

CHARACTERIZATIONS OF STRICTLY SINGULAR OPERATORS ON BANACH LATTICES

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ABSTRACT. New characterizations of strictly singular operators between Banach lattices are given. It is proved that for Banach lattices X and Y , such that X has finite cotype and Y satisfies a lower 2-estimate, an operator $T : X \rightarrow Y$ is strictly singular if and only if it is disjointly strictly singular and ℓ_2 -singular. Moreover, if T is regular the same equivalence holds provided Y is just order continuous. Furthermore, it is shown that these results fail if the conditions on the lattices are relaxed.

INTRODUCTION

Strictly singular operators were introduced by T. Kato [18] in connection with the perturbation theory of Fredholm operators. Recall that an operator $T : X \rightarrow Y$ between Banach spaces is *strictly singular* if it is not an isomorphism when restricted to any infinite dimensional (closed) subspace of X . Strictly singular operators constitute a closed two-sided operator ideal which contains the ideal of compact operators. Moreover, an operator $T : X \rightarrow Y$ is strictly singular if and only if for every infinite dimensional subspace M of X , there exists an infinite dimensional subspace N of M such that the restriction $T|_N$ is compact.

In the context of Banach lattices a weaker notion is the following: given a Banach lattice X , a Banach space Y and an operator $T : X \rightarrow Y$, we say that T is *disjointly strictly singular* if it is not an isomorphism when restricted to the closed linear span of any disjoint sequence in X . This notion is a quite useful tool in the study of strictly singular operators on Banach lattices, for example, in the context of domination problems for positive operators (cf. [10]), and for comparing structures of rearrangement invariant spaces (cf. [13],[14]). Several properties of disjointly strictly singular operators have been studied in [8], [9] and [11].

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In this paper we are interested in giving characterizations of the strict singularity of operators acting between Banach lattices. Since strictly singular operators are disjointly strictly singular, we are mainly interested in converse statements.

Our motivation stems from the following facts. First, it is well known that an endomorphism of $L_p = L_p[0, 1]$, with $1 \leq p < \infty$, is strictly singular if and only if it is ℓ_p -singular and ℓ_2 -singular ([22], [25]). In other words, an endomorphism T on L_p is strictly singular if and only if it is disjointly strictly singular and ℓ_2 -singular. Recall that an operator between Banach spaces is called ℓ_p -singular, for some $1 \leq p \leq \infty$, if it is not an isomorphism when restricted to any subspace isomorphic to ℓ_p . For recent results on ℓ_p -singular operators we refer to [16].

Given an order continuous Banach lattice X , if an operator $T : X \rightarrow Y$ is disjointly strictly singular and ℓ_p -singular, for every $1 \leq p \leq 2$, then T is strictly singular. This can be seen using the Kadec-Pelczynski disjointification method and Aldous' theorem on subspaces of L_1 ([2]). Furthermore, in the special case of X (or Y) being a Banach lattice with type 2, if $T : X \rightarrow Y$ is disjointly strictly singular and ℓ_2 -singular, then T is strictly singular. A similar statement also holds for inclusion operators between rearrangement invariant spaces ([12]).

One of our main results in this direction is the following.

Theorem A. *Let X and Y be Banach lattices such that X has finite cotype and Y satisfies a lower 2-estimate. Then an operator $T : X \rightarrow Y$ is strictly singular if and only if it is both disjointly strictly singular and ℓ_2 -singular.*

Then, we consider the class of *regular* operators, i.e. those which are a difference of positive operators, proving that for this class the equivalence given above in Theorem A is also true under much weaker conditions on the lattices.

Theorem B. *Let X and Y be Banach lattices such that X has finite cotype and Y is order continuous. Then a regular operator $T : X \rightarrow Y$ is strictly singular if and only if it is both disjointly strictly singular and ℓ_2 -singular.*

Both Theorem A and B are obtained by means of the following general result: *If X is a Banach lattice with finite cotype, Y a Banach space, and $T : X \rightarrow Y$ is disjointly strictly singular and AM-compact, then T is strictly singular* (see Theorem 2.4). Recall that, for a Banach lattice X , an operator $T : X \rightarrow Y$ is called *AM-compact* if the image of every order interval is a relatively compact set. The connection between AM-compact operators and ℓ_2 -singular operators is studied in Section 2 (see Propositions 2.5 and 2.6). Let us remark that the motivation of this kind of result dates from Rosenthal [24], where it was proved that for endomorphisms on L_1 spaces, being AM-compact and ℓ_2 -singular are equivalent notions.

As an application of these characterizations a domination result for positive strictly singular operators is easily obtained, improving a result of [10]. Precisely, given two

operators $0 \leq R \leq T : X \rightarrow Y$, with T strictly singular, then we have that R is also strictly singular provided that X has finite cotype and Y is order continuous (Corollary 2.8).

In Section 3 we prove that the hypothesis in Theorem A on the range lattice Y cannot be weakened, in the sense that the result is no longer true for Banach lattices Y with a lower q -estimate for some $q > 2$. To this end we consider the Banach lattice $L_r(\ell_q)$, which consists of sequences $x = (x_1, x_2, \dots)$ of elements in L_r such that

$$\|x\|_{L_r(\ell_q)} = \left\| \left(\sum_{i=1}^{\infty} |x_i|^q \right)^{1/q} \right\|_{L_r} < \infty.$$

Theorem C. *Consider the Banach lattices L_p and $L_r(\ell_q)$, where $1 < r < p < 2 < q < \infty$. For each $p < s < 2$, there exists an operator $T : L_p \rightarrow L_r(\ell_q)$ such that it is ℓ_p -singular and ℓ_2 -singular, but not ℓ_s -singular.*

In particular, the operator T is disjointly strictly singular and ℓ_2 -singular, but not strictly singular.

The proof of this fact requires some preliminary results. First, we present some technical Lemmas which make use of known estimates for independent and identically distributed (i.i.d.) s -stable random variables, for $1 < s < 2$, given in [15] (see Proposition 3.1 below). We consider the atomic lattice representation H_r of the space L_r , associated to the unconditional Haar basis (h_i) , in order to define a suitable operator R from H_r to $L_r(\ell_q)$ which restricted to L_p satisfies the required conditions. More precisely, let (w_n) denote a block basis of the Haar basis (h_i) of the form

$$w_n = \sum_{i=q_{n-1}+1}^{q_n} a_i h_i,$$

which is equivalent to a sequence of i.i.d. s -stable random variables in both L_p and in L_r (for $1 < r < p < s < 2$). We can consider the operator $R : H_r \rightarrow L_r(\ell_q)$, defined by

$$R \left((c_i)_{i=1}^{\infty} \right) = \left(\sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right)_{n=1}^{\infty}$$

The operator $T_s : L_p \rightarrow L_r(\ell_q)$ is now defined as the composition RLJ , where J is the canonical inclusion $L_p \hookrightarrow L_r$ and L is the isomorphism between H_r and L_r mapping each sequence in H_r to the corresponding expansion as a series with respect to the Haar system.

We also show that the characterization for regular operators given in Theorem B fails if the order continuity of the range lattice is missing.

1. PRELIMINARIES

Let us start with some notation and definitions. We refer the reader to the monographs [4], [20], [21] and [28] for unexplained terminology from Banach lattices and positive operator theory.

Throughout, we will write *SS* (resp. *DSS*) for strictly singular (resp. disjointly strictly singular) operators. Let us recall that a Banach space X has *cotype* q , for some $2 \leq q < \infty$, if there exists a constant $C < \infty$ so that, for every finite set of vectors $(x_j)_{j=1}^n$ in X , we have

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q} \leq C \int_0^1 \left\| \sum_{j=1}^n r_j(t)x_j \right\| dt,$$

where r_j denotes the j^{th} Rademacher function.

A Banach lattice Y is *q-concave*, for some $1 \leq q < \infty$, if there exists a constant $C < \infty$, so that, for every choice of vectors $(y_j)_{j=1}^n$ in Y , we have

$$\left(\sum_{j=1}^n \|y_j\|^q \right)^{1/q} \leq C \left\| \left(\sum_{j=1}^n |y_j|^q \right)^{1/q} \right\|.$$

Every q -concave Banach lattice with $q \geq 2$ is of cotype q .

A Banach lattice Y satisfies a *lower q-estimate* for some $1 < q < \infty$ if there exists a constant $C < \infty$ such that for every choice of pairwise disjoint elements $(y_j)_{j=1}^n$ in Y , we have

$$\left(\sum_{j=1}^n \|y_j\|^q \right)^{1/q} \leq C \left\| \sum_{j=1}^n y_j \right\|.$$

Banach lattices with finite cotype are q -concave for some $q < \infty$ and have order continuous norm. In the sequel, any separable (or with a weak unit) order continuous Banach lattice will be represented as a Köthe function space, i.e. it is lattice isomorphic to a (in general not closed) ideal of $L_1(\mu)$, for some probability space (Ω, Σ, μ) (cf. [20, Theorem 1.b.14]). Let us recall the Kadec-Pelczyński disjointification method for order continuous Banach lattices [7, Theorem 4.1]:

Proposition 1.1. *Let X be an order continuous Banach lattice with a weak unit, and $j : X \hookrightarrow L_1(\mu)$ the formal inclusion. For any (closed) subspace $Y \subset X$ one of the following holds:*

- (1) *There exists a normalized almost disjoint sequence $(x_n) \subset Y$, (i.e. there exists a disjoint sequence $(z_n) \subset X$ such that $\|z_n - x_n\| \rightarrow 0$ when $n \rightarrow \infty$).*
- (2) *Y is isomorphic to a closed subspace of $L_1(\mu)$ (in fact, $j : X \hookrightarrow L_1$ is an isomorphism when restricted to Y), in this case we say that Y is a strongly embedded subspace.*

Note that more can be said if instead of a subspace we consider a normalized sequence $(x_n) \subset X$; now the alternatives are

- (1) either $(\|x_n\|_{L_1})$ is bounded away from zero,
- (2) or there exist a subsequence (x_{n_k}) and a disjoint sequence $(z_k) \subset X$ such that $\|z_k - x_{n_k}\| \rightarrow 0$ when $k \rightarrow \infty$.

Recall that a subset M of an order continuous Banach lattice X is *equi-integrable* if $\sup_{f \in M} \|f\chi_A\| \rightarrow 0$ when $\mu(A) \rightarrow 0$. This concept has an analogue for general Banach lattices: a bounded subset M of a Banach lattice X is *L-weakly compact* if for every $\varepsilon > 0$ there exists $x \in X_+^a$ such that $M \subset [-x, x] + \varepsilon B_X$ (where X^a denotes the maximal order ideal in X on which the induced norm is order continuous, and B_X the closed unit ball of X). Note that a bounded sequence (g_k) in an order continuous Köthe function space (over a probability space) is L-weakly compact if and only if it is equi-integrable.

We will make use of the following standard facts (cf. e. g. [10, Lemmas 3.2 and 3.3]):

Lemma 1.2. *Let $T : X \rightarrow Y$ be a regular operator from a Banach lattice X into a Banach lattice Y with order continuous norm. If $A \subset X$ is L-weakly compact, then $T(A)$ is L-weakly compact.*

Lemma 1.3. *Let X be an order continuous Banach lattice with a weak unit, hence representable as an order ideal in $L_1(\mu)$ for some probability space. A norm bounded sequence (g_n) in X is convergent to zero if and only if (g_n) is equi-integrable and $\|g_n\|_{L_1}$ -convergent to zero.*

A Banach lattice X with an order continuous norm satisfies the *subsequence splitting property* ([15, Chapter 6] and [26]) if for every bounded sequence (f_n) in X there exist a subsequence (f_{n_k}) and sequences $(g_k), (h_k)$ in X with $|g_k| \wedge |h_k| = 0$ and $f_{n_k} = g_k + h_k$ for all k , such that (g_k) is equi-integrable in X , and (h_k) is disjoint. It is known that every Banach lattice with finite cotype has the subsequence splitting property (see [26, Theorem 2.5]).

We will make use of the following fact.

Lemma 1.4. *Let (f_n) be a weakly null normalized sequence in $L_p(\mu)$, for some finite measure μ and $1 < p < \infty$. Suppose that (f_n) is uniformly bounded (i.e. there exists $M < \infty$ such that $|f_n| \leq M$). Then, there is a subsequence (f_{n_k}) equivalent to the unit vector basis of ℓ_2 .*

Proof. Since the sequence (f_n) is uniformly bounded, it is in particular equi-integrable. Thus, since it is a normalized sequence in $L_p(\mu)$, by Lemma 1.3 it follows that $\inf \|f_n\|_{L_1} > 0$. Hence, (f_n) is a seminormalized sequence in every $L_q(\mu)$, $1 \leq q \leq \infty$. Moreover, as the sequence (f_n) is weakly null in $L_p(\mu)$, it has to be weakly null

in every $L_q(\mu)$ with $1 \leq q < \infty$. Therefore, for every $1 < q < \infty$, passing to a subsequence, (f_n) will be equivalent to a block basis of the Haar system in $L_q(\mu)$ (cf. [19, Proposition 1.a.12]). Hence, since the Haar system forms an unconditional basis of $L_q(\mu)$ [20, Theorem 2.c.5], and the blocks of an unconditional basis are also unconditional, then, passing to a further subsequence, (f_{n_k}) can be assumed to be an unconditional basic sequence in $L_q(\mu)$.

Let $1 < q \leq p$ with $1 < q \leq 2$, and consider a subsequence (f_{n_k}) which is seminormalized and unconditional in both $L_q(\mu)$ and $L_p(\mu)$. Thus, for every finite sequence of scalars $(a_i)_{i=1}^n$, we have

$$\begin{aligned} K_1 \left\| \sum_{i=1}^n a_i f_{n_i} \right\|_{L_q} &\geq \int_0^1 \left\| \sum_{i=1}^n r_i(t) a_i f_{n_i} \right\|_{L_q} dt \\ &\geq C_1 \left(\sum_{i=1}^n \|a_i f_{n_i}\|_{L_q}^2 \right)^{\frac{1}{2}} \\ &\geq C_1 (\inf \|f_j\|_{L_q}) \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where C_1 is the cotype 2 constant of L_q and K_1 is the unconditional constant of (f_{n_i}) in $L_q(\mu)$.

On the other hand, we have

$$\begin{aligned} \left\| \sum_{i=1}^n a_i f_{n_i} \right\|_{L_p} &\leq K_2 \int_0^1 \left\| \sum_{i=1}^n r_i(t) a_i f_{n_i} \right\|_{L_p} dt \\ &\leq K_2 C_2 \left\| \left(\sum_{i=1}^n |a_i f_{n_i}|^2 \right)^{\frac{1}{2}} \right\|_{L_p} \\ &\leq 2K_2 C_2 M \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

where C_2 is the constant appearing in [20, Theorem 1.d.6], and K_2 is the unconditional constant of (f_{n_i}) in $L_p(\mu)$. This finishes the proof. \square

2. PROOFS OF THEOREMS A AND B

In the proofs of Theorems A and B we will make use of the following:

Proposition 2.1. *If E is a Banach lattice with a lower 2-estimate, then every strictly singular operator T from ℓ_2 to E is compact.*

Proof. Since E satisfies a lower 2 estimate, in particular it is order continuous and, by [20, Proposition 1.a.9], we can consider an ideal with a weak unit containing $T(\ell_2)$. By [20, Theorem 1.b.14], this ideal can be represented as an order dense ideal in $L_1(\mu)$, for some probability measure μ . Thus, let us consider the operator T as an operator into $L_1(\mu)$.

Let us see that $T : \ell_2 \rightarrow E$ is compact. It clearly suffices to prove that $\|Te_n\|_E \rightarrow 0$, where (e_n) is any weakly null normalized sequence in ℓ_2 . Suppose not, by Proposition 1.1, either $(\|Te_n\|_{L_1})$ is bounded away from zero or (Te_n) has a subsequence equivalent to a disjoint sequence.

Assume first that the sequence $(\|Te_n\|_{L_1})$ is bounded away from zero, then by [3, §6.Theorem], there exists a constant $\delta > 0$, and a subsequence (Te_{n_k}) such that for scalars $(a_k)_{k=1}^m$ we have

$$C \|T\| \left(\sum_{k=1}^m |a_k|^2 \right)^{\frac{1}{2}} \geq \left\| \sum_{k=1}^m a_k Te_{n_k} \right\|_E \geq \left\| \sum_{k=1}^m a_k Te_{n_k} \right\|_{L_1} \geq \delta \left(\sum_{k=1}^m |a_k|^2 \right)^{\frac{1}{2}},$$

where $C > 0$ is the equivalence constant between the sequence (e_n) and the unit vector basis of ℓ_2 .

On the other hand, if (Te_{n_j}) were equivalent to a disjoint sequence, then we would have

$$C \|T\| \left(\sum_{j=1}^k |a_j|^2 \right)^{\frac{1}{2}} \geq \left\| \sum_{j=1}^k a_j T(e_{n_j}) \right\|_E \geq M^{-1} (\inf \|Te_n\|_E) \left(\sum_{j=1}^k |a_j|^2 \right)^{\frac{1}{2}},$$

where M is the constant appearing in the lower 2-estimate of E . Thus, in both cases we see that T is not strictly singular, and we reach a contradiction. \square

The following fact is well known.

Lemma 2.2. *Let X be a Banach lattice, Y a Banach space, and $T : X \rightarrow Y$ an AM-compact operator. For every L -weakly compact set A in X , it holds that $T(A)$ is relatively compact.*

We will make use of the following fact that relates ℓ_1 -singular operators with operators not preserving a disjoint copy of ℓ_1 (compare with [5] and [23]).

Lemma 2.3. *Let X be a Banach lattice with the subsequence splitting property, Y a Banach space, and $T : X \rightarrow Y$ an operator. If T is an isomorphism on a subspace of X isomorphic to ℓ_1 , then there exists a disjoint sequence (h_j) in X equivalent to the unit vector basis of ℓ_1 , such that T restricted to the span $[h_n]$ is an isomorphism.*

Proof. Let (x_n) be a normalized sequence in X which is equivalent to the unit vector basis of ℓ_1 , and such that T restricted to $[x_n]$ is an isomorphism. By the subsequence

splitting property, there exist a subsequence (x_{n_k}) , and sequences (g_k) , (h_k) , with (g_k) equi-integrable, (h_k) disjoint, $|g_k| \wedge |h_k| = 0$ and $x_{n_k} = g_k + h_k$ for every $k \in \mathbb{N}$.

Now, by Rosenthal's Lemma, the sequence (Th_k) either has a weakly Cauchy subsequence, or a subsequence equivalent to the unit vector basis of ℓ_1 . Suppose that (Th_k) is weakly Cauchy. Then, since (g_k) is equi-integrable, it has a weakly convergent subsequence, say (g_{k_j}) . This would imply that $Tx_{n_{k_j}} = Tg_{k_j} + Th_{k_j}$ is a weakly Cauchy sequence, in contradiction with the fact that T is an isomorphism on the span $[x_n]$.

Therefore, passing to a subsequence we can assume that (Th_k) is equivalent to the unit vector basis of ℓ_1 . Thus, for scalars $(a_k)_{k=1}^n$ and some constant $C > 0$ we have

$$C \sum_{k=1}^n |a_k| \leq \left\| \sum_{k=1}^n a_k Th_k \right\| \leq \|T\| \left\| \sum_{k=1}^n a_k h_k \right\| \leq \|T\| \sum_{k=1}^n |a_k|.$$

This finishes the proof. \square

Theorem 2.4. *Let X be a Banach lattice with finite cotype and Y a Banach space. If an operator $T : X \rightarrow Y$ is disjointly strictly singular and AM-compact, then it is strictly singular.*

Proof. Suppose $T : X \rightarrow Y$ is DSS, AM-compact and not SS. Then, using ([20, Theorem 1.c.9]), we have that T is an isomorphism when restricted to the span of some unconditional basic sequence (u_n) in X . Now, by Rosenthal's Lemma, we can assume that some subsequence (u_{n_k}) is weakly Cauchy, otherwise T would be an isomorphism on a subspace isomorphic to ℓ_1 and, by Lemma 2.3, T would preserve a disjoint copy of ℓ_1 , in contradiction with the fact that T is DSS. Hence, taking differences, the sequence $(u_{n_k} - u_{n_{k+1}})$ is weakly null and seminormalized. Thus, there is a subsequence equivalent to a block basis of (u_n) , which is an unconditional basic sequence (see [19, Proposition 1.a.12 and page 19]). Let us denote by (f_n) this weakly null, unconditional sequence in $[u_n]$.

Let $\alpha > 0$ be such that for every sequence of scalars (a_n) we have

$$\left\| T \left(\sum_{n=1}^{\infty} a_n f_n \right) \right\| \geq \alpha \left\| \sum_{n=1}^{\infty} a_n f_n \right\|.$$

Since X has the subsequence splitting property, we can extract a subsequence (still denoted (f_n)) and sequences (g_n) and (h_n) such that $|g_n|, |h_n| \leq |f_n|$, $f_n = g_n + h_n$, and $|g_n| \wedge |h_n| = 0$, where (g_n) is equi-integrable in X and (h_n) is disjoint.

Suppose first the case $\|h_n\| \rightarrow 0$; then, passing to a subsequence if needed, the sequence (f_n) would inherit the equi-integrability from the sequence (g_n) . Therefore, since the operator T is AM-compact, by Lemma 2.2, the sequence (Tf_n) would have a convergent subsequence, but since T is invertible on $[f_n]$, this would imply that

(f_n) also has a convergent subsequence. This is a contradiction with the fact that (f_n) is weakly null and normalized.

Alternatively, let us suppose $\|h_n\| \geq \rho > 0$. We consider the operator $V : [f_n] \rightarrow X$ defined by

$$V\left(\sum_{n=1}^{\infty} a_n f_n\right) = \sum_{n=1}^{\infty} a_n h_n,$$

which is bounded. Indeed, since $|h_n| \leq |f_n|$, and X has finite cotype, for some constant $C > 0$ we have (cf. [20, Theorem 1.d.6])

$$\begin{aligned} \left\| V\left(\sum_{i=1}^n a_i f_i\right) \right\| &= \left\| \sum_{i=1}^n a_i h_i \right\| \\ &= \left\| \left(\sum_{i=1}^n |a_i|^2 |h_i|^2 \right)^{\frac{1}{2}} \right\| \\ &\leq \left\| \left(\sum_{i=1}^n |a_i f_i|^2 \right)^{\frac{1}{2}} \right\| \\ &\leq C \left\| \int_0^1 \left| \sum_{i=1}^n a_i f_i r_i(t) \right| dt \right\| \\ &\leq C \int_0^1 \left\| \sum_{i=1}^n a_i f_i r_i(t) \right\| dt. \end{aligned}$$

But now, if K denotes the unconditional constant of (f_n) , we have

$$\int_0^1 \left\| \sum_{i=1}^n a_i f_i r_i(t) \right\| dt = 2^{-n} \sum_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^n \varepsilon_i a_i f_i \right\| \leq K \left\| \sum_{i=1}^n a_i f_i \right\|.$$

Hence, the operator V is bounded with $\|V\| \leq KC$.

Therefore, the restriction operator $T|_{[f_n]} : [f_n] \rightarrow Y$ can be decomposed as

$$T|_{[f_n]} = TV + T(I_{[f_n]} - V),$$

where $I_{[f_n]} : [f_n] \hookrightarrow X$ is the identity inclusion. It holds that the operator $TV : [f_n] \rightarrow Y$ is strictly singular. Indeed, if it were not the case, then TV would be invertible on some subspace $M \subset [f_n]$. Then, T would also be invertible in the subspace $V(M) \subset [h_n]$. Now, arguing as in the first paragraph of the proof we can find a normalized weakly null sequence in $V(M)$, which will be equivalent to a block basis of $[h_n]$, which is disjoint. More precisely, by [19, Proposition 1.a.12], there exists a normalized sequence (v_n) in $V(M)$ with $\|v_n - \sum_{i=m_n+1}^{m_{n+1}} c_i h_i\|_X \rightarrow 0$

for some increasing sequence (m_n) and scalars (c_i) . If we denote $v'_n = \sum_{i=m_n+1}^{m_{n+1}} c_i h_i$, then this sequence is clearly disjoint and also satisfies $\|Tv_n - Tv'_n\|_Y \rightarrow 0$, so by the perturbation argument, passing to a further subsequence, (Tv'_n) is equivalent to (Tv_n) , and by the invertibility of T on $V(M)$, this sequence is equivalent to (v_n) , and hence to (v'_n) . This is a contradiction with the fact that T is DSS.

Hence, the operator TV is strictly singular, and, since $T|_{[f_n]}$ is an isomorphism we have, by [19, Proposition 2.c.10], that the operator $T(I_{[f_n]} - V)$ has finite dimensional kernel and closed range.

Now, by Lemma 2.2, the sequence $T(I_{[f_n]} - V)(f_n) = T(g_n)$ has a convergent subsequence. Consider the decomposition

$$[f_n] = \text{Ker}(TI_{[f_n]} - V) \oplus Z.$$

Since $T(I_{[f_n]} - V)$ has closed range, the operator $T(I_{[f_n]} - V)$ is invertible on Z . Hence the sequence (f_n) would also have a convergent subsequence. Again, this is a contradiction with the fact that (f_n) is weakly null and normalized. \square

Recall that an operator $T : X \rightarrow Y$ is ℓ_2 -singular if it is not an isomorphism when restricted to any subspace isomorphic to ℓ_2 .

Proposition 2.5. *Let X and Y be Banach lattices such that X has finite cotype. If $T : X \rightarrow Y$ is an ℓ_2 -singular operator, then T is also AM-compact under any of the following conditions:*

- (1) Y satisfies a lower 2-estimate.
- (2) Y is order continuous and T is regular.

Proof. Let $x \in X_+$ be fixed and denote by E_x the closed ideal of X generated by x . Since X is q -concave for some $1 < q < \infty$, we have $L_q(\mu) \hookrightarrow E_x \hookrightarrow L_1(\mu)$ for a certain probability measure μ (see [15, p. 14]). First let us prove that (in both cases) for every positive constant M the set $T[-M, M]$ is relatively compact. To this end, let (f_n) be such that $|f_n| \leq M$. Since the order intervals in X are weakly compact, without loss of generality we can assume that $f_n \rightarrow 0$ weakly.

Let us consider some $p > \max\{q, 2\}$, clearly the sequence (f_n) must be also weakly null and seminormalized in $L_p(\mu)$, otherwise, the sequence (Tf_n) would have a convergent subsequence and that would finish the proof. Moreover, passing to a further subsequence (still denoted (f_n)) and using Lemma 1.4, the sequence (f_n) spans in $L_p(\mu)$ a subspace isomorphic to ℓ_2 . Therefore, since a normalized disjoint sequence in $L_p(\mu)$ spans a subspace isomorphic to ℓ_p , by Proposition 1.1 $[f_n]$ has to be strongly embedded in $L_p(\mu)$, which means that on $[f_n]$ the topology of $L_p(\mu)$ and $L_1(\mu)$ coincide.

Thus, for certain constants $\alpha, \beta > 0$, and any scalars $(a_i)_{i=1}^n$ we have

$$\alpha \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{i=1}^n a_i f_i \right\|_{L_1} \leq \left\| \sum_{i=1}^n a_i f_i \right\|_X \leq \left\| \sum_{i=1}^n a_i f_i \right\|_{L_p} \leq \beta \left(\sum_{i=1}^n |a_i|^2 \right)^{\frac{1}{2}}$$

Thus, (f_n) has a subsequence whose span in X is isomorphic to ℓ_2 .

On the one hand, suppose that Y satisfies a lower 2-estimate. Then $(T(f_n))$ must have a subsequence that tends to zero in norm. Otherwise, the operator T restricted to the span $[f_{n_k}]$ which is isomorphic to ℓ_2 , would not be a compact operator, and by Proposition 2.1, T would not be ℓ_2 -singular. This is a contradiction.

Consider now the case where Y is order continuous and the operator T is regular. Then, the sequence $(T(f_n))$ is equi-integrable by Lemma 1.2. Hence, if $(T(f_n))$ is not convergent to zero in Y , then by Lemma 1.3, $(\|T(f_n)\|_{L_1})$ is bounded away from zero. This implies that the operator $R : \ell_2 \rightarrow L_1(\nu)$, defined by the following diagram

$$\begin{array}{ccc} \ell_2 & \xrightarrow{R} & L_1(\nu) \\ \downarrow i & & \uparrow j \\ [f_{n_k}] & \xrightarrow{T} & Y \end{array}$$

is not compact. Here i is just an isomorphism, and j the formal inclusion of Y in some $L_1(\nu)$ (or of an ideal with weak unit containing $T(f_n)$) ([20, Theorem 1.b.14]). By Proposition 2.1 we see that T is not ℓ_2 -singular. This is a contradiction.

So far, we have shown that, in both cases, $T[-M, M]$ is a relatively compact set for every positive constant M . Now, we use the density of $L_\infty(\mu)$ in X , which follows from the representability of E_x as a function space between $L_q(\mu)$ and $L_1(\mu)$. Hence, given $\varepsilon > 0$ we can consider $M_\varepsilon < \infty$ such that $x \in [0, M_\varepsilon] + \varepsilon \|T\|^{-1} B_X$. Therefore, $T[-x, x] \subset T[-M_\varepsilon, M_\varepsilon] + \varepsilon B_Y$ and since $T[-M_\varepsilon, M_\varepsilon]$ is relatively compact, so is $T[-x, x]$. This finishes the proof. \square

Proposition 2.5 has a partial converse:

Proposition 2.6. *Let X be a Banach lattice and Y a Banach space. Suppose X has finite cotype and does not contain any sequence of disjoint elements which span a subspace isomorphic to ℓ_2 . If an operator $T : X \rightarrow Y$ is AM-compact, then T is also ℓ_2 -singular.*

Proof. Suppose $T : X \rightarrow Y$ is AM-compact, but not ℓ_2 -singular. Therefore, there exists a sequence (f_n) in X which spans a subspace isomorphic to ℓ_2 and such that T is an isomorphism when restricted to $[f_n]$.

Since X has the subsequence splitting property, passing to a subsequence we have $f_n = h_n + g_n$ with $|h_n| \wedge |g_n| = 0$, (g_n) equi-integrable, (h_n) disjoint, and $|h_n|, |g_n| \leq |f_n|$ for all n . Again, arguing as in the proof of Theorem 2.4, the operator $V : [f_n] \rightarrow$

X defined by $V(f_n) = h_n$ is bounded. Hence, since $[f_n]$ is isomorphic to ℓ_2 , all three sequences (f_n) , (g_n) , and (h_n) must be weakly null.

Thus, by Lemma 2.2, the sequence $(T(g_n))$ must have a subsequence that goes to zero in norm. Therefore, $(T(h_n))$ has a subsequence, say $(T(h_{n_k}))$, equivalent to $(T(f_{n_k}))$, which is equivalent to the unit vector basis of ℓ_2 . Then, there exist constants α and β , such that for any n and $(a_k)_{k=1}^n$ we have

$$\begin{aligned} \alpha \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} &\leq \left\| T \left(\sum_{k=1}^n a_k h_{n_k} \right) \right\| \leq \|T\| \left\| \sum_{k=1}^n a_k h_{n_k} \right\| \\ &\leq \|T\| \|V\| \left\| \sum_{k=1}^n a_k f_{n_k} \right\| \leq \beta \|T\| \|V\| \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

This means that $[h_{n_k}]$ is isomorphic to ℓ_2 , and this is impossible according to the hypothesis on X . \square

of Theorems A and B. Clearly, strictly singular operators are disjointly strictly singular and ℓ_2 -singular. Conversely, if the operator $T : X \rightarrow Y$ is ℓ_2 -singular, then, by Proposition 2.5, T must also be AM-compact. The conclusion now follows from Theorem 2.4. \square

Remark 2.7. Note that Theorem B is still true if X is just an order continuous Banach lattice with the subsequence splitting property.

As a consequence we obtain a domination result for positive strictly singular operators, which improves [10, Theorem 3.1].

Corollary 2.8. *Let X be a Banach lattice with finite cotype and Y an order continuous Banach lattice. Suppose that $R \leq T : X \rightarrow Y$ are positive operators; if T is strictly singular, then R is also strictly singular.*

Proof. Since the operator T is strictly singular, it is obviously disjointly strictly singular and ℓ_2 -singular. Now, since Y is order continuous, [9, Theorem 1.1] yields that R is disjointly strictly singular. Moreover, by Proposition 2.5 the operator T is AM-compact, and by [21, Proposition 3.7.2], it follows that R is also AM-compact. Hence, by Theorem 2.4, we conclude that the operator R is strictly singular. \square

See [17] for related results concerning the domination of complementably ℓ_p -singular operators.

3. PROOF OF THEOREM C

In this section we will construct operators in order to show that the conditions on Theorems A and B cannot be relaxed.

First, we need some preliminary Lemmas. The first of them will be deduced from the following result for independent identically distributed p -stable random variables given in [15, Lemma 3.10].

Proposition 3.1. *Let X be a finite dimensional Banach space with a 1-unconditional basis $(x_i)_{i=1}^n$ and let Y be a q -concave Banach lattice, for some $1 < q < \infty$. Let T be an isomorphism from X into Y and let $(g_i)_{i=1}^n$ be a sequence of independent and identically distributed (i.i.d.) p -stable random variables over a probability space (Ω, Σ, μ) , for some $1 < p < 2$. Then, for every scalar sequence $(a_i)_{i=1}^n$ for which*

$$\int_{\Omega} \left\| \sum_{i=1}^n a_i g_i(\omega) x_i \right\|_X d\mu(\omega) \leq \left\| \sum_{i=1}^n a_i x_i \right\|_X,$$

the following inequalities hold

$$K \left\| \sum_{i=1}^n a_i x_i \right\|_X \leq \left\| \max_{1 \leq i \leq n} |a_i T x_i| \right\|_Y \leq \left\| \left(\sum_{i=1}^n |a_i T x_i|^p \right)^{1/p} \right\|_Y \leq \|T\| \|g_1\|_{L_1}^{-1} \left\| \sum_{i=1}^n a_i x_i \right\|_X,$$

for certain constant K (depending only on the q -concavity constant of Y).

In what follows, the expression $\| \sum_{n=1}^k a_n f_n \| \sim \| \sum_{n=1}^k a_n g_n \|$ will mean as usual that there exist constants $c, C > 0$ such that for any $k \in \mathbb{N}$ and any $(a_n)_{n=1}^k$ we have

$$c \left\| \sum_{n=1}^k a_n f_n \right\| \leq \left\| \sum_{n=1}^k a_n g_n \right\| \leq C \left\| \sum_{n=1}^k a_n f_n \right\|.$$

Lemma 3.2. *Let $1 < q < s < 2$, and let $T : \ell_s \rightarrow L_q(\mu)$ be an isomorphic embedding. If $f_n = T(e_n)$ denotes the image under T of (e_n) , the canonical basis of ℓ_s , then for every $2 \leq r < \infty$ we have*

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \left\| \left(\sum_{n=1}^k |a_n f_n|^r \right)^{1/r} \right\|_{L_q} \sim \left\| \max_{1 \leq n \leq k} |a_n f_n| \right\|_{L_q},$$

for any scalar sequence $(a_n)_{n=1}^k$.

Proof. Take $n \in \mathbb{N}$ and let $X = \ell_s^n$, $Y = L_q(\mu)$, and $T = T|_X$ in Proposition 3.1. If $(g_i)_{i=1}^n$ is a sequence of i.i.d. p -stable random variables with $s < p < 2$, then

$$\begin{aligned} \int_{\Omega} \left\| \sum_{i=1}^n a_i g_i(\omega) e_i \right\|_{\ell_s} d\mu(\omega) &= \int_{\Omega} \left(\sum_{i=1}^n |a_i g_i(\omega)|^s \right)^{1/s} d\mu(\omega) \\ &\leq \left(\int_{\Omega} \sum_{i=1}^n |a_i g_i(\omega)|^s d\mu(\omega) \right)^{1/s} \\ &= \left(\sum_{i=1}^n |a_i|^s \int_{\Omega} |g_i(\omega)|^s d\mu(\omega) \right)^{1/s} \\ &= \|g_1\|_{L_s} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_s}. \end{aligned}$$

Since $\|g_1\|_{L_s} < \infty$, using Proposition 3.1 we get

$$\begin{aligned} K \|g_1\|_{L_s}^{1/2} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_s} &\leq \left\| \max_{1 \leq i \leq n} |a_i f_i| \right\|_{L_q} \\ &\leq \left\| \left(\sum_{i=1}^n |a_i f_i|^p \right)^{1/p} \right\|_{L_q} \\ &\leq \|T\| \|g_1\|_{L_1}^{-1} \|g_1\|_{L_s}^{1/2} \left\| \sum_{i=1}^n a_i e_i \right\|_{\ell_s} \end{aligned}$$

for certain constant K independent of n .

Since

$$\left\| \max_{1 \leq i \leq n} |a_i f_i| \right\|_{L_q} \leq \left\| \left(\sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\|_{L_q} \leq \left\| \left(\sum_{i=1}^n |a_i f_i|^p \right)^{1/p} \right\|_{L_q},$$

by the previous inequality, we immediately get

$$\left\| \max_{1 \leq i \leq n} |a_i f_i| \right\|_{L_q} \sim \left\| \left(\sum_{i=1}^n |a_i f_i|^2 \right)^{1/2} \right\|_{L_q} \sim \left\| \left(\sum_{i=1}^n |a_i f_i|^p \right)^{1/p} \right\|_{L_q}.$$

On the other hand, since (f_n) is an unconditional basic sequence we have

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \int_0^1 \left\| \sum_{n=1}^k r_n(t) a_n f_n \right\|_{L_q} dt.$$

Thus, [20, Theorem 1.d.6(i)] yields

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \left\| \left(\sum_{n=1}^k |a_n f_n|^2 \right)^{1/2} \right\|_{L_q}.$$

Therefore, for every $2 \leq r \leq \infty$, we have

$$\left\| \sum_{n=1}^k a_n f_n \right\|_{L_q} \sim \left\| \left(\sum_{n=1}^k |a_n f_n|^r \right)^{1/r} \right\|_{L_q}.$$

□

The next result shows why Lemma 3.2 cannot be extended to the case $s = 2$. Given functions $f, g : \mathbb{N} \rightarrow \mathbb{R}$, by $f = o(g)$ we mean, as usual, that $\frac{f(m)}{g(m)} \rightarrow 0$ when $m \rightarrow \infty$.

Lemma 3.3. *Let $1 < p < 2$. If $T : \ell_2 \rightarrow L_p(\mu)$ is a bounded operator, and $f_n = T(e_n)$ where (e_n) denotes the canonical basis of ℓ_2 , then for each natural number m , we have*

$$\inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} = o(\sqrt{m}).$$

Moreover, for any $2 < q < \infty$,

$$\inf_{|A|=m} \left\| \left(\sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} = o(\sqrt{m}).$$

Proof. By Krivine's theorem (cf. [20, Theorem 1.f.14]), given any finite family $(x_i)_{i=1}^n$ of vectors in ℓ_2 we have

$$\left\| \left(\sum_{i=1}^n |Tx_i|^2 \right)^{1/2} \right\|_{L_p} \leq K_G \|T\| \left\| \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \right\|_{\ell_2} = K_G \|T\| \left(\sum_{i=1}^n \|x_i\|_{\ell_2}^2 \right)^{1/2}$$

where K_G is Grothendieck's constant. Now, by Maurey's factorization theorem (cf. [1, Theorem 7.1.2]), there exists a density function h on Ω (i.e. $h > 0$ and $\int h d\mu = 1$) such that

$$\begin{array}{ccc} \ell_2 & \xrightarrow{T} & L_p(\mu) \\ \tilde{T} \downarrow & & \uparrow J \\ L_2(hd\mu) & \xrightarrow{i} & L_p(hd\mu) \end{array}$$

where $\tilde{T}(x) = h^{-1/p} T(x)$ for every $x \in \ell_2$, i denotes the canonical inclusion, and J is the isometry mapping each $f \in L_p(hd\mu)$ to $J(f) = fh^{1/p}$.

Let us denote $\widetilde{f}_n = \widetilde{T}(e_n) \in L_2(hd\mu)$. Since $L_2(hd\mu)$ satisfies the subsequence splitting property, there exist a subsequence (\widetilde{f}_{n_k}) and sequences (g_k) , (h_k) , with (h_k) disjoint and (g_k) equi-integrable in $L_2(hd\mu)$ such that $\widetilde{f}_{n_k} = g_k + h_k$ and $|g_k| \wedge |h_k| = 0$. Therefore, the sequence $(J(h_k))$ is disjoint in $L_p(\mu)$, and for scalars $(a_k)_{k=1}^n$ we have

$$\begin{aligned} \left\| \sum_{k=1}^n a_k J(h_k) \right\|_{L_p} &\leq \left\| \sum_{k=1}^n a_k h_k \right\|_{L_2(h)} \\ &= \left\| \left(\sum_{k=1}^n |a_k h_k|^2 \right)^{1/2} \right\|_{L_2(h)} \\ &\leq \left\| \left(\sum_{k=1}^n |a_k \widetilde{f}_{n_k}|^2 \right)^{1/2} \right\|_{L_2(h)} \\ &\leq K_G \|\widetilde{T}\| \left\| \left(\sum_{k=1}^n |a_k e_{n_k}|^2 \right)^{1/2} \right\|_{\ell_2} \\ &= C \left(\sum_{k=1}^n |a_k|^2 \right)^{1/2}, \end{aligned}$$

with a constant C independent of n (cf. [20, Theorem 1.f.14]). Now, if $\inf_k \|J(h_k)\|_{L_p} > 0$, then for some constant c and for all $(a_k)_{k=1}^n$ we have

$$c \left(\sum_{k=1}^n |a_k|^p \right)^{1/p} \leq \left\| \sum_{k=1}^n a_k J(h_k) \right\|_{L_p} \leq C \left(\sum_{k=1}^n |a_k|^2 \right)^{1/2},$$

which is impossible (since $1 < p < 2$).

Thus, passing to a subsequence we can assume that $\|J(h_k)\|_{L_p} \rightarrow 0$ as $k \rightarrow \infty$. Now, for each $m \in \mathbb{N}$, let us denote

$$\phi(m) = \left\| \max_{1 \leq k \leq m} |g_k| \right\|_{L_2(h)},$$

and let us take disjoint measurable sets $A_1^m, A_2^m, \dots, A_m^m$ in Ω such that

$$\max_{k \leq m} |g_k| = \sum_{k=1}^m |g_k| \chi_{A_k^m}.$$

Claim: We have $\frac{\phi(m)}{\sqrt{m}} \rightarrow 0$ when $m \rightarrow \infty$.

Assume the contrary; then, there exist $\varepsilon > 0$ and an increasing sequence $m_n \rightarrow \infty$ such that

$$\frac{\phi(m_n)}{\sqrt{m_n}} \geq \varepsilon.$$

That is, for all $n \in \mathbb{N}$ we can choose an integer m_n , and disjoint sets $A_1^{m_n}, A_2^{m_n}, \dots, A_{m_n}^{m_n}$ such that

$$(1) \quad \frac{1}{m_n} \sum_{k=1}^{m_n} \int_{A_k^{m_n}} |g_k|^2 h d\mu = \frac{1}{m_n} \left\| \sum_{k=1}^{m_n} |g_k| \chi_{A_k^{m_n}} \right\|_{L_2(h)}^2 = \left(\frac{\phi(m_n)}{\sqrt{m_n}} \right)^2 \geq \varepsilon^2$$

for every natural n . From this fact we conclude that for every $N \in \mathbb{N}$ we can find B_1, \dots, B_N disjoint sets in Ω such that

$$\sup_k \left\{ \int_{B_n} |g_k|^2 h d\mu \right\} \geq \frac{\varepsilon^2}{2},$$

for all $n = 1, \dots, N$.

Otherwise, suppose that there exists N such that for n large enough the set

$$S_n = \{k \leq m_n : \int_{A_k^{m_n}} |g_k|^2 h d\mu \geq \varepsilon^2/2\}$$

always has cardinality less than N . Then for n large enough so that $\frac{m_n - N}{m_n} < 1$ and $\frac{N \sup_k \|g_k\|_{L_2}}{m_n} < \frac{\varepsilon^2}{2}$, we have

$$\frac{1}{m_n} \sum_{k=1}^{m_n} \int_{A_k^{m_n}} |g_k|^2 h d\mu < \frac{1}{m_n} [(m_n - N) \frac{\varepsilon^2}{2} + N \sup_k \|g_k\|_{L_2}] < \varepsilon^2$$

which is a contradiction with (1).

Hence, by [27, III.C.12 Theorem], we reach a contradiction with the fact that (g_k) is equi-integrable in $L_2(hd\mu)$. Therefore, $\frac{\phi(m)}{\sqrt{m}} \rightarrow 0$ when $m \rightarrow \infty$, and the claim is proved.

Now, since $\|J(h_k)\|_{L_p} \rightarrow_{k \rightarrow \infty} 0$, for every $m \in \mathbb{N}$ we have that, for every $\varepsilon > 0$ there exists a set $A_\varepsilon = \{k_1, k_2, \dots, k_m\}$ of natural numbers, such that $\|J(h_k)\|_{L_p} < \varepsilon/m$ for all $k \in A_\varepsilon$. Therefore,

$$\begin{aligned} \inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} &\leq \inf_{|A|=m} \left\| \max_{j \in A} J(|g_j| + |h_j|) \right\|_{L_p} \\ &\leq \phi(m) + \left\| \max_{j \in A_\varepsilon} J(|h_j|) \right\|_{L_p} \\ &\leq \phi(m) + \varepsilon, \end{aligned}$$

and since this inequality holds for all $\varepsilon > 0$ we get that $\inf_{|A|=m} \|\max_{j \in A} |f_j|\|_{L_p} \leq \phi(m)$, which implies

$$\inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p} = o(\sqrt{m}).$$

Now, the second assertion of the Lemma is obtained by a Hölder type inequality ([20, Proposition 1.d.2]). Indeed, given $m \in \mathbb{N}$, for any $A \subset \mathbb{N}$ with $|A| = m$,

$$\begin{aligned} \left\| \left(\sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} &\leq \left\| \left(\left(\sum_{j \in A} |f_j|^2 \right)^{1/2} \right)^\theta \left(\max_{j \in A} |f_j| \right)^{1-\theta} \right\|_{L_p} \\ &\leq \left\| \left(\sum_{j \in A} |f_j|^2 \right)^{1/2} \right\|_{L_p}^\theta \left\| \max_{j \in A} |f_j| \right\|_{L_p}^{1-\theta}, \end{aligned}$$

for $\theta = \frac{2}{q} \in (0, 1)$. Now, by the first part of the Lemma, the function

$$\varphi(m) = \inf_{|A|=m} \left\| \max_{j \in A} |f_j| \right\|_{L_p},$$

satisfies $\frac{\varphi(m)}{\sqrt{m}} \rightarrow 0$ as $m \rightarrow \infty$. Moreover, by passing to a subsequence, (f_j) can be assumed to be unconditional, so we have

$$\left\| \left(\sum_{j \in A} |f_j|^2 \right)^{1/2} \right\|_{L_p} \sim \left\| \sum_{j \in A} f_j \right\|_{L_p} \leq \|T\| \sqrt{m}.$$

Thus, for some constant $C < \infty$, and for any $q > 2$

$$\inf_{|A|=m} \left\| \left(\sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} \leq (C \|T\| \sqrt{m})^\theta (\varphi(m))^{1-\theta},$$

and clearly

$$\frac{(C \|T\| \sqrt{m})^\theta \varphi(m)^{1-\theta}}{\sqrt{m}} \rightarrow 0,$$

as $m \rightarrow \infty$. □

Lemma 3.4. *Let $1 < p < 2$. If a sequence $(f_n) \subset L_p$ satisfies*

$$\left\| \left(\sum_{n=1}^{\infty} |a_n|^2 |f_n|^2 \right)^{1/2} \right\|_{L_p} \leq C \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2},$$

for some constant $C > 0$ and every finitely non zero scalar sequence (a_n) , then for $2 < q < \infty$,

$$\inf_{|A|=m} \left\| \left(\sum_{j \in A} |f_j|^q \right)^{1/q} \right\|_{L_p} = o(\sqrt{m}).$$

Proof. Let us consider the operator $T : \ell_2 \rightarrow L_p([0, 1] \times [0, 1])$ defined by $T(e_n) = f_n \otimes r_n$, where (e_n) denotes the canonical basis of ℓ_2 , (r_n) are the Rademacher functions on $[0, 1]$, and $f_n \otimes r_n(s, t) = f_n(s)r_n(t)$. By Khintchine's inequality, for scalars (a_n) we have

$$\begin{aligned} \left\| \sum_{n=1}^k a_n f_n \otimes r_n \right\|_{L_p} &= \left(\int_0^1 \int_0^1 \left| \sum_{n=1}^k a_n f_n(s) r_n(t) \right|^p dt \right)^{\frac{1}{p}} \\ &\leq C_p \left\| \left(\sum_{n=1}^{\infty} |a_n|^2 |f_n|^2 \right)^{1/2} \right\|_{L_p}, \end{aligned}$$

for certain constant C_p . Therefore, by the hypothesis of the Lemma, this operator is bounded, and hence by Lemma 3.3, we have

$$\inf_{|A|=m} \left\| \left(\sum_{j \in A} |f_j \otimes r_j|^q \right)^{1/q} \right\|_{L_p} = o(\sqrt{m}).$$

But, since $|f_j| = |f_j \otimes r_j|$ we are done. \square

We are now in position to state and prove the main result of this Section. We shall consider the Banach lattice $L_r(\ell_q)$, for $1 \leq r, q < \infty$, defined as the set of all sequences $x = (x_1, x_2, \dots)$ of elements of L_r such that

$$\|x\|_{L_r(\ell_q)} = \sup_n \left\| \left(\sum_{i=1}^n |x_i|^q \right)^{1/q} \right\|_{L_r} < \infty,$$

(cf. [20, pp. 46-47]).

Theorem C. *Let us consider the Banach lattices L_p and $L_r(\ell_q)$, where $1 < r < p < 2 < q < \infty$. For each $p < s < 2$, there exists an operator $T : L_p \rightarrow L_r(\ell_q)$ such that it is ℓ_p -singular and ℓ_2 -singular, but not ℓ_s -singular. In particular, the operator T is disjointly strictly singular and ℓ_2 -singular, but not strictly singular.*

Proof. Since $1 < r < p < 2$, we can consider the formal inclusion $J : L_p[0, 1] \hookrightarrow L_r[0, 1]$. Let us denote by H_r the atomic Banach lattice whose lattice structure comes from the unconditional Haar basis (h_n) in $L_r[0, 1]$, and which is isomorphic to $L_r[0, 1]$. Let $L : L_r[0, 1] \rightarrow H_r$ be this isomorphism.

Now, given $p < s < 2$, consider a sequence (f_n) of i.i.d. s -stable random variables in $L_p[0, 1]$, hence the span $[f_n]$ is isometrically isomorphic to ℓ_s both in $L_p[0, 1]$ and

$L_r[0, 1]$. Now, since $f_n \rightarrow 0$ weakly in $L_p[0, 1]$ there exists a block basis (w_n) of (h_n) , of the form

$$w_n = \sum_{i=q_{n-1}+1}^{q_n} a_i h_i,$$

with (q_n) an increasing sequence of natural numbers, such that (w_n) is equivalent to a subsequence of (f_n) (still denoted (f_n)), in fact we have that $\|f_n - w_n\|_{L_p} \rightarrow 0$, and since $r < p$ we also have $\|f_n - w_n\|_{L_r} \rightarrow 0$. Hence, passing to a further subsequence, we have that (f_n) and (w_n) are equivalent both in $L_p[0, 1]$ and $L_r[0, 1]$.

Let us consider now the operator R defined by

$$\begin{aligned} R : \quad H_r &\longrightarrow L_r(\ell_q) \\ (c_i)_{i=1}^\infty &\longmapsto \left(\sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right)_{n=1}^\infty \end{aligned}$$

which is clearly bounded. Indeed, since $q > 2$ and the Haar basis (h_n) is unconditional in $L_r[0, 1]$, we have

$$\begin{aligned} \left\| R(c_i) \right\|_{L_r(\ell_q)} &= \sup_k \left\| \left(\sum_{n=1}^k \left| \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right|^q \right)^{1/q} \right\|_{L_r} \\ &\leq \sup_k \left\| \left(\sum_{n=1}^k \left| \sum_{i=q_{n-1}+1}^{q_n} c_i h_i \right|^2 \right)^{1/2} \right\|_{L_r} \\ &\leq C \left\| \sum_{i=1}^\infty c_i h_i \right\|_{L_r} = C \|(c_i)\|_{H_r}, \end{aligned}$$

for certain constant $C > 0$.

Let us consider now the operator $T : L_p[0, 1] \rightarrow L_r(\ell_q)$ defined by

$$\begin{array}{ccc} L_p & \xrightarrow{T} & L_r(\ell_q) \\ \downarrow J & & \uparrow R \\ L_r & \xrightarrow{L} & H_r \end{array}$$

The operator $T = RLJ$ is not ℓ_s -singular. Indeed, T is an isomorphism when restricted to the subspace $[w_n]$ in L_p , which is isomorphic to ℓ_s , since, by Lemma 3.2, we have

$$\begin{aligned}
\left\| T \left(\sum_{n=1}^{\infty} b_n w_n \right) \right\|_{L_r(\ell_q)} &= \sup_k \left\| \left(\sum_{n=1}^k b_n \sum_{i=q_{n-1}+1}^{q_n} a_i h_i \right)^q \right\|_{L_r}^{1/q} \\
&\sim \left\| \sum_{n=1}^{\infty} b_n w_n \right\|_{L_r} \sim \left\| \sum_{n=1}^{\infty} b_n w_n \right\|_{L_p} \\
&\sim \left(\sum_{n=1}^{\infty} |b_n|^s \right)^{1/s}.
\end{aligned}$$

In particular T is not strictly singular. On the other hand, the operator T is ℓ_p -singular because so is the inclusion $J : L_p \hookrightarrow L_r$.

Let us prove now that T does not preserve an isomorphic copy of ℓ_2 . To see this it suffices to show that RL preserves no isomorphic copy of ℓ_2 . Indeed, if this were not the case, let (g_n) be a sequence equivalent to the unit vector basis of ℓ_2 in L_r , so that $(RL(g_n))$ is also equivalent to it. Since $g_n \rightarrow 0$ weakly, without loss of generality we can suppose that (g_n) is a block basis of the Haar system. In fact, we can extract a subsequence (still denoted (g_n)) such that

$$g_n = \sum_{k=p_{n-1}+1}^{p_n} \psi_k^n,$$

where each $\psi_k^n \in [h_{q_{j_k-1}+1}, \dots, h_{q_{j_k}}]$ for certain increasing sequence (j_k) in \mathbb{N} (notice that the sequence (q_n) has already been fixed in the definition of the operator R).

Now, the sequence $(\psi_k^n)_{n=1, k=p_{n-1}+1 \dots p_n}^{\infty}$ forms an unconditional basic sequence since it is a sequence of blocks of the Haar basis, which is unconditional in $L_r(0, 1)$. Therefore, for every finitely non zero sequence of scalars (a_n) we have

$$\begin{aligned}
\left\| \left(\sum_{n=1}^{\infty} \sum_{k=p_{n-1}+1}^{p_n} |a_n \psi_k^n|^2 \right)^{1/2} \right\|_{L_r} &\sim \left\| \sum_{n=1}^{\infty} \sum_{k=p_{n-1}+1}^{p_n} a_n \psi_k^n \right\|_{L_r} \\
&= \left\| \sum_{n=1}^{\infty} a_n g_n \right\|_{L_r} \\
&\sim \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2}
\end{aligned}$$

(see [20, Theorem 1.d.6]). Let us consider

$$f_n = \left(\sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \right)^{1/q}.$$

Since $q > 2$ we have

$$\begin{aligned} \left\| \left(\sum_{n=1}^{\infty} a_n^2 f_n^2 \right)^{1/2} \right\|_{L_r} &= \left\| \left(\sum_{n=1}^{\infty} a_n^2 \left(\sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \right)^{2/q} \right)^{1/2} \right\|_{L_r} \\ &\leq \left\| \left(\sum_{n=1}^{\infty} a_n^2 \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^2 \right)^{1/2} \right\|_{L_r} \sim \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2}. \end{aligned}$$

Now, by Lemma 3.4, we get

$$\inf_{|A|=m} \left\| \left(\sum_{n \in A} |f_n|^q \right)^{1/q} \right\|_{L_r} = o(\sqrt{m}).$$

However, by hypothesis, there exist some constant $C > 0$ such that

$$\begin{aligned} C\sqrt{m} &\leq \inf_{|A|=m} \left\| \sum_{n \in A} RLg_n \right\|_{L_r(\ell_q)} \\ &= \inf_{|A|=m} \left\| \left(\sum_{n \in A} \sum_{k=p_{n-1}+1}^{p_n} |\psi_k^n|^q \right)^{1/q} \right\|_{L_r} \\ &= \inf_{|A|=m} \left\| \left(\sum_{n \in A} |f_n|^q \right)^{1/q} \right\|_{L_r} = o(\sqrt{m}). \end{aligned}$$

This is a contradiction; thus, the operator RL is ℓ_2 -singular, and so is $T = RLJ$. \square

Remark 3.5. Note that if the sequence (q_n) , appearing in the definition of the operator $T : L_p \rightarrow L_r(\ell_q)$ given above, increases fast enough, then it can be seen that the operator T is not ℓ_{s_n} -singular, where (s_n) is a countable dense set in the interval $(p, 2)$. From this fact, using an approximation argument for s -stable random variables we can conclude that T is not ℓ_s -singular for any $s \in (p, 2)$.

Remark 3.6. The hypothesis of order continuity of the range Banach lattice Y in Theorem B cannot be removed.

Indeed, consider the operator $T : L_p \rightarrow L_r(\ell_q)$ constructed in Theorem C and the canonical isomorphic embedding $j : L_r(\ell_q) \rightarrow \ell_\infty$. Now, the composition $jT : L_p \rightarrow \ell_\infty$ is a regular operator (cf. [21, Theorem 1.5.11]), which is disjointly strictly singular and ℓ_2 -singular, but it is not strictly singular (because T is not strictly singular and j is an isomorphic embedding).

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