x_{k_n}^* \Big) \Big(C^{-1} \sum_{n=1}^{m} v_n \Big) \geqslant C^{-1} \sum_{n=1}^{m} |a_n| x_{k_n}^*(v_n) \\ &\geqslant C^{-1}(1-\varepsilon) \sum_{n=1}^{m} |a_n|. \end{aligned}$$

Hence, it follows that

$$C^{-1}(1-\varepsilon)\sum_{n=1}^{m}|a_n| \leqslant \Big\|\sum_{n=1}^{m} a_n x_{k_n}^*\Big\| \leqslant \sum_{n=1}^{m}|a_n|.$$

This yields that every disjoint sequence in E^* has a subsequence equivalent to the unit vector basis of ℓ_1. In particular E^* is disjointly homogeneous. $\square$

For general disjointly homogeneous spaces this duality is not true, as the following example shows.

Example. Given $1<q<\infty$, the Lorentz function space $L_{q,1}(0,1)$ is disjointly homogeneous, but the dual $L_{p,\infty}(0,1)$ is not (where $\frac{1}{p}+\frac{1}{q}=1$).

Proof. Indeed, every disjoint normalized sequence in $L_{q,1}$ has a subsequence equivalent to the unit vector basis of ℓ_1 (see [CD88, Lemma 2.1]). In contrast, every disjoint sequence in the order continuous part of $L_{p,\infty}$ (the closed linear span of the characteristic functions in $L_{p,\infty}$) has a subsequence equivalent to the unit vector basis of c_0 (see [NST]);yet $L_{p,\infty}$ contains disjoint sequences spanning ℓ_p.

Let us proof this last assertion. Consider the functions in $[0,1]$ defined by

$$f_n(t)=\frac{p-1}{p}(t-2^{-n})^{-\frac{1}{p}}\chi_{(2^{-(n+1)},2^{-n})}(t).$$

We claim that the closed linear span $[f_n]$ is isomorphic to ℓ_p.

Since $\|f\|_{L_{p,\infty}}=\sup_{s>0}s(\mu_f(s))^{\frac{1}{p}}$, where $\mu_f(s)=\mu\{t\in(0,1):|f(t)|>s\}$ is the distribution function, for each $n\in\mathbb{N}$, we have

$$\begin{aligned}
\mu_{f_n}(s) &= \mu\{t\in(2^{-(n+1)},2^{-n}):\frac{p-1}{p}(t-2^{-n})^{-\frac{1}{p}}>s\}\\
&= \mu\Big\{t\in(2^{-(n+1)},2^{-n}):t<2^{-n}+\Big(\frac{p-1}{p}\Big)^p\frac{1}{s^p}\Big\}\\
&= \begin{cases} 2^{-n}-2^{-(n+1)} & \text{if } s\leq \frac{p-1}{p(2^{-n}-2^{-(n+1)})^{\frac{1}{p}}},\\ \left(\frac{p-1}{p}\right)^p\frac{1}{s^p} & \text{if } s>\frac{p-1}{p(2^{-n}-2^{-(n+1)})^{\frac{1}{p}}}.\end{cases}
\end{aligned}$$

This clearly implies that (f_n) is a seminormalized sequence in $L_{p,\infty}$. Now, given scalars a,b let us see that $\|af_i+bf_j\|_{L_{p,\infty}}\sim(|a|^p+|b|^p)^{\frac{1}{p}}$, for $i\neq j$. Indeed, since f_i and f_j

are disjoint, we have

$$
\begin{aligned}
\|af_i + bf_j\|_{L_{p,\infty}} &= \sup_{s>0} s\Big(\mu_{f_i}(\frac{s}{|a|}) + \mu_{f_j}(\frac{s}{|b|})\Big)^{\frac{1}{p}} \\
&\geq s_0\Big(\mu_{f_i}(\frac{s_0}{|a|}) + \mu_{f_j}(\frac{s_0}{|b|})\Big)^{\frac{1}{p}} \\
&= s_0\Big[\Big(\frac{p-1}{p}\Big)^p\frac{|a|^p}{s_0^p} + \Big(\frac{p-1}{p}\Big)^p\frac{|b|^p}{s_0^p}\Big]^{\frac{1}{p}} \\
&= \frac{p-1}{p}(|a|^p + |b|^p)^{\frac{1}{p}}
\end{aligned}
$$

where s_0 is any number greater than $\max\Big\{\frac{|a|p-1}{p(2^{-(i+1)}-2^{-i})^{\frac{1}{p}}}, \frac{|b|p-1}{p(2^{-(j+1)}-2^{-j})^{\frac{1}{p}}}\Big\}$. Moreover, since $L_{p,\infty}$ satisfies an upper p-estimate [Cr81], we also get $\|af_i + bf_j\|_{L_{p,\infty}} \leq C(|a|^p + |b|^p)^{\frac{1}{p}}$ for certain constant $C > 0$. The statement that $[f_n]$ is isomorphic to ℓ_p follows by induction. □

It remains as an open question wether every reflexive Banach lattice E is disjointly homogeneous if and only if E^* is disjointly homogeneous.

3. Regular operators on disjointly homogeneous Banach lattices

Recall that an operator on a Banach lattice is called ***disjointly strictly singular*** if its restriction to any subspace spanned by a disjoint sequence is not an isomorphism [HS89]. This class contains the class of strictly singular operators but in general they do not coincide.

Proposition 3.1. *If an operator $T : E \to F$ from a Banach lattice E to a KB-space F is not an isomorphism on any subspace isomorphic to ℓ_1, then it is weakly compact. In particular, if T is disjointly strictly singular, then it is weakly compact as well.*

Proof. Let $(x_n)_n$ be a normalized sequence in E. If (Tx_n) has no weakly Cauchy subsequence, then by Rosenthal's ℓ_1 theorem, there exists a subsequence $(Tx_{n_k})_k$ equivalent to the unit vector basis of ℓ_1. Therefore, T preserves an isomorphic copy of ℓ_1, which contradicts the hypothesis.

Hence, there is a weakly Cauchy subsequence (Tx_{n_k}) of (Tx_n). Since F is weakly sequentially complete, (Tx_{n_k}) is weakly convergent.

Since F is order continuous, it follows from [Chen99] (see, also, [FTT08, Theorem 2.7]) that every operator preserving an isomorphic copy of ℓ_1, also preserves a lattice copy of ℓ_1. Hence disjointly strictly singular operators into an order continuous Banach lattice are never an isomorphism on a subspace isomorphic to ℓ_1.

□

The following result improves the one obtained in [FTT08] in the setting of disjointly homogeneous Banach lattices.

Theorem 3.2. *Suppose that E is a disjointly homogeneous Banach lattice with order continuous norm and a weak unit. Suppose that S and T are two regular operators on E such that S is disjointly strictly singular and T is AM-compact.*

(i) *If E^* is order continuous then ST is compact.*

(ii) *If E^* is not order continuous then TS is compact.*

In particular, if R is disjointly strictly singular and regular, then STR is compact.

Proof. Since E is order continuous and has a weak unit, we can consider E is as an ideal in $L_1(\mu)$ for some probability measure μ, and extend T to a Dunford-Pettis operator $\widetilde{T}\colon L_1(\mu) \to L_1(\mu)$.

(i) Suppose that E^* is order continuous but ST is not compact. Then there exists a normalized sequence (u_n) such that (STu_n) has no convergent subsequences. It follows that (u_n) has no convergent subsequences. Since E^* is order continuous, E doesn't contain a copy of ℓ_1, so by Rosenthal's ℓ_1-theorem, we may assume that (u_n) is weakly Cauchy. Since (STu_n) has no convergent subsequences, we can assume by passing to a further subsequence that there exists an $\delta > 0$ such that $\|STu_n - STu_m\|_E > \delta$ whenever $m \neq n$. For every $n \in \mathbb{N}$ put $x_n = u_{n+1} - u_n$, $y_n = Tx_n$, and $z_n = Sy_n = STx_n$. Then (z_n) is seminormalized, hence (x_n) and (y_n) are seminormalized as well. Also, (x_n) is weakly null, so that (y_n) and (z_n) are weakly null as well.

Since (x_n) is also weakly null in $L_1(\mu)$, and $\widetilde{T}$ is Dunford-Pettis, it follows that $\|y_n\|_1 \to 0$. However, (y_n) is seminormalized in E, hence the sequence (y_n) is not contained in any Kadec-Pełczyński set $M(\varepsilon)$ for any $\varepsilon > 0$. After passing to a subsequence of (x_n) we may assume that (y_n) is equivalent to a disjoint sequence (v_n) and $\|y_n - v_n\|_E \to 0$. By passing to subsequences we may assume that $\|y_n - v_n\|_E < 2^{-n}$.

Since S is regular, $\widetilde{S}$ is bounded, so that $\|z_n\|_1 \to 0$. Similarly, we may assume that (z_n) is equivalent to a disjoint sequence (w_n) and $\|z_n - w_n\|_E \to 0$. Since (v_n) and (w_n) are disjoint seminormalized sequences and E is disjointly homogeneous, by passing to further subsequences we may assume that they are equivalent.

Since S is disjointly strictly singular, we can find a normalized block sequence (h_k) of (v_n) such that $Sh_k \to 0$. Suppose that $h_k = \sum_{n=m_k+1}^{m_{k+1}} \alpha_n v_n$. Since (v_n) is a basic sequence, there exists a positive real C such that $|\alpha_n| < C$. Let $g_k = \sum_{n=m_k+1}^{m_{k+1}} \alpha_n y_n$

for all k, then

$$\|h_k - g_k\|_E \leqslant \sum_{n=m_k+1}^{m_{k+1}} |\alpha_n| \|v_n - y_n\| \leqslant C2^{-m_k} \to 0,$$

so that $\|Sg_k\|_E \leqslant \|Sh_k\|_E + \|S\| \|h_k - g_k\|_E \to 0$. On the other hand, since (z_n) and (w_n) are equivalent, we have

$$\|Sg_k\|_E = \Big\| \sum_{n=m_k+1}^{m_{k+1}} \alpha_n z_n \Big\|_E \geqslant C_1 \Big\| \sum_{n=m_k+1}^{m_{k+1}} \alpha_n v_n \Big\|_E = \|h_k\|_E = 1;$$

a contradiction.

(ii) Suppose that E^* is not order continuous, hence not a KB-space. Then Proposition 2.1 yields that E is a KB-space and is 1-disjointly homogeneous. Hence, S is weakly compact by Proposition 3.1. Since $\widetilde{T} : L_1 \to L_1$ is Dunford-Pettis, the composition

$$E \xrightarrow{S} E \hookrightarrow L_1(\mu) \xrightarrow{\widetilde{T}} L_1(\mu)$$

is a compact operator. If TS is not compact, there exists a normalized sequence (x_n) in E such that the sequence $(TS(x_n))$ is not contained in any $M(\varepsilon)$. Therefore, $(T(x_n))$ has a subsequence which is equivalent to a disjoint sequence in E. Hence, this sequence must have a subsequence equivalent to the unit vector basis of ℓ_1, because E is 1-disjointly homogeneous. However, this implies that TS must preserve an isomorphic copy of ℓ_1, which is impossible since S is weakly compact.

□

Observe that Theorem 3.2(ii) remains valid in the case that S is not regular. Also, it remains valid if, instead of being disjointly strictly singular, S is only assumed to be weakly compact.

Corollary 3.3. *Let E be a disjointly homogeneous Banach lattice. If $T : E \to E$ is regular, disjointly strictly singular, and AM-compact, then T^2 is compact.*

Corollary 3.3 together with Lomonosov's Theorem [Lom73] immediately yield the following result.

Corollary 3.4. *Let E be a disjointly homogeneous Banach lattice. If $T : E \to E$ is regular, disjointly strictly singular and AM-compact. Then T has a hyperinvariant subspace.*

A subset S of an order continuous Banach lattice of functions over a measure space (Ω, Σ, μ) is called *equi-integrable* if

$$\sup_{f\in S}\|f\chi_A\| \to 0 \quad \text{when} \quad \mu(A)\to 0.$$

We will make use of the following well-known fact (see [FH02, Lemma 3.3] for a proof).

Lemma 3.5. *Let E be an order continuous Banach lattice which is continuously included, as a dense ideal, in $L_1(\mu)$ for some probability measure μ. A norm bounded sequence (g_n) in E is convergent to zero if and only if (g_n) is equi-integrable and convergent to zero in the norm of L_1.*

Recall that an order continuous Banach lattice E has the subsequence splitting property [Wei89] if for every bounded sequence (f_n) there exist a disjoint sequence (h_k), an equi-integrable sequence (g_k) and a subsequence (f_{n_k}) such that $f_{n_k} = g_k + h_k$ with g_k and h_k disjoint for all k. For positive operators on a disjointly homogeneous Banach lattice with the subsequence splitting property, the conclusion of Corollary 3.3 can be improved as follows. Compare with the results in [CG87] for L_p spaces.

Theorem 3.6. *Let E be a disjointly homogeneous Banach lattice with the subsequence splitting property, such that E^* is order continuous. If $T : E \to E$ is a regular operator which is disjointly strictly singular and AM-compact, then T is compact.*

Proof. Let (x_n) be a norm bounded sequence in E. Since E has the subsequence splitting property, passing to a subsequence we have $x_{n_k} = g_k + h_k$ with (g_k) equi-integrable and (h_k) a disjoint sequence. Since (g_k) is equi-integrable, for some subsequence (still denoted (g_k)) we must have $g_k \to g$ weakly for some $g \in E$ [AB85].

Since E^* is order continuous, $|h_k|$ tends weakly to zero. Thus, so does $|T|(|h_k|)$ which is positive. Since $E \hookrightarrow L_1$, we have that $|T|(|h_k|)$ tends to zero weakly in L_1, hence $\|T(h_k)\|_{L_1} \leq \||T|(|h_k|)\|_{L_1} \to 0$.

Let us apply now Kadec-Pełczyński dichotomy to the sequence (Th_k) in E [FJT75]. Suppose first that (Th_k) is not contained in any $M(\varepsilon)$, then there is a subsequence (Th_{k_j}) equivalent to a disjoint sequence. Hence, since the sequence (h_k) is disjoint, and E is disjointly homogeneous passing to a further subsequence we have that (Th_{k_j}) and (h_{k_i}) are equivalent basic sequences. This implies that T is an isomorphism when restricted to the span of (h_{k_i}). However, this is a contradiction, because T is disjointly strictly singular.

Therefore, (Th_k) is contained in some $M(\varepsilon)$, but then $\|Th_k\|_E \to 0$ since $\|Th_k\|_1 \to 0$. Moreover, since T is AM-compact, $Tg_k \to Tg$ in $L_1(\mu)$ [FTT08, Theorem 2.2]. Now, since (Tg_k) is equi-integrable in E, by Lemma 3.5, it follows that $Tg_k \to Tg$ in E; thus, $Tx_k = Th_k + Tg_k \to Tg$, so T is compact. □

Notice that Theorem 3.6 need not be true if E^* is not order continuous, even if the operator is positive, as the following example shows.

Example. There exists a positive operator $T : L_1 \to L_1$ which is disjointly strictly singular and AM-compact, but not compact.

Proof. Let (f_n) be a sequence of pairwise disjoint, positive, normalized functions in $L_1(0,1)$. Clearly, the sequence (f_n) generates a complemented subspace isomorphic to ℓ_1. Let $P : L_1(0,1) \to \ell_1$ denote this projection, which is clearly positive. Now consider the operator $R : \ell_1 \to L_1$ defined by $R(e_{2n}) = r_n^+$ and $R(e_{2n+1}) = r_n^-$, where (e_n) denotes the canonical basis of ℓ_1 and (r_n) denotes the Rademacher functions on $(0,1)$.

Let us consider the operator $T = RP$, which is also positive. Since the order intervals in ℓ_1 are compact, and P is positive, T is AM-compact. Moreover, T is disjointly strictly singular, because every disjoint sequence in L_1 is equivalent to ℓ_1 and T factors through ℓ_2. However, T is not compact because the sequence $(f_{2n} - f_{2n+1})$ is norm bounded, and its image $T(f_{2n} - f_{2n+1}) = r_n$ does not have any convergent subsequence. □

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Department of Applied Mathematics, Escet, Universidad Rey Juan Carlos, 28933, Móstoles, Madrid, Spain.
E-mail address: `julio.flores@urjc.es`

Department of Mathematical Analysis, Universidad Complutense de Madrid, 28040, Madrid, Spain.
E-mail address: `tradacete@mat.ucm.es`

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB, T6G 2G1. Canada.
E-mail address: `vtroitsky@math.ualberta.ca`