POWERS OF OPERATORS DOMINATED BY STRICTLY SINGULAR OPERATORS

JULIO FLORES, FRANCISCO L. HERNÁNDEZ, AND PEDRO TRADACETE

Abstract. It is proved that every positive operator R on a Banach lattice E dominated by a strictly singular operator $T : E \to E$ satisfies that the fourth power R^4 is strictly singular. Moreover, if E is order continuous then the square R^2 is already strictly singular.

Introduction

A classical question in the setting of positive operators between Banach lattices is the "domination problem": if R and T are positive operators between Banach lattices E and F, such that $R \leq T : E \to F$, what properties of T does the operator R inherit?

Recall that for compact operators P. G. Dodds and D. H. Fremlin [5] proved that if E and F are Banach lattices, such that E^* and F are order continuous, then $0 \leq R \leq T : E \to F$ with T compact implies that R is also compact. In the same direction, A. W. Wickstead showed in [18] that if E^* or F are order continuous and T is weakly compact, then R is also weakly compact. In addition, N. Kalton and P. Saab proved in [11] that if F is order continuous and T is Dunford-Pettis, then R is also Dunford-Pettis.

In the particular case that $E = F$, hence R and T are endomorphisms on E, it is interesting and useful to study whether some power of R inherits properties of T, under no assumptions on the Banach lattice E. This is called the "power problem" relative to a certain operator class. This approach was developed by C. D. Aliprantis and O. Burkinshaw in [2] and [3], where the following results for compact and weakly compact operators were obtained.

Theorem 1. ([2]) *Let E be a Banach lattice. If $0 \leq R \leq T : E \to E$ and T is compact, then R^3 is also compact.*

Theorem 2. ([3]) *Let E be a Banach lattice. If $0 \leq R \leq T : E \to E$ and T is weakly compact, then R^2 is also weakly compact.*

For the class of Dunford-Pettis operators, N. Kalton and P. Saab proved the following.

Theorem 3. ([11]) *Let E be a Banach lattice. If $0 \leq R \leq T : E \to E$ and T is Dunford-Pettis, then R^2 is also Dunford-Pettis.*

These results are optimal in the sense that it is possible to produce counterexamples when the powers are lower.

Our aim here is to study the domination and power problems for strictly singular operators. Recall that an operator $T : X \to Y$ between Banach spaces is said to

Key words and phrases. Banach lattices, strictly singular operators, positive operators.

be *strictly singular* (or *Kato*) if for every infinite dimensional (closed) subspace M of X, the restriction $T|_M$ is not an isomorphism into Y. This class of operators forms a closed operator ideal (in the sense of Pietsch), which properly contains the ideal of compact operators. Moreover, it is well known that an operator $T : X \to Y$ between Banach spaces is strictly singular if and only if, for every infinite dimensional subspace M of X there exists another infinite dimensional subspace $N \subset M$ such that the restriction $T|_N$ is compact.

The domination problem for strictly singular operators has been studied by the first two authors in [8] and [9], where positive results were obtained for a large class of Banach lattices. In this paper, using factorization techniques we are able to improve some of the results given in [9] in two directions. Firstly, we give new domination results for strictly singular operators between Banach lattices E and F. Secondly, we obtain a power domination result for strictly singular endomorphisms *without* any assumption on the Banach lattice involved. Precisely, our main results are the following.

Theorem 4. *Let E be a Banach lattice with the subsequence splitting property, and F an order continuous Banach lattice. If $0 \leq R \leq T : E \to F$ with T strictly singular, then R is strictly singular.*

Theorem 5. *Let E be a Banach lattice and $0 \leq R \leq T : E \to E$ two positive operators. If T is strictly singular, then R^4 is also strictly singular.*

Moreover, if E is order continuous, then R^2 is strictly singular.

The proof of this result will be obtained as a consequence of the following more general result for composition of operators.

Theorem 6. *Let*

$$E_1 \underset{R_1}{\overset{T_1}{\rightrightarrows}} E_2 \underset{R_2}{\overset{T_2}{\rightrightarrows}} E_3 \underset{R_3}{\overset{T_3}{\rightrightarrows}} E_4 \underset{R_4}{\overset{T_4}{\rightrightarrows}} E_5$$

be operators between Banach lattices, such that $0 \leq R_i \leq T_i$ for $i = 1, 2, 3, 4$.

If T_1, T_3 are strictly singular, and T_2, T_4 are order weakly compact, then $R_4R_3R_2R_1$ is also strictly singular.

The paper is organized as follows: in the first section we introduce the terminology and tools needed for the proofs. The second section is mainly devoted to the proof of two domination theorems for strictly singular operators that will be used afterwards. In the third section we present the proof of Theorem 6 as well as some consequences and remarks.

We refer to [4], [13] and [14] for unexplained terminology.

1. Tools

Given a Banach lattice E and a Banach space Y, an operator $T : E \to Y$ is *order weakly compact* if $T[-x, x]$ is relatively weakly compact for every $x \in E_+$.

Order weakly compact operators can be characterized as those operators which fail to be invertible on any sublattice isomorphic to c_0 with an order bounded unit ball (see [14, Cor. 3.4.5]). Also, if X is a Banach space and F a Banach lattice, an operator $T : X \to F$ does not preserve an isomorphic copy of ℓ_1 complemented in F if and only if its adjoint T^* is order weakly compact (see [14, Thm. 3.4.14].

We now recall two basic constructions of factorization for positive operators, which are in a sense dual to each other (see [10] and [4]).

Theorem 1.1. [10, Thm. I.2] *Let E_1, E_2 be Banach lattices and operators $0 \leq R \leq T : E_1 \to E_2$. Then there exists a Banach lattice F, a lattice homomorphism $\phi : E_1 \to F$ and operators $0 \leq R^F \leq T^F$ with $T = T^F\phi$ and $R = R^F\phi$*

$$\begin{tikzcd}
E_1 \arrow[rr, "T"] \arrow[rr, dashed, "R"'] \arrow[dr, "\phi"'] & & E_2 \\
& F \arrow[ur, "T^F"] \arrow[ur, dashed, "R^F"'] &
\end{tikzcd}$$

such that F is order continuous if and only if $T : E_1 \to E_2$ is order weakly compact. Moreover, if E_2 does not contain an isomorphic copy of c_0 neither does F.

Theorem 1.2. [10, Thm. I.6] *Let E_1, E_2 be Banach lattices and operators $0 \leq R \leq T : E_1 \to E_2$. Then there exist a Banach lattice G, a lattice homomorphism $\psi : G \to E_2$ and operators $0 \leq R^G \leq T^G$ with $T = \psi T^G$ and $R = \psi R^G$*

$$\begin{tikzcd}
E_1 \arrow[rr, "T"] \arrow[rr, dashed, "R"'] \arrow[dr, "T^G"] \arrow[dr, dashed, "R^G"'] & & E_2 \\
& G \arrow[ur, "\psi"'] &
\end{tikzcd}$$

such that G^* is order continuous if and only if $T^* : E_2^* \to E_1^*$ is order weakly compact.

Recall that the Banach lattice F is obtained by completing the normed lattice E_1/I where $I = \{x \in E_1 : T|x| = 0\}$, under the norm $q_T(x + I) = \|T|x|\|$. On the other hand, the Banach lattice G is obtained by interpolating E_2 with its norm and the Minkowski functional of the solid convex hull of $T(B_{E_1})$. See [10] for details.

We will also make use of the *Kadeč-Pełczyński disjointification method* in the setting of order continuous Banach lattices (see [7]).

Theorem 1.3. *Let X be any subspace of an order continuous Banach lattice E. Then, either*

(1) *X contains an almost disjoint normalized sequence, that is , there exist a normalized sequence $(x_n)_{n=1}^\infty \subset X$ and a disjoint sequence $(z_n)_{n=1}^\infty \subset E$ such that $\|z_n - x_n\| \to 0$, or,*
(2) *X is isomorphic to a closed subspace of $L_1(\Omega, \Sigma, \mu)$.*

Notice that if X is separable, then it can be included in some ideal I of E with a weak order unit (see [13, 1.a.9]). Therefore, this ideal has a representation as a Köthe function space over a finite measure space (Ω, Σ, μ) [13, Thm. 1.b.14], and in this case the previous dichotomy says that either X contains an almost disjoint sequence or the natural inclusion $j : I \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is an isomorphism when restricted to X.

Recall that, given a Banach lattice E and a Banach space Y, an operator $T : E \to Y$ is called *disjointly strictly singular* if it is not invertible on any subspace of E generated by a disjoint sequence. Clearly, every strictly singular operator is also disjointly strictly singular. Although this class is not an operator ideal, it only lacks being closed by composition from the right.

The following domination result for disjointly strictly singular operators will be used.

Theorem 1.4. ([8]) *Let E and F be Banach lattices such that F is order continuous. If T is disjointly strictly singular and $0 \leq R \leq T : E \to F$, then R is also disjointly strictly singular.*

Freudenthal's theorem states that, under certain conditions, an operator R, such that $|R| \leq T$, can be approximated in the sense of order by components of T (see [14, Section 1.2]). This means that there exists a sequence $(S_n)_{n=1}^\infty$ of components of T such that $0 \leq R - S_n \leq \frac{1}{n}T$ for each natural number n.

Under some extra properties of the operator T, it is possible to replace the previous order approximation with an approximation in norm. Recall that an operator T has *order continuous norm* whenever every sequence of positive operators with $|T| \geq T_n \downarrow 0$ in $\mathcal{L}(E,F)$ satisfies $\|T_n\| \downarrow 0$ (here $\mathcal{L}(E,F)$ denotes the space of bounded linear operators between E and F endowed with the natural norm). Let

$$I_T := \{S \in \mathcal{L}(E,F) : \text{there exists } n \in \mathbb{N} \text{ such that } |S| \leq n|T|\},$$

and denote by $Ring(T)$ the closure of the set of operators in $\mathcal{L}(E,F)$ of the form $\sum_{i=1}^n R_i T S_i$ with $S_i \in \mathcal{L}(E)$, $R_i \in \mathcal{L}(F)$.

Theorem 1.5. [4, Thm. 18.18] *Let E be a Banach lattice which is either σ-Dedekind complete or has a quasi-interior point, and let F be a Dedekind complete Banach lattice. If T has order continuous norm, then*

$$I_T \subseteq Ring(T).$$

2. Domination results

In this section we present new domination results for strictly singular operators between Banach lattices improving some others obtained in [9]. In addition, they will be used in next section for the power problem.

The following is a well-known fact, whose proof we include for the convenience of the reader.

Lemma 2.1. *Let (Ω, Σ, μ) be a finite measure space and $(f_n)_{n=1}^\infty$ be a weakly convergent sequence in $L_1(\mu)$. If $(f_n)_{n=1}^\infty$ converges to zero in measure, then it converges to zero in norm.*

Proof. Assume $\mu(\Omega) = 1$. The sequence $(f_n)_{n=1}^\infty$ is equi-integrable since it is weakly convergent (cf. [6, Cor. IV.8.11]). Hence for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\chi_B f_n\|_1 < \varepsilon/2$ for every integer n and every $B \in \Sigma$ with $\mu(B) < \delta$. Consider $B_n = \{t \in \Omega : |f_n(t)| > \varepsilon/2\}$. By assumption there exists an integer n_0 such that $\mu(B_n) < \delta$ for $n \geq n_0$. Thus, for $n \geq n_0$ we have

$$\|f_n\|_1 = \int_{B_n} |f_n| + \int_{\Omega \backslash B_n} |f_n| \leq \|\chi_{B_n} f_n\|_1 + \frac{\varepsilon}{2}\mu(\Omega \backslash B_n) < \varepsilon.$$

□

Recall that an operator between Banach lattices $T : E \to F$ is *M-weakly compact* if $\|Tx_n\| \to 0$ for every norm bounded disjoint sequence $(x_n)_{n=1}^\infty$ in E.

A Banach lattice has the *positive Schur property* if every positive, weakly null sequence is convergent. Some examples of Banach lattices with the positive Schur property (but not the Schur property) are the $L_1(\mu)$ spaces, the Orlicz function spaces $L^{x\log^p(1+x)}[0,1]$ for $p>0$, and the Lorentz function spaces $L^{p,1}[0,1]$ for $1<p<\infty$ (cf. [19]).

Proposition 2.1. *Let E and F be Banach lattices such that F has the positive Schur property. Given operators $0\le R\le T: E\to F$ with T strictly singular, then R is strictly singular.*

Proof. Note that F cannot contain an isomorphic copy of c_0. Indeed, if this were not the case F would contain a sequence of positive, pairwise disjoint elements $(e_n)_{n=1}^\infty$ equivalent to the unit vector basis of c_0 [13, pp. 34-35], hence weakly null and yet not convergent in norm. Since F has the positive Schur property we obtain a contradiction. In particular, F is order continuous [13, pp. 6-8].

Suppose that R is not strictly singular. Then there exists an infinite dimensional (separable) subspace X in E such that $R|_X$ is an isomorphism. From the lines above it follows that $R(X)$ cannot contain an isomorphic copy of c_0. Moreover, if $R(X)$ contained an isomorphic copy of ℓ_1, then R would be an isomorphism on the span of a disjoint sequence equivalent to the canonical basis of ℓ_1 ([16]); but this would be a contradiction to Theorem 1.4 and the fact that T is disjointly strictly singular and F order continuous. Therefore, $R(X)$, hence X, must be reflexive [13, Thm 1.c.5.].

Consider now the ideal E_X generated by X in E. We claim that the restriction $T|_{E_X}$ is M-weakly compact. Indeed, by Theorem 1.1 we have the factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[dr, "\phi"'] & & F \\ & H \arrow[ur, "T^H"'] & \end{tikzcd}$$

where ϕ is a lattice homomorphism and the Banach lattice H does not contain an isomorphic copy of c_0. Let B_X denote the closed unit ball of X, which is a weakly compact set. Thus, $\phi(B_X)$ is also weakly compact, and [4, Thm. 13.8] implies that the solid hull $so(\phi(B_X))$ is also weakly compact. Since ϕ is a lattice homomorphism, the inclusion $\phi(so(B_X))\subset so(\phi(B_X))$ holds, and therefore $\phi(so(B_X))$ must be relatively weakly compact. So, if $(x_n)_{n=1}^\infty$ is a normalized positive disjoint sequence in E_X, the sequence $(\phi(x_n))_{n=1}^\infty$, which is pariwise disjoint as ϕ is a lattice homomorphism, must have a weakly convergent subsequence which in fact converges weakly to zero by [13, Thm 1.b.14] and Lemma 2.1. Since T is positive and F has the positive Schur property it follows that $(Tx_n)_{n=1}^\infty$ converges in norm to zero. This proves that $T|_{E_X}$ is M-weakly compact, as claimed.

Consider now $\hat{X}$, the sublattice of E generated by X, which is also separable. The restriction operator

$$T|_{\hat{X}}:\hat{X}\to F$$

is clearly M-weakly compact, hence has order continuous by [5, Thm. 5.1]. Moreover, since F is Dedekind complete and $\hat{X}$ has a quasi-interior point being separable, we get by Theorem 1.5 that $R|_{\hat{X}}\in Ring(T|_{\hat{X}})$. Thus $R|_{\hat{X}}$ is strictly singular because so is $T|_{\hat{X}}$. But then R cannot be an isomorphism when restricted to X. This finishes the proof. ☐

Before stating our main domination result we recall some facts. A bounded subset A of a Banach lattice E is said to be *L-weakly compact* if $\|x_n\|\to 0$ for every disjoint sequence $(x_n)_{n=1}^{\infty}$ contained in the solid hull of A. The following holds ([9, Lemma 3.2]).

Lemma 2.2. *Let T be a regular operator from a Banach lattice E into a Banach lattice F with order continuous norm. If $A \subset E$ is L-weakly compact, then $T(A)$ is L-weakly compact.*

If E is a Banach function space with an order continuous norm defined on a finite measure space (Ω, Σ, μ), a bounded subset $A \subset E$ is *equi-integrable* if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|f\chi_B\|_E < \varepsilon$ for every $B \in \Sigma$ with $\mu(B) < \delta$ and every $f \in A$.

Lemma 2.3. *Let E be a Banach lattice with order continuous norm and a weak unit, and hence representable as an order ideal in $L^1(\Omega, \Sigma, \mu)$ for some probability space (Ω, Σ, μ).*

a) A bounded subset of E is equi-integrable if and only if it is L-weakly compact.

b) A norm bounded sequence $(g_n)_{n=1}^{\infty}$ in E is convergent to zero if and only if $(g_n)_{n=1}^{\infty}$ is equi-integrable and $\| \|_1$-convergent to zero.

Proof. ([9, Lemma 3.3.]) □

A Banach lattice E with an order continuous norm satisfies the *subsequence splitting property* ([7], [17]) if for every bounded sequence $(f_n)_{n=1}^{\infty}$ included in E there is a subsequence $(n_k)_{k=1}^{\infty}$ and sequences $(g_k)_{k=1}^{\infty}$, $(h_k)_{k=1}^{\infty}$ in E with $|g_k| \wedge |h_k| = 0$ and $f_{n_k} = g_k + h_k$ such that $(g_k)_{k=1}^{\infty}$ is equi-integrable and $|h_k| \wedge |h_l| = 0$ if $k \neq l$. It is known that every p-concave Banach lattice ($p < \infty$) has the subsequence splitting property ([7]).

Theorem 2.1. *Let E be a Banach lattice with the subsequence splitting property, and F an order continuous Banach lattice. If $0 \leq R \leq T : E \to F$ with T strictly singular, then R is strictly singular.*

Proof. Since T is strictly singular, in particular, the adjoint T^* is order weakly compact, so by Theorem 1.2 we obtain the factorization

$$\begin{tikzcd} E \arrow[rr, "T"] \arrow[rr, dashed, "R"'] \arrow[dr, "T^G"] \arrow[dr, dashed, "R^G"'] & & F \\ & G \arrow[ur, "\psi"'] & \end{tikzcd}$$

where G^* is order continuous. Moreover, since F is order continuous, by [10, Prop. I.4.d] it follows that G is also order continuous.

We claim that the operator $T^G : E \to G$ is strictly singular. Indeed, since T is strictly singular, for every infinite dimensional subspace M of E there exists another infinite dimensional subspace N of M such that T restricted to N is compact. This means that the set $T(B_N)$ is precompact in F, and, by [4, Thm. 17.19], this implies that $T^G(B_N)$ is precompact in G (since $T(B_N)$ is contained in the solid convex hull of $T(B_E)$). Hence, T^G is strictly singular.

Note that the operator $R^G : E \to G$ is disjointly strictly singular; indeed, since the operator T^G is disjointly strictly singular and G is order continuous, Theorem 1.4 implies that R^G is also disjointly strictly singular.

We claim that R^G is strictly singular. Suppose the contrary, that is, R^G is an isomorphism when restricted to some separable subspace H of E. Consider $\widehat{H}$, the sublattice of E generated by H, which is also separable. By [13, Prop. 1.a.9.], $R^G(\widehat{H})$ is contained in some ideal A of G. Now, if j denotes the formal inclusion $j : A \hookrightarrow L_1$, Proposition 2.1 together with the fact that T^G is strictly singular yield that the operator

$$jR^G|_{\widehat{H}} : \widehat{H} \to A \hookrightarrow L_1$$

is strictly singular. Thus, we can consider an infinite dimensional subspace H' of H such that $jR^G|_{H'}$ is compact. Because of the order continuity of E there exists an unconditional basic sequence contained in this subspace [13, Thm. 1.c.9]. Let us denote by X the span of this sequence, where R^G is invertible and jR^G is compact.

Consider the subspace $R^G(X)$ of G, and let us apply Theorem 1.3. If the norms of G and L_1 were equivalent on $R^G(X)$, then the operator $jR^G : E \to L_1$ would be an isomorphism when restricted to X. However, this is impossible since jR^G is compact when restricted to X. Therefore, by Theorem 1.3, $R^G(X)$ contains an almost disjoint sequence.

Let us denote this sequence by $(R^G(f_n))$ and suppose it normalized. Since G^* is order continuous, $R^G(f_n) \to 0$ weakly. Hence, as R^G is an isomorphism on X, $(f_n)_{n=1}^\infty$ is also a weakly null, unconditional basic sequence in X which is bounded away from zero. Moreover, by the compactness of $jR^G|_X$ it follows that

$$\|R^G(f_n)\|_1 \to 0.$$

Since E has the subsequence splitting property, we can extract a subsequence (still denoted $(f_n)_{n=1}^\infty$) and sequences $(g_n)_{n=1}^\infty$ and $(h_n)_{n=1}^\infty$ such that

$$|g_n|,\ |h_n| \le |f_n|, \quad \text{and} \quad f_n = g_n + h_n,$$

with $\{g_n\}_{n=1}^\infty$ equi-integrable in X, and $(h_n)_{n=1}^\infty$ disjoint. There is no loss of generality if we suppose that $\|g_n\| > \delta$ for some $\delta > 0$ and every n, because if $\|g_n\| \to 0$ then the operator R^G is easily seen to be invertible on the span $[h_n]$; however, this is not possible because R^G is disjointly strictly singular.

Now, if the sequence of absolute values $(|h_n|)_{n=1}^\infty$ has no weak Cauchy subsequence, then, by Rosenthal's Theorem (cf. [4, Thm. 14.24]), it has a subsequence $(|h_{n_k}|)_{k=1}^\infty$ which is equivalent to the unit vector basis of ℓ_1. Hence, for scalars $(a_k)_{k=1}^\infty$ we have:

$$\begin{aligned}
\left\|\left(\sum_{k=1}^\infty a_k f_{n_k}\right)\right\| &\ge A_1 K^{-1} \left\|\left(\sum_{k=1}^\infty |a_k|^2 |f_{n_k}|^2\right)^{\frac12}\right\| \\
&\ge A_1 K^{-1} \left\|\left(\sum_{k=1}^\infty |a_k|^2 |h_{n_k}|^2\right)^{\frac12}\right\| \\
&= A_1 K^{-1} \left\|\sum_{k=1}^\infty a_k |h_{n_k}|\right\| \\
&\ge A_1 K^{-1} C \sum_{k=1}^\infty |a_k|
\end{aligned}$$

where A_1 is a universal constant (see [13, Thm. 1.d.6]), K is the unconditional constant of $(f_n)_{n=1}^\infty$, and C is the equivalence constant between $(|h_{n_k}|)_{k=1}^\infty$ and the unit vector basis of ℓ_1. Hence, $(f_{n_k})_{k=1}^\infty$ is equivalent to the unit vector basis of ℓ_1, and the operator R^G preserves an isomorphic copy of ℓ_1. However, this implies that R^G preserves a lattice copy of ℓ_1 (see [14, Thm. 3.4.17.], [16]), but since R^G is disjointly strictly singular, this is a contradiction. Thus, $(|h_n|)_{n=1}^\infty$ has a weakly Cauchy subsequence.

Since L_1 is weakly sequentially complete, the sequence $(|h_n|)_{n=1}^\infty$ has a weakly convergent sequence in L_1. Since it is disjoint this sequence converges to zero in measure, so Lemma 2.1 yields that

$$\|h_n\|_1 \to 0.$$

Similarly, one can prove that

$$\|R^G(h_n)\|_1 \to 0.$$

Note that $(h_n)_{n=1}^\infty$ does not converge to zero in E. Otherwise, the sequence $(f_n)_{n=1}^\infty$ would inherit the equi-integrability of the sequence $(g_n)_{n=1}^\infty$; since R^G is positive, the sequence $(R^G(f_n))_{n=1}^\infty$ would also be equi-integrable by Lemma 2.2. But since this sequence is also almost disjoint, this would imply that $R^G(f_n) \to 0$ in the norm of G, which is a contradiction. Therefore, we can assume that $\|h_n\| > \rho$ for some $\rho > 0$.

We claim that $(R^G(f_n))_{n=1}^\infty$ and $(R^G(h_n))_{n=1}^\infty$ are equivalent basic sequences in G. Indeed, the sequence $(g_n)_{n=1}^\infty$ is norm bounded since $|g_n| \leq |f_n|$ for all n. Moreover, it is equi-integrable, and by Lemma 2.2, $(R^G(g_n))_{n=1}^\infty$ is equi-integrable too. On the other hand, we have that

$$\|R^G(g_n)\|_1 = \|R^G(f_n) - R^G(h_n)\|_1 \leq \|R^G(f_n)\|_1 + \|R^G(h_n)\|_1 \to 0.$$

Hence, $R^G(g_n)$ goes also to zero in the norm of G (cf. Lemma 2.3). Thus, by passing to a subsequence, we may assume that $\sum\limits_{n=1}^{\infty}\|R^G(f_n)-R^G(h_n)\|$ is a convergent series. The perturbation result (cf. [12, Prop.1.a.9]) gives a constant $\alpha > 0$ such that

$$\alpha^{-1}\Big\|\sum_{n=1}^\infty a_n R^G(h_n)\Big\| \leq \Big\|\sum_{n=1}^\infty a_n R^G(f_n)\Big\| \leq \alpha\Big\|\sum_{n=1}^\infty a_n R^G(h_n)\Big\|.$$

Hence, we have

$$\begin{aligned}
\Big\|R^G\Big(\sum_{n=1}^\infty a_n h_n\Big)\Big\| &= \Big\|\sum_{n=1}^\infty a_n R^G(h_n)\Big\| \geq \alpha^{-1}\Big\|\sum_{n=1}^\infty a_n R^G(f_n)\Big\| \\
&\geq \beta\alpha^{-1}\Big\|\sum_{n=1}^\infty a_n f_n\Big\| \geq \beta\alpha^{-1}A_1K^{-1}\Big\|\Big(\sum_{n=1}^\infty |a_n|^2|f_n|^2\Big)^{\frac{1}{2}}\Big\| \\
&\geq \beta\alpha^{-1}A_1K^{-1}\Big\|\Big(\sum_{n=1}^\infty |a_n|^2|h_n|^2\Big)^{\frac{1}{2}}\Big\| = \beta\alpha^{-1}A_1K^{-1}\Big\|\sum_{n=1}^\infty a_n h_n\Big\|
\end{aligned}$$

where A_1 is a universal constant (mentioned above), K is the unconditional constant of (f_n), and β is a lower bound for the operator R^G restricted to X. Therefore, R^G is an isomorphism when restricted to the span of the disjoint sequence $(h_n)_{n=1}^\infty$. This is a contradiction to the fact that R^G is disjointly strictly singular.

Hence, R^G cannot be an isomorphism when restricted to any subspace of E, that is R^G is strictly singular; thus, so is R and the proof is finished. □

Note that the above result improves [9, Thm 3.1], removing the order continuity of E^*.

3. Powers of dominated operators

In this section we study the power problem for strictly singular endomorphisms. The key result is the following.

Theorem 3.1. *Let*

$$\begin{tikzcd} E_1 \arrow[r, shift left, "T_1"] \arrow[r, dashed, shift right, "R_1"'] & E_2 \arrow[r, shift left, "T_2"] \arrow[r, dashed, shift right, "R_2"'] & E_3 \arrow[r, shift left, "T_3"] \arrow[r, dashed, shift right, "R_3"'] & E_4 \arrow[r, shift left, "T_4"] \arrow[r, dashed, shift right, "R_4"'] & E_5 \end{tikzcd}$$

be operators between Banach lattices such that $0 \leq R_i \leq T_i$ *for* $i = 1, 2, 3, 4$.

If T_1, T_3 *are strictly singular, and* T_2, T_4 *are order weakly compact, then* $R_4R_3R_2R_1$ *is also strictly singular.*

Proof. Suppose that $R_4R_3R_2R_1$ is not strictly singular. Then there exists an infinite dimensional subspace M of E_1 such that $R_4R_3R_2R_1|_M$ is an isomorphism. Clearly we can suppose that M is separable.

Since T_2 is an order weakly compact operator, by Theorem 1.1, we have the factorizations

$$\begin{tikzcd} E_1 \arrow[r] & E_2 \arrow[rr, shift left, "T_2"] \arrow[rr, dashed, shift right, "R_2"'] \arrow[dr, "\phi"'] & & E_3 \arrow[r] & E_4 \arrow[r] & E_5 \\ & & F \arrow[ur, shift left, "T_2^F"] \arrow[ur, dashed, shift right, "R_2^F"'] \arrow[d, "P"] & & & \\ & & A \arrow[r, hook, "j"] & L_1(\mu) & & \end{tikzcd}$$

where F is an order continuous Banach lattice, ϕ is a lattice homomorphism and $0 \leq R_2^F \leq T_2^F$.

Consider the subspace $X = \phi R_1(M) \subset F$, which is separable, hence is contained in a closed ideal $A \subset F$ with weak order unit which is complemented in F by a positive projection, say $P : F \to A$. Therefore A , as an order continuous Banach lattice with weak unit, can be represented as a dense ideal of $L_1(\Omega, \Sigma, \mu)$ for some probability measure μ so that the formal inclusion $j : A \hookrightarrow L_1(\Omega, \Sigma, \mu)$ is continuous.

We now apply the Kadeč-Pełczyński method (Theorem 1.3) to $X \subset F$. Then either there exist a normalized sequence $(x_n)_{n=1}^\infty \subset X$ and a disjoint sequence $(w_n)_{n=1}^\infty \subset F$ such that $\|w_n - x_n\| \to 0$ or X is a closed subspace of $L_1(\Omega, \Sigma, \mu)$.

Suppose first that X is a closed subspace of $L_1(\Omega, \Sigma, \mu)$. Then we have the operators

$$0 \leq jP\phi R_1 \leq jP\phi T_1 : E_1 \to L_1(\Omega, \Sigma, \mu).$$

Since T_1 is strictly singular, then so is $jP\phi T_1$. Now, since $L_1(\Omega, \Sigma, \mu)$ has the positive Schur property, by Proposition 2.1, we get that the operator $jP\phi R_1$ is also strictly singular. According to the remark following Theorem 1.3, jP is an isomorphism restricted to X; therefore, ϕR_1 cannot be an isomorphism when restricted to M. This is a contradiction to the assumption that $R_4R_3R_2R_1|_M$ is an isomorphism. This finishes the proof in this case.

Alternatively, assume that there exist a sequence $(x_n)_{n=1}^\infty$ in X, and a disjoint sequence $(w_n)_{n=1}^\infty$ in F such that $\|w_n - x_n\| \to 0$. Passing to a subsequence, if needed, we can suppose that they are equivalent basic sequences.

Since the operator T_4 is order weakly compact, by Theorem 1.1 there exists an order continuous Banach lattice H such that the following factorizations hold

$$\begin{tikzcd}
E_1 \arrow[r] & E_2 \arrow[rr] \arrow[dr] & & E_3 \arrow[r] & E_4 \arrow[rr, shift left, "T_4"] \arrow[rr, dashed, shift right, "R_4"'] \arrow[dr, "\varphi"'] & & E_5 \\
& & F \arrow[ur] & & & H \arrow[ur, shift left, "T_4^H"] \arrow[ur, dashed, shift right, "R_4^H"'] &
\end{tikzcd}$$

where $0 \leq R_4^H \leq T_4^H$.

Now, let us consider the operator

$$\widetilde{T} = \varphi T_3 T_2^F : F \to H,$$

which is strictly singular because so is T_3, in particular $\widetilde{T}$ is disjointly strictly singular. But clearly $\widetilde{T}$ dominates the operator

$$\widetilde{R} = \varphi R_3 R_2^F : F \to H$$

and since H is order continuous, Theorem 1.4 implies that $\widetilde{R}$ is disjointly strictly singular.

However, we are assuming that the restriction of the operator $R_4 R_3 R_2 R_1|_M$ is an isomorphism, so $\widetilde{R}|_{\phi R_1(M)}$ is also an isomorphism (since it is a factor of the former operator). Since $\|w_n - x_n\| \to 0$, we can find a subsequence of natural numbers $(n_j)_{j=1}^\infty$ such that

$$\sum_{j=1}^\infty \|w_{n_j} - x_{n_j}\| < \frac{1}{2} C_1,$$

and

$$\sum_{j=1}^\infty \|\widetilde{R} w_{n_j} - \widetilde{R} x_{n_j}\| < \frac{1}{2} C_2,$$

where C_1 and C_2 are respectively the basis constants of $(x_n)_{n=1}^\infty$ and $(\widetilde{R} x_n)_{n=1}^\infty$. This implies that the operator $\widetilde{R}$ is invertible on the span of the disjoint sequence $(w_{n_j})_{j=1}^\infty$, in contradiction with the fact that $\widetilde{R}$ is disjointly strictly singular. The proof is finished. $\square$

As a consequence we get the following.

Corollary 3.1. *Let E be a Banach lattice, and consider operators $0 \leq R \leq T : E \to E$. If T is strictly singular, then R^4 is also strictly singular.*

Proof. Since T is strictly singular, it cannot preserve an isomorphic copy of c_0, so, in particular, it is order weakly compact. Therefore, it suffices to apply Theorem 3.1 to $E_i = E$, $R_i = R$ and $T_i = T$ for all i. $\square$

Corollary 3.2. *Let $0 \leq R \leq T : E \to F$, and $0 \leq S \leq V : F \to G$. If F and G are order continuous Banach lattices, and T and V are strictly singular operators, then SR is strictly singular.*

In particular, if $0 \leq R \leq T : E \to E$, with T strictly singular and E order continuous, then R^2 is strictly singular.

Proof. Since F is order continuous, the identity $I_F : F \to F$ is order weakly compact. Consider the Banach lattices

$$E_1 = E,\ E_2 = F,\ E_3 = F,\ E_4 = G \text{ and } E_5 = G,$$

and the operators

$$T_1 = T,\ T_2 = I_F,\ T_3 = V \text{ and } T_4 = I_G.$$

Then, by Theorem 3.1, we obtain that $I_G S I_F R = SR$ is strictly singular. □

The last assertion of this corollary was proved under stronger assumptions in [9, Thm. 3.17].

Notice that, in general, the domination problem for strictly singular endomorphisms is nontrivial ([9]).

Example. *There exist operators $0 \le R \le T : L^2[0,1] \oplus \ell_\infty \to L^2[0,1] \oplus \ell_\infty$ such that T is strictly singular but R is not.*

Indeed, consider the rank one operator $Q : L^1[0,1] \to \ell_\infty$ defined by $Q(f) = (\int f, \int f, \ldots)$. Take also an isometry $S : L^1[0,1] \to \ell_\infty$ given by $S(f) = (h'_n(f))_{n=1}^\infty$, where $(h_n)_{n=1}^\infty$ is a dense sequence in the unit ball of $L^1[0,1]$, and $(h'_n)_{n=1}^\infty$ is a sequence of norm one functionals such that $h'_n(h_n) = \|h_n\|$ for all $n \in \mathbb{N}$. If $J : L^2[0,1] \hookrightarrow L^1[0,1]$ denotes the canonical inclusion, then the operator $SJ : L^2[0,1] \to \ell_\infty$ is not strictly singular.

Since ℓ_∞ is Dedekind complete we have that $|SJ|$, $(SJ)^+$ and $(SJ)^-$ are also continuous operators between $L^2[0,1]$ and ℓ_∞. It is easy to see that $|SJ| \le QJ$. Since SJ is not strictly singular, we must have that either $(SJ)^+$ or $(SJ)^-$ is not strictly singular, so let us assume without loss of generality that $(SJ)^+$ is not strictly singular. Now consider the operator matrices

$$R = \begin{pmatrix} 0 & 0 \\ (SJ)^+ & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ QJ & 0 \end{pmatrix},$$

which clearly define operators with the required properties.

Note that $L^2[0,1] \oplus \ell_\infty$ is not an order continuous Banach lattice. But the square R^2 is the zero operator, which is obviously strictly singular.

We now give some domination results under weaker conditions on the Banach lattices by imposing extra conditions on the dominating operator.

Proposition 3.1. *Let E and F be Banach lattices with F order continuous, and operators $0 \le R \le T : E \to F$. If T is both weakly compact and Dunford-Pettis, then R is strictly singular.*

Proof. The domination theorems for weakly compact operators [18] and Dunford-Pettis operators [11] give us that $R : E \to F$ is both weakly compact and Dunford-Pettis (because of the order continuity of F). And this implies that R is strictly singular. Indeed, suppose that there exists a subspace X in E such that the restriction $R|_X$ is an isomorphism. Since R is weakly compact, for every bounded sequence $(x_n)_{n=1}^\infty$ in X we can find a subsequence $(x_{n_k})_{k=1}^\infty$ such that $(Rx_{n_k})_{k=1}^\infty$ is weakly convergent. Since $R|_X$ is an isomorphism, this implies that $(x_{n_k})_{k=1}^\infty$ is already weakly convergent; but R is Dunford-Pettis, and therefore $(Rx_{n_k})_{k=1}^\infty$ is norm convergent. Thus, the sequence $(x_{n_k})_{k=1}^\infty$ must be norm convergent since the restriction $R|_X$ is an isomorphism. We have shown that every bounded sequence in X has a convergent subsequence, so X must be finite dimensional. □

Notice that the previous Proposition is not true without the order continuity (see Example above).

Corollary 3.3. *Let E be a Banach lattice, and $0 \leq R \leq T : E \to E$ positive operators. If T is weakly compact and Dunford-Pettis, then R^2 is strictly singular. In particular, if T is compact, then R^2 is strictly singular.*

Proof. Since T is weakly compact, in particular it is order weakly compact, so by Theorem 1.1 we have the factorization

$$\begin{tikzcd} E \arrow[r, shift left, "T"] \arrow[r, dashed, shift right, "R"'] & E \arrow[r, shift left, "T"] \arrow[r, dashed, shift right, "R"'] \arrow[dr, "\phi"'] & E \\ & & F \arrow[u, shift left, "T^F"] \arrow[u, dashed, shift right, "R^F"'] \end{tikzcd}$$

with F an order continuous Banach lattice, and $0 \leq \phi R \leq \phi T : E \to F$. Since T is compact, ϕT is weakly compact and Dunford-Pettis. By the previous Proposition, ϕR is strictly singular, and so is R^2. $\square$

Note that along similar lines, Theorems 1.1 and 1.2, together with the Dodds-Fremlin domination theorem for compact operators [5], provide an alternative proof for Theorem 1.

Two natural questions remain open: Do there exist an order continuous Banach lattice E, and operators $0 \leq R \leq T : E \to E$ such that T is strictly singular but R is not? Do there exist a (non order continuous) Banach lattice E and operators $0 \leq R \leq T : E \to E$ such that T is strictly singular but R^3 is not?

Aknowledgements

Ministerio de Educación y Ciencia (MTM2005-00082), AP-2004-4841; BSCH-UCM (PR27/05-14045).

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Department of Applied Mathematics. Escet. Universidad Rey Juan Carlos, 28933, Móstoles, Madrid, Spain.
E-mail address: julio.flores@urjc.es

Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040, Madrid, Spain.
E-mail address: pacoh@mat.ucm.es

Departamento de Análisis Matemático, Universidad Complutense de Madrid, 28040, Madrid, Spain.
E-mail address: tradacete@mat.ucm.es