TWISTED HILBERT TRANSFORMS VS KAKEYA SETS OF DIRECTIONS

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ABSTRACT. Given a discrete group G and an orthogonal action $\gamma: G \to O(n)$ we study L_p convergence of Fourier integrals which are frequency supported on the semidirect product $\mathbb{R}^n \rtimes_{\gamma} G$. Given a unit $u \in \mathbb{R}^n$ and 1 , our $main result shows that the twisted (directional) Hilbert transform <math>H_u \rtimes_{\gamma} id_G$ is L_p -bounded iff the orbit $\mathcal{O}_{\gamma}(u)$ is finite. This is in sharp contrast with twisted Riesz transforms $R_u \rtimes_{\gamma} id_G$, which are always bounded. Our result characterizes Fourier summability in L_p for this class of groups. We also extend de Leeuw's compactification theorem to this setting and obtain stronger estimates for functions with "lacunary" frequency support.

INTRODUCTION

Given a *p*-integrable function $f : \mathbb{R}^n \to \mathbb{C}$ with p > 1 and a family of bounded functions m_R which converge pointwise to 1, the classical L_p convergence problem for Fourier integrals consists in determining for which families of m_R 's do we find a vanishing limit

$$\lim_{R \to \infty} \int_{\mathbb{R}^n} \left| f(x) - \int_{\mathbb{R}^n} m_R(\xi) \widehat{f}(\xi) \exp(2\pi i \langle \xi, x \rangle) \, d\xi \right|^p dx = 0.$$

This problem is typically studied for dilations $m_R(\xi) = m(\xi/R)$ of a fixed function $m:\mathbb{R}^n\to\mathbb{C}.$ In this case, the convergence problem reduces to deciding when the Fourier multiplier map associated to m turns out to be L_p -bounded. If m is smooth enough, the Hörmander-Mihlin criterium suffices to obtain the L_p convergence. The problem is more interesting when the multiplier is less smooth, like $m = \chi_{\Omega}$ for some (convex) open set Ω containing the origin. The L_p -boundedness of directional Hilbert transforms shows immediately that Ω can always be taken to be a convex polyhedron. On the other hand, a consequence of Fefferman's multiplier theorem for the ball [16] is that the boundary of Ω must be flat. These considerations also apply to periodic functions $f: \mathbb{T}^n \to \mathbb{C}$ if we replace Fourier integrals by Fourier series, with frequency group given by the integer lattice \mathbb{Z}^n . Moreover, according to de Leeuw's theorem [13], the same discussion is valid for the Bohr compactification of \mathbb{R}^n , whose frequency group is \mathbb{R}^n with the discrete topology $\mathbb{R}^n_{\text{disc}}$. What happens for other frequency groups? We will give necessary and sufficient conditions on convex polyhedra so that there is Fourier summability in L_p for semidirect products of the classical groups above with arbitrary discrete groups acting on them. Other classes of groups will be also considered.

The group von Neumann algebra associated to a locally compact group is a noncommutative analog of the algebra of essentially bounded functions on the dual group of a given abelian group. As basic models of quantum groups, they play a prominent role in noncommutative geometry and operator algebra [8, 27]. Harmonic analysis on these algebras puts the reference group on the frequency side, in contrast with the vast literature generalizing classical groups on the spatial side, like in the case of nilpotent Lie groups [33, 34, 35, 36, 37]. Our dual approach is inspired by the ground-breaking results of Cowling/Haagerup [12, 19] on the approximation property and Fourier multipliers on group von Neumann algebras. This paper is part of an effort to extend Fourier analysis to discrete group von Neumann algebras [6, 20, 21, 23, 25]. We will study directional Hilbert transforms (semispace Fourier multipliers) and the L_p convergence of Fourier series in this setting. This is a natural continuation of [21, 23], where smooth multipliers were investigated. We will work with semidirect products $\mathbb{R}^n \rtimes \mathcal{G}$, $\mathbb{R}^n_{\text{disc}} \rtimes \mathcal{G}$ and $\mathbb{Z}^n \rtimes \mathcal{G}$ with an arbitrary discrete group G, which are basic models of nonabelian groups which contain a Euclidean subgroup embedded in a nontrivial way. This class of groups is sufficiently rich to exhibit barriers to Fourier summability not encountered in the classical case. It includes well-known examples (via subgroups of the affine group) which have been studied so far on the spatial side but not on the frequency side. Our methods also establish sufficient conditions for L_p convergence on more general nonabelian discrete groups.

Let G be a discrete group with left regular representation $\lambda_{G} : G \to \mathcal{B}(\ell_{2}(G))$ given by $\lambda_{G}(g)\delta_{h} = \delta_{gh}$, where the δ_{g} 's form the unit vector basis of $\ell_{2}(G)$. Write $\mathcal{L}(G)$ for its group von Neumann algebra, the weak operator closure of the linear span of $\lambda_{G}(G)$ in $\mathcal{B}(\ell_{2}(G))$. Consider the standard trace $\tau_{G}(\lambda_{G}(g)) = \delta_{g=e}$ where e denotes the identity of G. Any $f \in \mathcal{L}(G)$ has a Fourier series expansion of the form $\sum_{g} \widehat{f}(g)\lambda_{G}(g)$ with $\tau_{G}(f) = \widehat{f}(e)$. Define

$$L_p(\widehat{\mathbf{G}}) = L_p(\mathcal{L}(\mathbf{G}), \tau_{\mathbf{G}}) \equiv \text{Closure of } \mathcal{L}(\mathbf{G}) \text{ wrt } \|f\|_{L_p(\widehat{\mathbf{G}})} = \left(\tau_{\mathbf{G}}[|f|^p]\right)^{\frac{1}{p}},$$

the natural L_p space over the noncommutative measure space $(\mathcal{L}(\mathbf{G}), \tau_{\mathbf{G}})$. We invite the reader to check that $L_p(\mathcal{L}(\mathbb{Z}^n), \tau_{\mathbb{Z}^n}) = L_p(\mathbb{T}^n)$, after identifying $\lambda_{\mathbb{Z}^n}(k)$ with $e^{2\pi i \langle k, \cdot \rangle}$. In general, the (unbounded) operator $|f|^p$ is obtained from functional calculus on the Hilbert space $\ell_2(\mathbf{G})$. Assume now that \mathbf{G} acts on \mathbb{R}^n by orthogonal transformations and let

$$\gamma: \mathbf{G} \to O(n)$$

stand for the corresponding action. Let us write $\mathbb{R}^n_{\text{disc}}$ for the *n*-dimensional Euclidean space equipped with the discrete topology. As a discrete abelian group we find $\mathcal{L}(\mathbb{R}^n_{\text{disc}}) \simeq L_{\infty}(\mathbb{R}^n_{\text{bohr}})$, the algebra of essentially bounded functions on the Bohr compactification. The semidirect product $\Gamma_{\text{disc}} = \mathbb{R}^n_{\text{disc}} \rtimes_{\gamma} G$ is still discrete and elements of $\mathcal{L}(\Gamma_{\text{disc}})$ are formally given by

$$f \sim \sum_{\xi \in \mathbb{R}^n_{\text{disc}}} \sum_{g \in \mathcal{G}} \widehat{f}(\xi, g) \lambda_{\Gamma_{\text{disc}}}(\xi \rtimes_{\gamma} g) \sim \sum_{g \in \mathcal{G}} f_g \rtimes_{\gamma} \lambda_{\mathcal{G}}(g)$$

with $f_g \sim \sum_{\xi} \widehat{f}(\xi, g)$ b-exp $_{\xi}$ and b-exp $_{\xi}$ the ξ -th character on $\mathbb{R}^n_{\text{bohr}}$. Its restriction to \mathbb{R}^n is the standard character $\exp_{\xi}(x) = \exp(2\pi i \langle \xi, x \rangle)$. The formal equivalence follows from the isometric isomorphism $\mathcal{L}(\Gamma_{\text{disc}}) \simeq \mathcal{L}(\mathbb{R}^n_{\text{disc}}) \rtimes_{\gamma} G$ with the cross product algebra, whose main operations are recalled below:

- $(f \rtimes_{\gamma} \lambda_{\mathcal{G}}(g))^* = \gamma_{g^{-1}}(f^*) \rtimes_{\gamma} \lambda_{\mathcal{G}}(g^{-1}),$
- $(f \rtimes_{\gamma} \lambda_{\mathcal{G}}(g))(f' \rtimes_{\gamma} \lambda_{\mathcal{G}}(g')) = f \gamma_g(f') \rtimes_{\gamma} \lambda_{\mathcal{G}}(gg'),$

•
$$\tau_{\mathcal{L}(\mathbb{R}^n_{\operatorname{disc}})} \rtimes_{\gamma} \tau_{\operatorname{G}}(f \rtimes_{\gamma} \lambda_{\operatorname{G}}(g)) = \delta_{g=e} \int_{\mathbb{R}^n_{\operatorname{bohr}}} f(x) d\mu(x),$$

with $\gamma_g f(x) = f(\gamma_{g^{-1}}x)$ and μ the normalized Haar measure on $\mathbb{R}^n_{\text{bohr}}$. Restoring the usual topology on \mathbb{R}^n , the algebra $\mathcal{L}(\Gamma)$ for $\Gamma = \mathbb{R}^n \rtimes_{\gamma} G$ can still be represented as $\mathcal{L}(\mathbb{R}^n) \rtimes_{\gamma} G$, where $\mathcal{L}(\mathbb{R}^n) \simeq L_{\infty}(\mathbb{R}^n)$ with the Lebesgue measure and formally we have

$$f \sim \int_{\mathbb{R}^n} \sum_{g \in \mathcal{G}} \widehat{f}(\xi, g) \lambda_{\Gamma}(\xi \rtimes_{\gamma} g) \, d\xi.$$

Given a convex polyhedron K containing the origin of \mathbb{R}^n , the question is whether we still have L_p convergence of Fourier series/integrals along dilations of K. In other words, under which conditions do we have

Q1) $\lim_{R \to \infty} \left\| f - \int_{R\mathbf{K}} \sum_{g \in \mathbf{G}} \widehat{f}(\xi, g) \lambda_{\Gamma}(\xi \rtimes_{\gamma} g) \, d\xi \right\|_{L_{p}(\widehat{\Gamma})} = 0,$

Q2)
$$\lim_{R \to \infty} \left\| f - \sum_{\xi \in R\mathsf{K}} \sum_{g \in \mathsf{G}} \widehat{f}(\xi, g) \lambda_{\Gamma_{\mathrm{disc}}}(\xi \rtimes_{\gamma} g) \right\|_{L_{p}(\widehat{\Gamma_{\mathrm{disc}}})} = 0,$$

Q3) L_p convergence of Fourier series for more general groups.

According to de Leeuw's theorem [13], the $L_p(\mathbb{R}^n_{\text{bohr}})$ -boundedness of

$$H_u: \sum_{\xi} \widehat{f}(\xi) \text{ b-exp}_{\xi} \mapsto -i \sum_{\xi} \operatorname{sgn}\langle \xi, u \rangle \widehat{f}(\xi) \text{ b-exp}_{\xi} \qquad (u \in \mathbb{S}^{n-1})$$

is equivalent to the L_p -boundedness of $\widehat{H_u}f(\xi) = -i\mathrm{sgn}\langle \xi, u\rangle \widehat{f}(\xi)$, the *u*-directional Hilbert transform in \mathbb{R}^n with the usual topology. The map $\exp_k \mapsto -i\mathrm{sgn}\langle k, u\rangle \exp_k$ is also L_p bounded for $\mathcal{G} = \mathbb{Z}^n$ by transference arguments. In the context of Lie groups, we may consider similar operators by means of the exponential map with the vector *u* living in the corresponding Lie algebra. Motivated by our questions above, we are interested in directional Hilbert transforms for the class of groups Γ and Γ_{disc} . We characterize the L_p -boundedness of $H_u \rtimes_{\gamma} id_{\mathrm{G}}$ for 1on both algebras.

Theorem A. Consider the operator densely defined by

$$H_u \rtimes_{\gamma} id_{\mathcal{G}} : \sum_g f_g \rtimes_{\gamma} \lambda_{\mathcal{G}}(g) \mapsto \sum_g H_u(f_g) \rtimes_{\gamma} \lambda_{\mathcal{G}}(g).$$

If $1 and <math>u \in \mathbb{S}^{n-1}$, the following properties are equivalent

- i) $H_u \rtimes_{\gamma} id_G$ is bounded on $L_p(\widehat{\Gamma})$,
- ii) $H_u \rtimes_{\gamma} id_G$ is bounded on $L_p(\widehat{\Gamma_{\text{disc}}})$,
- iii) The orbit $\mathcal{O}_{\gamma}(u) = \{\gamma_g(u) \mid g \in G\}$ is finite,
- iv) The following matrix inequality holds

$$\int_{\mathbb{R}^n} \left\| \left(H_{\gamma_{g^{-1}}(u)}(f_{g,h})(x) \right) \right\|_{S_p(\mathbf{G})}^p dx \le c_p \int_{\mathbb{R}^n} \left\| \left(f_{g,h}(x) \right) \right\|_{S_p(\mathbf{G})}^p dx.$$

We will also prove $L_1 \to L_{1,\infty}$ and $L_{\infty} \to BMO$ type estimates for finite orbits.

Our arguments combine Kakeya type constructions, ergodic methods, geometric group theory and noncommutative Littlewood-Paley decompositions. The main application of Theorem A is that we find complete answers to Questions 1,2 and partial answers to Question 3 in Corollary C below. Although we refer (for obvious reasons) to these operators as γ -twisted Hilbert transforms, we have not found a close relation to other twisted convolution operators [18, 33, 50]. Of course, twisted Hilbert transforms are always L_2 bounded since they can be realized as Fourier multipliers on $\mathcal{L}(\Gamma)/\mathcal{L}(\Gamma_{\text{disc}})$ with an ℓ_{∞} multiplier. As an extension of H_u to a larger space, is it conceivable that $H_u \rtimes_{\gamma} id_{\text{G}}$ should remain bounded in L_p for 1 . In fact, as recently proven in [23] this is exactly what happens forRiesz transforms

$$R_u \rtimes_{\gamma} id_{\mathcal{G}} : \sum_g f_g \rtimes \lambda_{\mathcal{G}}(g) \mapsto \sum_g R_u(f_g) \rtimes \lambda_{\mathcal{G}}(g).$$

In contrast, Theorem A establishes a surprisingly rigid characterization in terms of the γ -orbit of u. This rigidity has led us to analyze the behavior of $(H_u \rtimes_{\gamma} id_G)f$ when the Fourier spectrum of f is supported on a subset Λ of G leading to infinite but lacunary γ -suborbits of u, see Theorem B below. Our notion of lacunarity partly relies on suitable length functions/cocycles on G, emphasizing after [23] the role of cohomology theory in our approach.

We will prove Theorem A by showing i) \Rightarrow ii) \Rightarrow iii) \Rightarrow i). The additional equivalence with iv) and the endpoint estimates will be proven later. The core of the proof is ii) \Rightarrow iii). Roughly, when $\mathcal{O}_{\gamma}(u)$ is not finite, our strategy is to construct a Littlewood-Paley type decomposition determined by a sequence g_1, g_2, \ldots in a group amplification of G so that $\gamma_{g_1}(u), \gamma_{g_2}(u), \ldots$ admits Kakeya sets of directions in some sense. The idea is then to show that L_p -boundedness of $H_u \rtimes_{\gamma} id_G$ implies a 'twisted Meyer inequality' in the compactified space $L_p(\mathbb{R}^n_{\text{bohr}})$,

$$\left\| \left(\sum_{j=1}^{\infty} |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \right)^{\frac{1}{2}} \right\|_p \le c_p \max\left\{ \left\| \left(\sum_{j=1}^{\infty} |f_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p, \left\| \left(\sum_{j=1}^{\infty} |\gamma_{g_j^{-1}} f_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p \right\}$$

whenever p > 2; a dual inequality arises for p < 2. The maximum on the right comes from the row/column nature of noncommutative square functions, see [23, 32, 44, 46] for more on Littlewood-Paley estimates in this context. See also Lemma 3.1 for more on Meyer type inequalities and Remark 3.4 for twisted generalizations. A less tractable square function estimate in $L_p(\mathbb{R}^n)$ is obtained by decompactification. The goal is then to pick directions $\gamma_{g_i}(u)$ distributed in a way so that such an inequality cannot occur for a suitable choice of the functions f_{g_i} . This is reminiscent of Fefferman's estimates in the solution of the ball multiplier problem [16]. The additional term on the right-hand side is estimated by ergodic arguments. We will finally disprove such an inequality for sets of directions that have a 2-dimensional projection which admits Besicovitch's sprout triangle construction. Let us say that such a set *admits Kakeya shadows*. Using further group theoretical tools, we will study the geometry of $\mathcal{O}_{\gamma}(u)$ and show that infinite orbits always admit Kakeya shadows. Our choice of functions f_{g_j} adapts Fefferman's construction to higher dimensions in a way which is compatible with the second (new) term in the maximum considered above. On the other hand, the boundedness of $H_{\mu} \rtimes_{\gamma} id_{\rm G}$ in L_p for finite orbits can be proved by standard methods.

The equivalence i) \Leftrightarrow ii) may be regarded as a twisted form of K. de Leeuw's compactification theorem [13] for the multiplier $H_u \rtimes_{\gamma} id_G$. Some other results along this line will appear in [26, 41] as a byproduct of noncommutative transference methods. According to the sketched argument —a decompactification process and a Kakeya type set construction— the implication ii) \Rightarrow i) in the twisted form of de Leeuw's theorem is now reduced to the straightforward one iii) \Rightarrow i). Our interest in ii) \Leftrightarrow iv) was motivated by the Neuwirth/Ricard transference argument [39] which provides a slightly weaker result for amenable G. Instead, we notice the equivalence iii) \Leftrightarrow iv) for general G, which follows easily from our previous approach. Although not closely related, it is somehow amusing to compare this with the Bateman/Thiele results [2, 3] on Hilbert transforms along one-variable vector fields, see also the work of Lacey/Li [28, 29]. They consider operators on \mathbb{R}^2 of the form $Tf(x,y) = H_{u(x)}f(x,y)$ so that the directions change only with the first variable. In our case, the operator lives in the algebra of matrix-valued functions and the directions change only with the rows $\langle e_g, Tf(x)e_h \rangle = H_{\gamma_g(u)}f_{g,h}(x)$. In the case of finite orbits, we also provide the associated weak type L_1 inequality for amenable groups and certain $L_{\infty} \to BMO$ estimate for general discrete groups. The weak type inequality arises from [39, 40]. The latter requires a suitable choice of *u*-directional BMO space.

Our remaining results require some terminology. A set $\Omega = \{\omega_j \mid j \ge 1\}$ in the unit sphere \mathbb{S}^{n-1} will be called *radially lacunary* if there exists a limit point ω in the sphere such that

$$\sup_{j\geq 1}\frac{|\omega_{j+1}-\omega|}{|\omega_j-\omega|} < 1.$$

This 'radial' lacunarity is in some sense one-dimensional. In a recent paper [42] on directional maximal operators, we introduced a higher dimensional notion. Given Ω as above, set $d = \dim [\operatorname{span}(\Omega)]$ and $\Sigma(d) = \{(j,k) \mid 1 \leq j < k \leq d\}$. Divide Ω into lacunary segments with respect to an orthonormal basis e_1, e_2, \ldots, e_d

$$\Omega_{\sigma,i} = \left\{ \omega \in \Omega \mid 0 < \theta_{\sigma,i+1} < \left| \frac{\langle \omega, e_k \rangle}{\langle \omega, e_j \rangle} \right| \le \theta_{\sigma,i} \right\} \quad \text{for} \quad \sigma = (j,k) \in \Sigma(d)$$

with $\sup_{i\in\mathbb{Z}} \theta_{\sigma,i+1}/\theta_{\sigma,i} < 1$. Letting $\Omega_{\sigma,\infty} = \Omega \cap (e_j^{\perp} \cup e_k^{\perp})$ and $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$ we obtain a partition $\{\Omega_{\sigma,i}\}_{i\in\mathbb{Z}^*}$ of Ω for all $\sigma \in \Sigma(d)$. A dissection will be such a choice of $\binom{d}{2}$ partitions. The set of directions Ω is called HD-lacunary of order 0 if it consists of a single direction. Recursively, it is HD-lacunary of order L if there is a dissection for which the $\Omega_{\sigma,i}$'s are HD-lacunary of order $\leq L - 1$ for all $i \in \mathbb{Z}^*$ and $\sigma \in \Sigma(d)$ with uniformly bounded lacunary constants.

Given $\Lambda \subset G$, we say that (Λ, γ, u) is a *lacunary triple* when the γ -suborbit $\mathcal{O}_{\gamma}(\Lambda^{-1}, u) = \{\gamma_g(u) | g^{-1} \in \Lambda\}$ is radially lacunary and HD-lacunary of finite order. Note that neither notion of lacunarity is stronger than the other. We also introduce the space

$$L_{\Lambda,p}(\widehat{\Gamma_{\text{disc}}}) \,=\, \Big\{ f \sim \sum_{g \in \Lambda} f_g \rtimes_{\gamma} \lambda_{\mathcal{G}}(g) \in L_p(\widehat{\Gamma_{\text{disc}}}) \Big\}.$$

Theorem B. We have

$$H_u \rtimes_{\gamma} id_{\mathcal{G}} : L_{\Lambda,p}(\widehat{\Gamma_{\text{disc}}}) \to L_{\Lambda,p}(\widehat{\Gamma_{\text{disc}}})$$

for any $\Lambda \subset G$ for which (Λ, γ, u) is a lacunary triple and for any 1 .

In other words, this result gives a sufficient condition on Λ for L_p -boundedness of γ -twisted Hilbert transforms acting on functions $f = \sum_{g \in \Lambda} f_g \rtimes_{\gamma} \lambda_G(g)$ whose Fourier spectrum lies in Λ . In particular, it provides infinite dimensional subspaces on which $H_u \rtimes_{\gamma} id_G$ is bounded when $\mathcal{O}_{\gamma}(u)$ is not finite. It is quite simple to construct specific examples. According to Lemma 3.1 and recent estimates for the directional maximal function [42], HD-lacunarity is used to avoid the presence of Kakeya shadows and radial lacunarity is used to apply Littlewood-Paley type estimates. The analog of Theorem B with Γ in place of Γ_{disc} only requires the adaptation of the Littlewood-Paley estimates in [23] to the (non-discrete) group Γ . More general notions of lacunarity provide generalizations of Theorem B, see Remark 3.3. A more in depth analysis would be related to some classical problems in harmonic analysis.

We have shown how Lie groups or semidirect products $\mathbb{R}^n \rtimes_{\gamma} G$ admit enough geometric structure to define directional Hilbert transforms on them. Given a general discrete group, there is no standard 'space of directions' to define Hilbert transforms on its group algebra, as we could do with \mathbb{R}^n or the corresponding Lie algebra. According to [23], the key point is to use a broader interpretation of tangent spaces in terms of length functions and cocycles. These tools provide natural forms of directional Hilbert transforms. Moreover, the problem of L_p convergence for Fourier series can be reformulated for general group von Neumann algebras in terms of cocycles. Given a conditionally negative length function $\psi : G \to \mathbb{R}_+$ with associated cocycle (\mathcal{H}, b, γ) —precise definitions can be found Section 1— consider any open convex bounded polytope K in the Hilbert space \mathcal{H} containing the origin. Typically we may think of K as a cube centered at the origin. The problem is then to determine conditions on K, for which truncation along dilations of K yields L_p convergence of the partial sums;

$$\lim_{R \to \infty} \left\| f - T_{R,\psi} f \right\|_{L_p(\widehat{\mathbf{G}})} = 0 \quad \text{where} \quad T_{R,\psi} f = \sum_{g: \ b(g) \in R\mathbf{K}} \widehat{f}(g) \lambda_{\mathbf{G}}(g).$$

Of course, inner cocycles are less interesting in this regard since partial sums are not finite truncations and the norm limit stabilizes in finite time. In the next result we provide sufficient conditions for L_p convergence on any pair (G, ψ) . Moreover, we give complete answers to Questions 1, 2 and partial answers to Question 3. Given $u \in \mathcal{H}$, define the (ψ, u) -Hilbert transform as

$$H_{\psi,u}: \lambda_{\mathcal{G}}(g) \mapsto -i\langle b(g), u \rangle_{\mathcal{H}} \lambda_{\mathcal{G}}(g).$$

Again, the space $L_{\Lambda,p}(\widehat{\mathbf{G}})$ is the closure in $L_p(\widehat{\mathbf{G}})$ of elements $f \sim \sum_{g \in \Lambda} \widehat{f}(g) \lambda_{\mathbf{G}}(g)$.

Corollary C. Let G be a discrete group equipped with a length function $\psi : G \to \mathbb{R}_+$ with associated (finite-dimensional) cocycle (\mathcal{H}, b, γ) . If 1 , then thefollowing results hold:

- a) $H_{\psi,u}: L_p(\widehat{\mathbf{G}}) \to L_p(\widehat{\mathbf{G}})$ if $\mathcal{O}_{\gamma}(u)$ is a finite orbit.
- b) $H_{\psi,u}: L_{\Lambda,p}(\widehat{\mathbf{G}}) \to L_{\Lambda,p}(\widehat{\mathbf{G}})$ if (Λ, γ, u) is a lacunary triple.
- c) $T_{R,\psi}f \to f$ in $L_p(\widehat{\mathbf{G}})$ whenever $|\mathcal{O}_{\gamma}(u_j)| < \infty$ for the normal directions u_1, u_2, \ldots, u_m to all the faces of K. Moreover, this condition is necessary for L_p convergence in $\Gamma = \mathbb{R}^n \rtimes_{\gamma} \mathbf{G}$ or $\Gamma_{\text{disc}} = \mathbb{R}^n_{\text{disc}} \rtimes_{\gamma} \mathbf{G}$.

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Note that, since K is a convex polyhedron, the directions u_1, u_2, \ldots, u_m must span \mathbb{R}^n . In particular, if the γ -orbit of the u_j 's is finite for all $1 \leq j \leq m$, we must have $N = \sup_{\xi \in \mathbb{R}^n} |\mathcal{O}_{\gamma}(\xi)| < \infty$. This might appear to be quite a rigid condition in the line of Theorem A, but we recall that the classical theory has only studied \mathbb{Z}^n and \mathbb{R}^n via the trivial cocycle, which is given by the identity inclusion map with the trivial action, so that orbits in that case are always of cardinality 1. A simple example for which we may find $1 < N < \infty$ is given by the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2 \simeq \mathbb{Z} \rtimes \mathbb{Z}_2$. On the other hand, this strong rigidity is surprisingly necessary for both Γ and Γ_{disc} . Additionally, for crossed products with the integer lattice $\mathbb{Z}^n \rtimes_{\gamma} G$ all possible actions fixing \mathbb{Z}^n must satisfy $N < \infty$. The proof of Corollary C follows from our previous results on Γ_{disc} together with an intertwining identity from [23]. The condition in a) no longer characterizes L_p boundedness of $H_{\psi,u}$ in general. Indeed, take $G = \mathbb{Z}$ with (\mathcal{H}, b, γ) given by

$$\mathcal{H} = \mathbb{C} \simeq \mathbb{R}^2, \quad b(k) = \exp_{\alpha}(k) - 1, \quad \gamma_k(z) = \exp_{\alpha}(k)z$$

for some $\alpha \in \mathbb{R}_+$. When $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the orbit of any $u \in \mathbb{S}^1$ is not finite, but the map $H_{\psi,u} : \exp_k \mapsto -i \operatorname{sgn} \langle \exp_\alpha(k) - 1, u \rangle \exp_k$ is L_p -bounded. This easily follows from de Leeuw's periodization and restriction theorems [13].

Corollary C should be compared with the L_p convergence problem of Fourier series/integrals for vector-valued functions, where directional Hilbert transforms are always L_p -bounded as long as we take values in a UMD Banach space. If we replace cross products by tensor products in our setting, we obtain

$$L_p(\mathcal{L}(\mathbb{R}^n) \otimes \mathcal{L}(\mathcal{G})) = L_p(\mathcal{L}(\mathbb{R}^n \times \mathcal{G})) = L_p(\mathbb{R}^n; L_p(\mathcal{L}(\mathcal{G})))$$

and L_p convergence for the group $\mathbb{R}^n \times \mathcal{G}$ follows from the vector-valued theory since noncommutative L_p spaces are UMD for any $1 . Our conditions in Corollary C for <math>\mathbb{R}^n \rtimes \mathcal{G}$ are (necessarily) much more demanding.

1. A LITTLEWOOD-PALEY TYPE THEOREM

The first ingredient for the proof of Theorem A will be a Littlewood-Paley type inequality for group von Neumann algebras. More concretely, let G be a discrete group equipped with a given length function $\psi : \mathbf{G} \to \mathbb{R}_+$ and consider a lacunary partition of $\mathbb{R}_+ = \bigcup_m \mathcal{I}_m$. Then, any $f \in L_p(\widehat{\mathbf{G}})$ can essentially be written as $f \sim \sum_m f_m$, where the Fourier spectrum of f_m lies in $\Lambda_m = \{g \in \mathbf{G} \mid \psi(g) \in \mathcal{I}_m\}$. One of the main results from [23] provides a norm equivalence between f and certain noncommutative square function associated to the f_m 's. In this section, we recall this Littlewood-Paley estimate and consider some natural length functions in \mathbf{G} related to γ . We also refer to Pisier's papers [44, 45] for more on lacunary type sets in discrete groups and noncommutative Littlewood-Paley inequalities.

1.1. Length functions and cocycles. An affine representation of G is a group homomorphism $G \to \mathcal{H} \rtimes O(\mathcal{H})$ into the affine group associated to a real Hilbert space \mathcal{H} . Affine representations are determined by a representation $\gamma : G \to O(\mathcal{H})$ together with a mapping $b : G \to \mathcal{H}$ satisfying $b(gh) = \gamma_g(b(h)) + b(g)$. The triple (\mathcal{H}, γ, b) is usually referred to as a cocycle of G. It is clear that affine representations and cocycles of G are in one-to-one correspondence. In this paper, we say that $\psi : G \to \mathbb{R}_+$ is a *length function* if it vanishes at the identity $e, \psi(g) = \psi(g^{-1})$ and $\sum_g \beta_g = 0 \Rightarrow \sum_{g,h} \overline{\beta}_g \beta_h \psi(g^{-1}h) \leq 0$. Those functions satisfying the last condition

are called conditionally negative. It is straightforward to show that length functions take values in \mathbb{R}_+ . Length functions are also in one-to-one correspondence with affine representations and cocycles. Namely, any cocycle (\mathcal{H}, γ, b) gives rise to the length function $\psi_b(g) = \langle b(g), b(g) \rangle_{\mathcal{H}}$, as it can be easily checked. Reciprocally, any length function ψ gives rise to a cocycle (\mathcal{H}, γ, b) . This is a standard application of Schoenberg's theorem [48], which claims that $\psi : \mathbf{G} \to \mathbb{R}_+$ is a length function if and only if the mappings $S_{\psi,t}(\lambda_{\mathbf{G}}(g)) = \exp(-t\psi(g))\lambda_{\mathbf{G}}(g)$ extend to a semigroup of unital completely positive maps on $\mathcal{L}(\mathbf{G})$. Let us collect these results.

Lemma 1.1. If $\psi : \mathbf{G} \to \mathbb{R}_+$ is a length function:

• The form

$$K_{\psi}(g,h) = \frac{\psi(g) + \psi(h) - \psi(g^{-1}h)}{2},$$

defines a positive matrix on $G \times G$ and leads to

$$\left\langle \sum_{g} a_{g} \delta_{g}, \sum_{h} b_{h} \delta_{h} \right\rangle_{\psi} = \sum_{g,h} a_{g} K_{\psi}(g,h) b_{h}$$

on the group algebra $\mathbb{R}[G]$ of finitely supported real functions on G.

• Let \mathcal{H} be the Hilbert space completion of

 $(\mathbb{R}[G]/N_{\psi}, \langle \cdot, \cdot \rangle_{\psi}) \quad with \quad N_{\psi} = null \ space \ of \ \langle \cdot, \cdot \rangle_{\psi}.$

If we consider the mapping $b: g \in \mathbf{G} \mapsto \delta_q + N_{\psi} \in \mathcal{H}$

$$\gamma_g \left(\sum_{h \in \mathbf{G}} a_h b(h) \right) = \sum_{h \in \mathbf{G}} a_h \left(b(gh) - b(g) \right)$$

determines an isometric action $\gamma : \mathbf{G} \to O(\mathcal{H})$ of \mathbf{G} on \mathcal{H} .

• The resulting triple (\mathcal{H}, γ, b) determines a cocycle of the group G.

The previous lemma allows the reader to consider a pseudo-metric on the discrete group G in terms of the length function ψ . Indeed, a short calculation leads to the crucial identity $\psi(g^{-1}h) = \langle b(g) - b(h), b(g) - b(h) \rangle_{\psi} = \|b(g) - b(h)\|_{\mathcal{H}}^2$. In particular we find that $\operatorname{dist}(g,h) = \sqrt{\psi(g^{-1}h)} = \|b(g) - b(h)\|_{\mathcal{H}}$ defines a pseudo-metric on G, which becomes a metric when the b is injective.

1.2. Littlewood-Paley estimates for length functions. Consider a family of operators $\Sigma = (\sigma_k)_{k\geq 1}$ acting on some Hilbert space. Then, the row and column square functions associated to Σ are respectively defined by

$$S_r(\Sigma) = \left(\sum_k \sigma_k \sigma_k^*\right)^{\frac{1}{2}}$$
 and $S_c(\Sigma) = \left(\sum_k \sigma_k^* \sigma_k\right)^{\frac{1}{2}}$.

Given a discrete group G and a family $\Sigma = (\sigma_k)_{k\geq 1}$ in $L_p(\widehat{\mathbf{G}})$, consider the norms $\|\Sigma\|_{L_p(\widehat{\mathbf{G}};\ell_2^r)} = \|\mathcal{S}_r(\Sigma)\|_p$ and $\|\Sigma\|_{L_p(\widehat{\mathbf{G}};\ell_2^r)} = \|\mathcal{S}_c(\Sigma)\|_p$. Both clearly coincide over commutative algebras. In general, certain combination is needed to obtain the noncommutative forms of classical results such as Khintchine, Burholder-Gundy or Littlewood-Paley type inequalities. It is now well-known that the right combination arises as follows

$$L_p(\widehat{\mathbf{G}}; \ell_{rc}^2) = \begin{cases} L_p(\widehat{\mathbf{G}}; \ell_2^r) + L_p(\widehat{\mathbf{G}}; \ell_2^c) & \text{if } 1 \le p \le 2, \\ L_p(\widehat{\mathbf{G}}; \ell_2^r) \cap L_p(\widehat{\mathbf{G}}; \ell_2^c) & \text{if } 2 \le p \le \infty. \end{cases}$$

In other words, we have

$$\|\Sigma\|_{L_p(\widehat{\mathbf{G}},\ell_{rc}^2)} = \begin{cases} \inf_{\Sigma=\Phi+\Psi} \|\mathcal{S}_r(\Phi)\|_p + \|\mathcal{S}_c(\Psi)\|_p & \text{if } 1 \le p \le 2, \\ \max \left\{ \|\mathcal{S}_r(\Sigma)\|_p, \|\mathcal{S}_c(\Sigma)\|_p \right\} & \text{if } 2 \le p \le \infty. \end{cases}$$

The following result can be found in [23].

Lemma 1.2. Let G be a discrete group equipped with a length function $\psi : G \to \mathbb{R}_+$ and assume that dim $\mathcal{H} = n < \infty$ for the cocycle Hilbert space. Let $k_n = [\frac{n}{2}] + 1$ and consider a family $h_m \in \mathcal{C}^{k_n}(\mathbb{R}_+ \setminus \{0\})$ satisfying

- $\sum_{m} |h_m(\zeta)|^2 = 1$,
- $\sum_{m} \left| \frac{d^j}{d\zeta} h_m(\zeta) \right|^2 \le c_n |\zeta|^{-2j} \text{ for } j \le \left[\frac{n}{2}\right] + 1.$

Then, the following holds for $f \in L_p(\widehat{\mathbf{G}})$ and 1

$$\|f\|_p \sim_{c_n} \left\|\sum_m f_m \otimes e_m\right\|_{L_p(\widehat{\mathbf{G}};\ell_{rc}^2)},$$

with the ψ -smooth Littlewood-Paley decomposition $f_m = \sum_g h_m(\psi(g))\widehat{f}(g)\lambda_G(g)$.

1.3. Length functions on G adapted to γ . It may be illustrative for non-experts to show how to construct natural length functions for those discrete groups which admit finite-dimensional orthogonal representations. This will be used below in the proof of Theorem B. Take

$$\psi_{O(n)}(\mathbf{A}) = \|\mathbf{A} - \mathbf{I}\|_{\mathrm{HS}}^2 = \sum_{jk} |\mathbf{A}_{jk} - \delta_{jk}|^2.$$

It is not difficult to check directly that $\psi_{O(n)}$ is a length function in O(n), but it is perhaps easier to note that $\psi_{O(n)}(A) = ||b(A)||_{\mathcal{H}}^2$ for the cocycle (\mathcal{H}, γ, b) which is determined by the $n \times n$ matrices with the Hilbert-Schmidt norm, the action $\gamma_A(B) = AB$ and the cocycle map $A \mapsto A - I$. Consider now a discrete group equipped with a orthogonal representation $\gamma : G \to O(n)$. Using that $\psi_{O(n)}$ is a length function, we may define $\psi_{\gamma} : G \to \mathbb{R}_+$ as follows

$$\psi_{\gamma}(g) = \psi_{O(n)}(\gamma_g) = \left\| \gamma_g - \mathbf{I} \right\|_{\mathrm{HS}}^2$$

It is now clear that ψ_{γ} defines a length function on the group G for any orthogonal representation γ . Alternatively, given any non-zero $\xi_0 \in \mathbb{R}^n$ we may also construct the length functions

$$\psi_{\gamma,\xi_0}(g) = \left\langle \gamma_g(\xi_0) - \xi_0, \gamma_g(\xi_0) - \xi_0 \right\rangle_{\mathbb{R}^n}.$$

Both choices of length functions correspond to inner cocycles $-b(g) = \gamma_g(\eta) - \eta$ for some $\eta \in \mathcal{H}$ — which are quotiented out in the formation of the corresponding cohomology group, so the reader could object that our length functions are singular in the sense of cohomology theory. This was already justified in [23], where inner cocycles turned out to be the most striking ones looking for pathological Fourier multipliers, even in \mathbb{R}^n . On the other hand, discrete groups satisfying Kazhdan's property (T) only admit inner cocycles. Thus, the information encoded by our length functions goes beyond the cohomology group, which is trivial for this class of groups. Bounded, integer valued lengths also arise regarding O(n) as a Coxeter group, counting the number of reflections in which γ_g decomposes.

2. Twisted Hilbert transforms vs Kakeya sets

In this section we prove Theorem A. Most of our efforts are devoted to proving the hardest implication ii) \Rightarrow iii). First, we use Littlewood-Paley estimates in a group amplification of G to obtain a square function inequality in $L_p(\mathbb{R}^n_{\text{bohr}})$, provided $H_u \rtimes_{\gamma} id_{\text{G}}$ is L_p -bounded. The group amplification is essential to provide enough room to disprove such an inequality for infinite orbits. Second, we will decompactify such an inequality adapting transference techniques in conjunction with ergodic type arguments, which yields an inequality in $L_p(\mathbb{R}^n)$. Third, we show that infinite orbits admit Kakeya shadows and disprove the latter Euclidean inequality. The proof of i) \Rightarrow ii) uses a slight variation of de Leeuw's compactification argument which adapts to the cross product setting. Finally, the L_p -boundedness for finite orbits is clear and we shall prove stronger endpoint estimates.

2.1. Littlewood-Paley methods. The following result plays a role similar to Meyer's lemma in Fefferman's solution of the disc conjecture [16]. By duality in Theorem A, it suffices to consider the case 2 . However, the inequality in the following lemma is not self-dual. One can formulate a (more intricate) version for <math>1 which will be easily guessed by the reader after the proof.

Lemma 2.1. If $2 and <math>H_u \rtimes_{\gamma} id_G : L_p(\widehat{\Gamma_{\text{disc}}}) \to L_p(\widehat{\Gamma_{\text{disc}}})$, then

$$\left\| \left(\sum_{j=1}^{\infty} |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \right)^{\frac{1}{2}} \right\|_p \le c_p \max\left\{ \left\| \left(\sum_{j=1}^{\infty} |f_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p, \left\| \left(\sum_{j=1}^{\infty} |\gamma_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p \right\}$$

for any sequence g_1, g_2, \ldots in G and any family of functions f_{g_1}, f_{g_2}, \ldots in $L_p(\mathbb{R}^n_{bohr})$.

Proof. Let $H = G \times \mathbb{Z}$ and set

$$\rho: (g,k) \in \mathbf{H} \mapsto \gamma_q \in O(n).$$

Clearly, ρ defines an orthogonal representation of H which yields

$$\mathbb{R}^n_{\operatorname{disc}} \rtimes_{\rho} \mathbf{H} \simeq (\mathbb{R}^n_{\operatorname{disc}} \rtimes_{\gamma} \mathbf{G}) \times \mathbb{Z}, \\ \mathcal{L}(\mathbb{R}^n_{\operatorname{disc}} \rtimes_{\rho} \mathbf{H}) \simeq \mathcal{L}(\mathbb{R}^n_{\operatorname{disc}} \rtimes_{\gamma} \mathbf{G}) \bar{\otimes} \mathcal{L}(\mathbb{Z}).$$

This group isomorphism $\xi \rtimes_{\rho} (g, k) \mapsto (\xi \rtimes_{\gamma} g, k)$ is additionally a homeomorphism since both sides are equipped with the discrete topology. The map $H_u \rtimes_{\rho} id_H$ factorizes as $(H_u \rtimes_{\gamma} id_G) \otimes id_{\mathbb{Z}}$, so that L_p -boundedness means

$$\int_{\mathbb{T}} \left\| (H_u \rtimes_{\gamma} id_{\mathcal{G}}) f(x) \right\|_p^p dx \le c_p \int_{\mathbb{T}} \|f(x)\|_p^p dx$$

for any $f \in L_p(\mathbb{T}; L_p(\widehat{\Gamma_{\text{disc}}}))$, which clearly holds by hypothesis. Once we know $H_u \rtimes_{\rho} id_{\mathrm{H}}$ is L_p -bounded, consider the length function on $\mathbb{R}^n_{\mathrm{disc}} \rtimes_{\rho} \mathrm{H}$ given by $\psi(\xi \rtimes_{\rho} (g, k)) = |k|^2$ for all $(g, k, \xi) \in \mathrm{G} \times \mathbb{Z} \times \mathbb{R}^n$. Recall that ψ gives rise to the trivial cocycle $b(\xi \rtimes_{\rho} (g, k)) = k$. Thus, we may apply Lemma 1.2 with n = 1. If we pick a small $\delta > 0$ and any nonnegative radially decreasing Schwartz function $\phi : \mathbb{R} \to \mathbb{R}_+$ which takes the value 1 for $|\zeta| \leq \frac{1+\delta}{2}$ and vanishes for $|\zeta| \geq 1 - \delta$, the family of functions

$$h_m(\zeta) = \left(\phi\left(\frac{2^{-m}\zeta}{2}\right) - \phi(2^{-m}\zeta)\right)^{\frac{1}{2}} \quad \text{for} \quad m \in \mathbb{Z}$$

trivially satisfy the hypotheses of the lemma. Now, given elements g_1, g_2, \ldots in G and functions f_{g_1}, f_{g_2}, \ldots in $L_p(\mathbb{R}^n_{\text{disc}})$, we set $f = \sum_j \phi_{g_j} \rtimes_\rho \lambda_{\text{H}}(g_j^{-1}, 2^j)$ with $\phi_{g_j} = \gamma_{g_j^{-1}} f_{g_j}$ and recall the identities

$$\sum_{j} \left(\phi_{g_{j}} \rtimes_{\rho} \lambda_{\mathrm{H}}(g_{j}^{-1}, 2^{j}) \right) \left(\phi_{g_{j}} \rtimes_{\rho} \lambda_{\mathrm{H}}(g_{j}^{-1}, 2^{j}) \right)^{*} = \left(\sum_{j} |\phi_{g_{j}}|^{2} \right) \rtimes_{\rho} \mathbf{1}_{\mathcal{L}(\mathrm{H})},$$
$$\sum_{j} \left(\phi_{g_{j}} \rtimes_{\rho} \lambda_{\mathrm{H}}(g_{j}^{-1}, 2^{j}) \right)^{*} \left(\phi_{g_{j}} \rtimes_{\rho} \lambda_{\mathrm{H}}(g_{j}^{-1}, 2^{j}) \right) = \left(\sum_{j} |\gamma_{g_{j}} \phi_{g_{j}}|^{2} \right) \rtimes_{\rho} \mathbf{1}_{\mathcal{L}(\mathrm{H})}.$$

According to Lemma 1.2, we find

$$\|f\|_{L_{p}(\mathcal{L}(\mathbb{R}^{n}_{\mathrm{disc}}\rtimes_{\rho}\mathrm{H}))} \sim \max\left\{ \left\| \left(\sum_{j} |f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p}, \left\| \left(\sum_{j} |\gamma_{g_{j}^{-1}}f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p} \right\}$$

Using the same norm equivalence for $(H_u \rtimes_{\rho} id_H)f = H_u[f]$, we also get

$$\|H_{u}[f]\|_{L_{p}(\mathcal{L}(\mathbb{R}^{n}_{\text{disc}}\rtimes_{\rho}\mathbf{H}))} \sim \max\Big\{\Big\|\Big(\sum_{j}|H_{u}\phi_{g_{j}}|^{2}\Big)^{\frac{1}{2}}\Big\|_{p}, \Big\|\Big(\sum_{j}|\gamma_{g_{j}}H_{u}\phi_{g_{j}}|^{2}\Big)^{\frac{1}{2}}\Big\|_{p}\Big\}.$$

By de Leeuw's theorem, H_u is bounded on $L_p(\mathbb{R}^n_{\text{disc}})$. Thus, the first term on the right hand side for $H_u[f]$ is dominated by the second term on the right hand side for f. Therefore, since $H_u \rtimes_{\rho} id_{\mathrm{H}}$ is L_p -bounded, the second term on the right hand side for $H_u[f]$ must be dominated by the maximum associated to f. Now, using the identity $\gamma_g H_u \phi = H_{\gamma_g(u)} \gamma_g \phi$ we recover the desired inequalities. \Box

Remark 2.2. The full strength of Lemma 1.2 is not necessary to prove Lemma 2.1. One can also combine Bourgain's extension of Littlewood-Paley estimates for UMD Banach spaces [5] together with Lust-Piquard/Pisier's noncommutative Khintchine inequalities [30, 31]. However, we will require Lemma 1.2 later to prove Theorem B, and we find our approach more intrinsic, which could help for future generalizations.

2.2. A partial decompactification. The inequality arising from Lemma 2.1 is stated for L_p -functions on the Bohr compactification of \mathbb{R}^n , with respect to the corresponding normalized Haar measure. In this paragraph we are interested in the form that such an inequality takes in $L_p(\mathbb{R}^n)$ with the Lebesgue measure. Given M > 0 and $f : \mathbb{R}^n \to \mathbb{C}$ supported by $[0, M] \times \cdots \times [0, M]$, we shall write $\pi_M f$ for its M-periodization in the axes directions $\pi_M f(x) = \sum_{k \in \mathbb{Z}^n} f(x - Mk)$. We also consider M-periodizations along g-lattices

$$\pi_{\mathbf{M}}^{g}f(x) = \sum_{k \in \mathbb{Z}^{n}} f\left(x - \mathbf{M}\gamma_{g^{-1}}(k)\right).$$

Let us also recall the L_p norm for almost periodic functions in \mathbb{R}^n

$$\left| \phi \right|_{p} = \left(\lim_{\Lambda \to \infty} \frac{1}{(2\Lambda)^{n}} \int_{[-\Lambda,\Lambda]^{n}} |\phi(x)|^{p} dx \right)^{\frac{1}{p}}.$$

Lemma 2.3. If $2 and <math>H_u \rtimes_{\gamma} id_G : L_p(\widehat{\Gamma_{\text{disc}}}) \to L_p(\widehat{\Gamma_{\text{disc}}})$, then

$$\left\| \left(\sum_{j=1}^{\infty} |H_{\gamma_{g_{j}}(u)}(f_{g_{j}})|^{2} \right)^{\frac{1}{2}} \right\|_{p}$$

$$\leq c_{p} \max \left\{ \left\| \left(\sum_{j=1}^{\infty} |f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p}, \lim_{M \to \infty} M^{\frac{n}{p}} \left\| \left(\sum_{j=1}^{\infty} |\pi^{g_{j}}_{M} \gamma_{g_{j}^{-1}} f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p} \right\}$$

for any sequence g_1, g_2, \ldots in G and any family of functions f_{g_1}, f_{g_2}, \ldots in $L_p(\mathbb{R}^n)$.

Proof. Our argument adapts de Leeuw's decompactification argument in [13]. By density, it suffices to prove such an inequality for a finite family $f_{g_1}, f_{g_2}, \ldots, f_{g_N}$ of compactly supported Schwartz functions. If S_p denotes the Schatten *p*-class, the left hand side is the norm in $L_p(\mathbb{R}^n; S_p)$ of $\sum_j H_{\gamma_j(u)}(f_{g_j}) \otimes e_{j_1}$. In particular, trace duality provides us with a (matrix-valued) compactly supported Schwartz function $h = \sum_j h_j \otimes e_{j_1}$ in the unit ball of $L_q(\mathbb{R}^n; S_q)$ for $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\begin{split} \left\| \left(\sum_{j=1}^{N} |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \right)^{\frac{1}{2}} \right\|_p &\sim \sum_{j=1}^{N} \int_{\mathbb{R}^n} H_{\gamma_{g_j}(u)}(f_{g_j})(x) \overline{h_j(x)} \, dx \\ &= \sum_{j=1}^{N} \int_{\mathbb{R}^n} H_{\widehat{\gamma_{g_j}(u)}(f_{g_j})}(\xi) \overline{\widehat{h_j}(-\xi)} \, d\xi \\ &= \sum_{j=1}^{N} \lim_{M \to \infty} \frac{1}{M^n} \sum_{k \in \frac{1}{M} \mathbb{Z}^n} -i \operatorname{sgn} \langle \gamma_{g_j}(u), k \rangle \widehat{f_{g_j}}(k) \overline{\widehat{h_j}(-k)} \end{split}$$

If $\operatorname{supp} f \subset [-\frac{M}{2}, \frac{M}{2}]^n$, we recall the identity

$$\widehat{f}(k) = \mathcal{M}^n \widehat{\pi_{\mathcal{M}} f}(k)$$

with the first Fourier transform calculated in \mathbb{R}^n and the second in $M\mathbb{T}^n$. This gives

$$\left\| \left(\sum_{j=1}^{N} |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \right)^{\frac{1}{2}} \right\|_p \sim -i \lim_{M \to \infty} M^n \sum_{j,k} \operatorname{sgn} \langle \gamma_{g_j}(u), k \rangle \widehat{\pi_M f_{g_j}}(k) \overline{\widehat{\pi_M h_j}(-k)}.$$

Let us write $\exp_{\xi} : \mathbb{R}^n \to \mathbb{T}$ for the usual characters of \mathbb{R}^n and $\operatorname{b-exp}_{\xi} : \mathbb{R}^n_{\operatorname{bohr}} \to \mathbb{T}$ for the characters of the Bohr compactification. Both families are indexed by the same set, but the latter are defined in a larger group. Define

$$\phi_{\mathbf{M},j} = \sum_{\xi \in \frac{1}{\mathbf{M}} \mathbb{Z}^n} \widehat{\pi_{\mathbf{M}} f_{g_j}}(\xi) \mathbf{b} \cdot \exp_{\xi} \quad \text{and} \quad \varphi_{\mathbf{M},j} = \sum_{\xi \in \frac{1}{\mathbf{M}} \mathbb{Z}^n} \widehat{\pi_{\mathbf{M}} h_j}(\xi) \mathbf{b} \cdot \exp_{\xi}.$$

This yields

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$$\begin{split} -i\sum_{k} \operatorname{sgn}\langle \gamma_{g_{j}}(u), k \rangle \widehat{\pi_{\mathrm{M}} f_{g_{j}}}(k) \overline{\widehat{\pi_{\mathrm{M}} h_{j}}(-k)} &= \sum_{\xi} H_{\gamma_{g_{j}}(u)}(\phi_{\mathrm{M},j})(\xi) \overline{\widehat{\varphi_{\mathrm{M},j}}(-\xi)} \\ &= \int H_{\gamma_{g_{j}}(u)}(\phi_{\mathrm{M},j})(x) \overline{\varphi_{\mathrm{M},j}(x)} \, d\mu(x), \end{split}$$

where now the Hilbert transforms must be understood as operators on the Bohr compactification and μ denotes the corresponding normalized Haar measure. On the other hand, trace duality in $L_2(\mathbb{R}^n_{\text{bohr}}; S_2)$ gives

$$\begin{split} \left| \sum_{j} \int H_{\gamma_{g_{j}}(u)}(\phi_{\mathrm{M},j})(x) \overline{\varphi_{\mathrm{M},j}(x)} \, d\mu(x) \right| \\ &\leq \left\| \sum_{j} H_{\gamma_{g_{j}}(u)}(\phi_{\mathrm{M},j}) \otimes e_{j1} \right\|_{L_{p}(\mathbb{R}^{n}_{\mathrm{bohr}};S_{p})} \left\| \sum_{j} \varphi_{\mathrm{M},j} \otimes e_{j1} \right\|_{L_{q}(\mathbb{R}^{n}_{\mathrm{bohr}};S_{q})} = \mathrm{AB}. \end{split}$$

According to Lemma 2.1, we know that

$$\mathbf{A} \leq c_p \max\left\{ \left\| \sum_{j} \phi_{\mathbf{M},j} \otimes e_{j1} \right\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}};S_p)}, \left\| \sum_{j} \gamma_{g_j^{-1}} \phi_{\mathbf{M},j} \otimes e_{j1} \right\|_{L_p(\mathbb{R}^n_{\mathrm{bohr}};S_p)} \right\}.$$

Let us now recall the identity for trigonometric polynomials

$$\left\|\sum_{\xi} \widehat{f}(\xi) \operatorname{b-exp}_{\xi}\right\|_{L_{p}(\mathbb{R}^{n}_{\operatorname{bohr}})} = \left\|\sum_{\xi} \widehat{f}(\xi) \operatorname{exp}_{\xi}\right\|_{p}$$

which relates Haar integration and mean values. Approximating by trigonometric polynomials and noticing again the identity between L_p -norms of column matrices and L_p -norms of square functions, we find

$$\begin{split} \left\| \sum_{j} \phi_{\mathrm{M},j} \otimes e_{j1} \right\|_{L_{p}(\mathbb{R}^{n}_{\mathrm{bohr}};S_{p})} &= \left\| \left(\sum_{j} |\pi_{\mathrm{M}} f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p} \\ &= \left\| \left(\sum_{j} |\pi_{\mathrm{M}} f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{L_{p}(\mathrm{MT}^{n})} \\ &= \frac{1}{\mathrm{M}^{\frac{n}{p}}} \left\| \left(\sum_{j} |f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{L_{p}(\mathbb{R}^{n})}. \end{split}$$

The second identity follows by M-periodicity and the third one is valid for M large enough, since the f_{g_j} 's are compactly supported. The same argument and the fact that $\sum_j h_j \otimes e_{j1}$ is in the unit ball of $L_q(\mathbb{R}^n; S_q)$ gives $\mathbf{B} = \mathbf{M}^{-\frac{n}{q}}$. Finally, since

$$\begin{split} \gamma_{g_j^{-1}}\phi_{\mathcal{M},j} &= \sum_{\mathcal{M}\xi\in\mathbb{Z}^n}\widehat{\pi_{\mathcal{M}}f_{g_j}}(\xi)\mathbf{b}\text{-}\mathrm{exp}_{\gamma_{g_j^{-1}}\xi} &= \sum_{\mathcal{M}\xi\in\mathbb{Z}^n}\frac{\widehat{f_{g_j}}(\xi)}{\mathcal{M}^n}\mathbf{b}\text{-}\mathrm{exp}_{\gamma_{g_j^{-1}}\xi} \\ &= \sum_{\mathcal{M}\xi\in\gamma_{g_j^{-1}}(\mathbb{Z}^n)}\frac{\widehat{\gamma_{g_j^{-1}}f_{g_j}}(\xi)}{\mathcal{M}^n}\mathbf{b}\text{-}\mathrm{exp}_{\xi} &= \sum_{\mathcal{M}\xi\in\gamma_{g_j^{-1}}(\mathbb{Z}^n)}\pi_{\mathcal{M}}^{\widehat{g_j}}\widehat{\gamma_{g_j^{-1}}}f_{g_j}(\xi)\mathbf{b}\text{-}\mathrm{exp}_{\xi}, \end{split}$$

we obtain the following identity as before

$$\left\|\sum_{j} \gamma_{g_{j}^{-1}} \phi_{\mathrm{M},j} \otimes e_{j1}\right\|_{L_{p}(\mathbb{R}^{n}_{\mathrm{bohr}};S_{p})} = \left\|\left(\sum_{j} |\pi_{\mathrm{M}}^{g_{j}} \gamma_{g_{j}^{-1}} f_{g_{j}}|^{2}\right)^{\frac{1}{2}}\right\|_{p}$$

The assertion now follows by combining the estimates obtained so far.

2.3. Distribution of points in lattice intersections. The decompactification of the last term in Lemma 2.3 requires a more careful analysis. Let us consider the two-dimensional lattice $M\rho_{\alpha}(\mathbb{Z}^2)$ given by an α -rotation of $M\mathbb{Z}^2$. We set π_M^{α} for the corresponding periodization operator and

$$\langle \beta \rangle = \begin{cases} \infty & \text{if } \beta \in \mathbb{R} \backslash \mathbb{Q}, \\ \sqrt{p^2 + q^2} & \text{if } \beta = p/q \in \mathbb{Q}, \end{cases}$$

where p/q is written in reduced form, so that p, q are relatively prime.

Lemma 2.4. Given $z_0 \in \mathbb{R}^2$ and $\delta > 0$

$$\lim_{\mathrm{M}\to\infty}\lim_{\Lambda\to\infty}\frac{\mathrm{M}^2}{4\Lambda^2}\Big|[-\Lambda,\Lambda]^2\cap\pi_{\mathrm{M}}(\mathrm{B}_{\delta}(z_0))\cap\pi_{\mathrm{M}}^{\alpha}(\mathrm{B}_{\delta}(z_0))\Big| \lesssim \frac{\delta^2}{\langle\tan\alpha\rangle}$$

Proof. Classical ergodic theory [14] gives that

$$\operatorname{Orb}_{\mathcal{M}}(\alpha) = \left\{ \left(x, y\right) \operatorname{mod} \mathcal{M} \times \mathcal{M} \mid (x, y) \in \pi^{\alpha}_{\mathcal{M}}(\{z_0\}) \right\}$$

is injective, dense and uniformly distributed in the unit cell $[0, M] \times [0, M]$ when $\tan \alpha \in \mathbb{R} \setminus \mathbb{Q}$. In particular, the following estimate holds for irrational slopes and sufficiently large M

$$\lim_{\Lambda \to \infty} \frac{\mathrm{M}^2}{4\Lambda^2} \Big| [-\Lambda, \Lambda]^2 \cap \pi_{\mathrm{M}}(\mathrm{B}_{\delta}(z_0)) \cap \pi_{\mathrm{M}}^{\alpha}(\mathrm{B}_{\delta}(z_0)) \Big| \lesssim \frac{\delta^4}{\mathrm{M}^2}.$$

Indeed, let \mathcal{Q}_{Λ} be a covering of $[-\Lambda, \Lambda]^2$ by disjoint adjacent $M \times M$ cubes with sides parallel to the axes. Note that $|\mathcal{Q}_{\Lambda}| \sim 4\Lambda^2/M^2$. On the other hand, if we pick a random point σ in $M\rho_{\alpha}(\mathbb{Z}^2)$, the probability that $B_{\delta}(z_0) \cap (B_{\delta}(z_0) + \sigma \mod M \times M)$ is not empty is the same that $z_0 + \sigma \mod M \times M$ belongs to the ball $B_{2\delta}(z_0)$, which in turn is comparable to $\frac{1}{M^2}|B_{2\delta}(z_0)|$ since the π_M^{α} -orbit of z_0 is uniformly distributed in the unit cell. In conclusion, we may rewrite the left-hand side as follows

$$\lim_{\Lambda \to \infty} \frac{1}{|\mathcal{Q}_{\Lambda}|} \sum_{Q \in \mathcal{Q}_{\Lambda}} \Big| Q \cap \pi_{\mathrm{M}}(\mathrm{B}_{\delta}(z_{0})) \cap \pi_{\mathrm{M}}^{\alpha}(\mathrm{B}_{\delta}(z_{0})) \Big|.$$

Translating everything to the unit cell, we get

$$\begin{split} \lim_{\Lambda \to \infty} \frac{1}{|\mathcal{Q}_{\Lambda}|} \sum_{Q \in \mathcal{Q}_{\Lambda}} \left| \mathbf{B}_{\delta}(z_{0}) \cap \left(\pi_{\mathbf{M}}^{\alpha}(\mathbf{B}_{\delta}(z_{0})) \cap Q \mod \mathbf{M} \times \mathbf{M} \right) \right| \\ \lesssim \quad |\mathbf{B}_{\delta}(z_{0})| \operatorname{Prob} \Big\{ \mathbf{B}_{\delta}(z_{0}) \cap (\mathbf{B}_{\delta}(z_{0}) + \sigma \mod \mathbf{M} \times \mathbf{M}) \neq \emptyset \Big\} \ \sim \ \frac{\delta^{4}}{\mathbf{M}^{2}} \end{split}$$

Taking limits in M, we conclude for $\tan \alpha \in \mathbb{R} \setminus \mathbb{Q}$. If $\tan \alpha = p/q$ is rational, set

$$e_1 = \frac{(p,q)}{\sqrt{p^2 + q^2}}$$
 and $e_2 = \frac{(-q,p)}{\sqrt{p^2 + q^2}}$.

Of course, we choose p, q relatively prime. Now we may write

$$\operatorname{Orb}_{\mathcal{M}}(\alpha) = \Big\{ \big(z_0 + j \operatorname{M} e_1 + k \operatorname{M} e_2 \big) \operatorname{mod} \mathcal{M} \times \mathcal{M} \mid j, k \in \mathbb{Z} \Big\}.$$

Therefore, the suborbits generated by Me_1 and Me_2 span $Orb_M(\alpha)$. Note that the suborbit generated by Me_j lives inside a union of segments Σ_k in the unit cell which point in the direction of e_j and such that

$$\sum_k |\Sigma_k| = \mathbf{M}\sqrt{p^2 + q^2}.$$

When this square root is irrational, $\operatorname{Orb}_{M}(\alpha)$ is again injective, dense and uniformly distributed, and the argument for irrational slopes still applies. When it is rational the orbit $\operatorname{Orb}_{M}(\alpha)$ is finite and its cardinality $J_{\alpha} = |\operatorname{Orb}_{M}(\alpha)|$ is independent of M. In particular, we set

$$\Delta_{M}(\alpha) = \inf \left\{ \operatorname{dist}(A, B) \mid A, B \in \operatorname{Orb}_{M}(\alpha) \right\} > 0.$$

Note that $\Delta_M(\alpha) \sim \frac{M}{M_0} \Delta_{M_0}(\alpha)$ for $M \geq M_0$ large enough, so that

$$\lim_{\mathbf{M}\to\infty}\lim_{\Lambda\to\infty}\frac{\mathbf{M}^2}{4\Lambda^2}\Big|[-\Lambda,\Lambda]^2\cap\pi_{\mathbf{M}}(\mathbf{B}_{\delta}(z_0))\cap\pi_{\mathbf{M}}^{\alpha}(\mathbf{B}_{\delta}(z_0))\Big|\lesssim\frac{|\mathbf{B}_{\delta}(z_0)|}{J_{\alpha}}\sim\frac{\delta^2}{J_{\alpha}}$$

since we eventually find $\Delta_{\mathrm{M}}(\alpha) > 2\delta$. However, the suborbit generated by Me₁ contains at least $\sqrt{p^2 + q^2}$ nonequivalent points, more if $\sqrt{p^2 + q^2} \in \mathbb{Q}\setminus\mathbb{Z}$. In particular, we always have $J_{\alpha} \geq \langle \tan \alpha \rangle$ and the proof is complete.

For the last term in Lemma 2.3, we will consider a finite family of pairwise commuting $g_j \in G$, with $1 \leq j \leq N$. In particular, the orthogonal maps γ_{g_j} will admit a simultaneous diagonal form. This means that we may find a direct sum decomposition

$$\mathbb{R}^n = \Delta_{+1} \oplus \Delta_{-1} \oplus \Theta_1 \oplus \Theta_2 \oplus \cdots \oplus \Theta_\ell,$$

where $\Delta_{\pm 1}$ is the direct sum of the eigenspaces with eigenvalues ± 1 and the Θ_k 's are 2-dimensional eigenspaces where the γ_{g_j} 's act by rotations. On the other hand, our functions f_{g_j} will be characteristic functions of prisms

$$A_j = R_j \times [-\lambda, \lambda]^{n-2}$$

with R_j disjoint rectangles living in Θ_1 and certain $\lambda > 0$ so that $\sum_j |A_j| \ge 1$. Let us write ρ_{g_j} and $\rho_{g_j,\perp}$ for the restriction of γ_{g_j} to Θ_1 and its orthocomplement respectively. We write α_{jk} for the rotation angle of the map $\rho_{q_j q_k^{-1}}$.

Lemma 2.5. Given $p_0 \ge 2$, we have

$$\lim_{M \to \infty} M^{\frac{n}{p_0}} \left| \left(\sum_{j=1}^{N} |\pi_M^{g_j} \gamma_{g_j^{-1}} \chi_{A_j}|^2 \right)^{\frac{1}{2}} \right|_{p_0} \sim \left| \bigcup_{j=1}^{N} A_j \right|^{\frac{1}{p_0}}.$$

provided that $\langle \tan \alpha_{jk} \rangle \ge N^{p_0+1} \lambda^{n-2} \operatorname{diam}^2 \left(\bigcup_{j=1}^N \rho_j^{-1}(R_j) \right)$ for any $1 \le j \ne k \le N$.

Proof. If M is large enough $\pi_{M}^{g_{j}}\gamma_{g_{j}}^{-1}\chi_{A_{j}} = \chi_{B_{M,j}}$, where $B_{M,j}$ is the periodization of $\gamma_{g_{j}}^{-1}(A_{j})$ along $M\gamma_{g_{j}}^{-1}(\mathbb{Z}^{n})$. If the periodized sets $B_{M,j}$ were pairwise disjoint for all $M \geq M_{0}$, the assertion would trivially follow by direct computation. In the presence of overlapping

$$\mathbf{M}^{\frac{n}{p_{0}}} \left\| \left(\sum_{j=1}^{N} |\pi_{\mathbf{M}}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p_{0}} \geq \left\| \bigcup_{j=1}^{N} A_{j} \right\|^{\frac{1}{p_{0}}}$$

for fixed M and $p_0 \ge 2$. In particular, it suffices to prove the reverse inequality. The idea is to show that the overlapping becomes small enough for large M. Consider the partition $[-\Lambda, \Lambda]^n = \Phi_{M,\Lambda} \cup \Psi_{M,\Lambda}$ with

$$\Psi_{\mathcal{M},\Lambda} = \bigcup_{1 \le j \ne k \le \mathcal{N}} \Big([-\Lambda,\Lambda]^n \cap \pi_{\mathcal{M}}^{g_j}(\gamma_{g_j^{-1}}(A_j)) \cap \pi_{\mathcal{M}}^{g_k}(\gamma_{g_k^{-1}}(A_k)) \Big),$$

the set where overlapping of the $B_{M,j}$'s occur in $[-\Lambda, \Lambda]^n$. This yields

$$\begin{split} \lim_{M \to \infty} M^{n} \left| \left(\sum_{j=1}^{N} |\pi_{M}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}|^{2} \right)^{\frac{1}{2}} \right|_{p_{0}}^{p_{0}} \\ &= \lim_{M \to \infty} \lim_{\Lambda \to \infty} \frac{M^{n}}{2^{n} \Lambda^{n}} \int_{\Phi_{M,\Lambda}} \sum_{j=1}^{N} \pi_{M}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}(x) \, dx \\ &+ \lim_{M \to \infty} \lim_{\Lambda \to \infty} \frac{M^{n}}{2^{n} \Lambda^{n}} \int_{\Psi_{M,\Lambda}} \left(\sum_{j=1}^{N} \pi_{M}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}(x) \right)^{\frac{p_{0}}{2}} \, dx \\ &\leq \sum_{j=1}^{N} |A_{j}| + N^{\frac{p_{0}}{2} - \frac{1}{2}} \lim_{M,\Lambda \to \infty} \frac{M^{n}}{2^{n} \Lambda^{n}} \int_{\Psi_{M,\Lambda}} \left(\sum_{j=1}^{N} \pi_{M}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}(x) \right)^{\frac{1}{2}} \, dx \\ &\leq \sum_{j=1}^{N} |A_{j}| + N^{\frac{p_{0}}{2} - \frac{1}{2}} \lim_{M,\Lambda \to \infty} \frac{M^{n}}{2^{n} \Lambda^{n}} |\Psi_{M,\Lambda}|^{\frac{1}{2}} \left(\int_{[-\Lambda,\Lambda]^{n}} \sum_{j=1}^{N} \pi_{M}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}(x) \, dx \right)^{\frac{1}{2}}, \end{split}$$

where we have used the absence of overlapping in $\Phi_{M,\Lambda}$ to eliminate the power $p_0/2$ and M-periodicity of the $\pi_M^{g_j} \gamma_{g_j}^{-1} \chi_{A_j}$'s (isolatedly) for the first term. We may also use periodicity to estimate the last term on the right-hand side

$$\begin{split} \lim_{\mathcal{M}\to\infty} \mathcal{M}^{n} \left| \left(\sum_{j=1}^{\mathcal{N}} |\pi_{\mathcal{M}}^{g_{j}} \gamma_{g_{j}^{-1}} \chi_{A_{j}}|^{2} \right)^{\frac{1}{2}} \right|_{p_{0}}^{p_{0}} \\ &\leq \sum_{j=1}^{\mathcal{N}} |A_{j}| + \mathcal{N}^{\frac{p_{0}}{2} - \frac{1}{2}} \left(\sum_{j=1}^{\mathcal{N}} |A_{j}| \right)^{\frac{1}{2}} \left(\lim_{\mathcal{M},\Lambda\to\infty} \frac{\mathcal{M}^{n}}{2^{n}\Lambda^{n}} |\Psi_{\mathcal{M},\Lambda}| \right)^{\frac{1}{2}} \\ &\leq \left[1 + \mathcal{N}^{\frac{p_{0}}{2} - \frac{1}{2}} \left(\lim_{\mathcal{M}\to\infty} \lim_{\Lambda\to\infty} \frac{\mathcal{M}^{n}}{2^{n}\Lambda^{n}} |\Psi_{\mathcal{M},\Lambda}| \right)^{\frac{1}{2}} \right] \left| \bigcup_{j=1}^{\mathcal{N}} A_{j} \right|, \end{split}$$

since $\sum_j |A_j| \ge 1$ and $A_j \cap A_k = \emptyset$ for $j \ne k$. We write $\pi_{\mathrm{M}}^{\alpha_j}$ and $\pi_{\mathrm{M},\perp}^j$ for the M-periodization operators in Θ_1 and its orthocomplement respectively associated to ρ_{g_j} (with rotation angle α_j) and $\rho_{g_j,\perp}$. According to the form of A_j we get the inequality

$$|\Psi_{\mathbf{M},\Lambda}| \leq \sum_{1 \leq j \neq k \leq \mathbf{N}} |\Psi^1_{\mathbf{M},\Lambda}(j,k)| \times |\Psi^2_{\mathbf{M},\Lambda}(j,k)|$$

with the sets on the right given by

$$\begin{split} \Psi^{1}_{\mathcal{M},\Lambda}(j,k) &= [-\Lambda,\Lambda]^{2} \cap \pi^{\alpha_{j}}_{\mathcal{M}}(\rho_{g_{j}}^{-1}(R_{j})) \cap \pi^{\alpha_{k}}_{\mathcal{M}}(\rho_{g_{k}}^{-1}(R_{k})), \\ \Psi^{2}_{\mathcal{M},\Lambda}(j,k) &= [-\Lambda,\Lambda]^{n-2} \cap \pi^{j}_{\mathcal{M},\perp}(\rho_{g_{j},\perp}^{-1}([-\lambda,\lambda]^{n-2})) \cap \pi^{k}_{\mathcal{M},\perp}(\rho_{g_{k},\perp}^{-1}([-\lambda,\lambda]^{n-2})). \end{split}$$

Pick $z_0 \in \Theta_1$ and $\delta > 0$ minimal so that $\rho_{g_j}^{-1}(R_j) \subset B_{\delta}(z_0)$ for all j. By Lemma 2.4 and the hypothesis,

$$\begin{split} \lim_{\mathbf{M}\to\infty} \lim_{\Lambda\to\infty} \frac{\mathbf{M}^2}{4\Lambda^2} |\Psi^1_{\mathbf{M},\Lambda}(j,k)| \\ &\leq \lim_{\mathbf{M}\to\infty} \lim_{\Lambda\to\infty} \frac{\mathbf{M}^2}{4\Lambda^2} \Big| [-\Lambda,\Lambda]^2 \cap \pi_{\mathbf{M}}^{\alpha_j}(\mathbf{B}_{\delta}(z_0)) \cap \pi_{\mathbf{M}}^{\alpha_k}(\mathbf{B}_{\delta}(z_0)) \Big| \lesssim \frac{1}{\mathbf{N}^{p_0+1}\lambda^{n-2}}. \end{split}$$

Moreover, eliminating the dependence on k for $\Psi^2_{M,\Lambda}(j,k)$ gives the upper bound

$$\lim_{M \to \infty} \lim_{\Lambda \to \infty} \frac{M^{n-2}}{2^{n-2}\Lambda^{n-2}} |\Psi_{M,\Lambda}^2(j,k)| \\
\leq \lim_{M \to \infty} \lim_{\Lambda \to \infty} \frac{M^{n-2}}{2^{n-2}\Lambda^{n-2}} \Big| [-\Lambda,\Lambda]^{n-2} \cap \pi_{M,\perp}^j \big(\rho_{g_j,\perp}^{-1}([-\lambda,\lambda]^{n-2}) \big) \Big| \lesssim \lambda^{n-2}.$$

Altogether, we have $\lim_{M,\Lambda} \frac{M^n}{2^n \Lambda^n} |\Psi_{M,\Lambda}| \lesssim N^{1-p_0}$ and the assertion follows.

2.4. Infinite orbits admit Kakeya shadows. We continue by introducing a class of sets in the unit sphere \mathbb{S}^{n-1} for which the inequality in Lemma 2.3 fails and show that infinite orbits of arbitrary discrete groups belong to this class. Our definition is motivated by Fefferman's construction [16]. Given a great circle ζ and any set of directions Ω in \mathbb{S}^{n-1} , we write π_{ζ} for the orthogonal projection onto the plane determined by ζ . Let

$$\Omega_{\zeta} \,=\, \Big\{ \frac{\pi_{\zeta}(\omega)}{|\pi_{\zeta}(\omega)|} \,\,\big|\,\,\omega \in \Omega \setminus \zeta^{\perp} \Big\} \,\subset\, \zeta$$

denote the geodesic projection of Ω onto ζ . Let $\mathcal{R}_{\Omega_{\zeta}}$ denote the collection of rectangles R in the plane determined by ζ with longest side pointing in a direction

of Ω_{ζ} . The expression 3R will refer to the rectangle with the same center and width as R, but with 3 times the length. We will say that Ω admits Kakeya shadows if there exists an absolute constant c_0 such that for each $m \geq 1$ we may find a great circle $\zeta(m)$, a measurable set E_m in the plane determined by $\zeta(m)$, and a finite collection of pairwise disjoint rectangles $\Sigma_{\Omega_{\zeta(m)}} \subset \mathcal{R}_{\Omega_{\zeta(m)}}$ so that

a)
$$|E_m| \leq \frac{c_0}{m} \sum_{R \in \Sigma_{\Omega_{\zeta(m)}}} |R|,$$

b) $|R| \leq c_0 |(3R \setminus R) \cap E_m|$ for each $R \in \Sigma_{\Omega_{\zeta(m)}}.$

The existence of sets admitting Kakeya shadows is a consequence of Besicovitch construction [4]. For instance, a set Ω admits Kakeya shadows whenever there exists a shadow Ω_{ζ} which is dense in the unit circle \mathbb{S}^1 . Let us now go back to the framework of our problem. Given a discrete group G, a unit vector $u \in \mathbb{R}^n$ and an orthogonal representation $\gamma : \mathbf{G} \to O(n)$, we are interested in knowing when the γ -orbit of u admits Kakeya shadows.

Lemma 2.6. The orbit $\mathcal{O}_{\gamma}(u)$ in \mathbb{S}^{n-1} is either finite or admits Kakeya shadows.

Proof. As described above, it suffices to prove the stronger statement that infinite orbits always admit a dense shadow. The argument is simple when n = 2. Assume the orbit $\mathcal{O}_{\gamma}(u)$ is not finite. By compactness, it must accumulate at some point $\sigma \in \mathbb{S}^1$ and we may find for each $\varepsilon > 0$ group elements $g_{\varepsilon}, h_{\varepsilon} \in G$ such that $|\gamma_{g_{\varepsilon}}(u) - \gamma_{h_{\varepsilon}}(u)| < \varepsilon$ and $\det \gamma_{g_{\varepsilon}} = \det \gamma_{h_{\varepsilon}}$. It follows that $\gamma_{g_{\varepsilon}^{-1}h_{\varepsilon}}$ is a rotation of angle $< \varepsilon$. Density of $\mathcal{O}_{\gamma}(u)$ follows iterating these maps for ε arbitrarily small. A similar argument applies when G is abelian and n is arbitrary. In that case, the γ_g 's are pairwise commuting maps and we may consider again the direct sum decomposition into common eigenspaces $\mathbb{R}^n = \Delta_{\pm 1} \oplus \Theta_1 \oplus \Theta_2 \oplus \cdots \oplus \Theta_{\ell}$, where the γ_g 's act by rotations on two-dimensional Θ_j 's. Let us decompose u as $v_{\delta} + \sum_j v_j$ with $(v_{\delta}, v_j) \in \Delta_{\pm 1} \times \Theta_j$. If $\mathcal{O}_{\gamma}(u)$ is not finite, there must exist $1 \leq j_0 \leq \ell$ with $v_{j_0} \neq 0$ and $|\mathcal{O}_{\gamma}(v_{j_0})| = \infty$. Taking ζ_0 to be the great circle in \mathbb{S}^{n-1} generating Θ_{j_0} , it is clear that the shadow $\mathcal{O}_{\gamma}(u)_{\zeta_0}$ contains $\mathcal{O}_{\gamma}(v_{j_0})$. Moreover, $\mathcal{O}_{\gamma}(v_{j_0})$ is an infinite orbit generated by rotations in Θ_{j_0} , so that it is dense in ζ_0 as in the n = 2 case considered above.

Now that we know the statement holds for discrete abelian groups, it suffices to show that infinite orbits of discrete groups always admit infinite suborbits generated by an abelian subgroup. To see this, consider the linear subspace

$$\mathcal{J}_{\gamma} = \left\{ \xi \in \mathbb{R}^n \, \big| \, \gamma_{g^k}(\xi) = \xi \text{ for all } g \in \mathcal{G} \text{ and some } k = k(g,\xi) \ge 1 \right\}.$$

If $u \notin \mathcal{J}_{\gamma}$, then we may find $g_0 \in \mathbf{G}$ such that $\gamma_{g_0}^j(u) \neq \gamma_{g_0}^k(u)$ for any pair of integers $j \neq k$. In that case, $\mathbf{A} = \langle g_0^k : k \in \mathbb{Z} \rangle \simeq \mathbb{Z}$ is an abelian subgroup of \mathbf{G} giving rise to an infinite suborbit $\mathcal{O}_{\gamma}(\mathbf{A}, u)$, as desired. On the other hand, if $u \in \mathcal{J}_{\gamma}$ we note that \mathcal{J}_{γ} is a \mathbf{G} -module. Indeed, given a pair $(g_0, \xi_0) \in \mathbf{G} \times \mathcal{J}_{\gamma}$ and any $g \in \mathbf{G}$, define $h_0 = g_0^{-1}gg_0$ and $k_0 = k(h_0, \xi_0)$. Then we have by definition

$$\gamma_{h_{0}^{k_{0}}}(\xi_{0}) = \xi_{0} \Rightarrow \gamma_{g^{k_{0}}}(\gamma_{g_{0}}(\xi_{0})) = \gamma_{g_{0}}(\xi_{0}).$$

This shows that $\gamma_{g_0}(\xi_0) \in \mathcal{J}_{\gamma}$ and proves our assertion above. In particular, since $u \in \mathcal{J}_{\gamma}$ the whole orbit $\mathcal{O}_{\gamma}(u)$ lies in \mathcal{J}_{γ} , which in turn is an invariant subspace

of the representation γ . Restricting to that subspace if necessary, we may assume that $\mathcal{J}_{\gamma} = \mathbb{R}^n$. Let e_1, e_2, \ldots, e_n denote an orthonormal basis of \mathbb{R}^n . Writing m = m(g) for the least common multiple of $k(g, e_1), k(g, e_2), \ldots, k(g, e_n)$ it is easily checked that $\gamma_{g^m}(\xi) = \xi$ for all $\xi \in \mathbb{R}^n$. Let us consider the normal subgroup $H = \{g \in G \mid \gamma_g(\xi) = \xi \text{ for all } \xi\}.$ Again, since $\gamma_{|_H}$ acts trivially on \mathbb{R}^n , we may restrict to the quotient group G/H or equivalently assume that $H = \{e\}$. This means that $q^m = e$ for all $q \in G$ and some exponent $m = m(q) \ge 1$, so that G is a torsion group. According to the Jordan-Schur theorem, any torsion subgroup of the group of $n \times n$ complex matrices is virtually abelian. In other words, G admits an abelian normal subgroup A of finite order. If we now consider the suborbit $\mathcal{O}_{\gamma}(\mathbf{A}, u)$, it must be infinite since $\mathcal{O}_{\gamma}(u)$ is infinite and A is of finite order.

Lemma 2.7. Let $2 , then <math>\mathcal{O}_{\gamma}(u)$ is finite if and only if

$$\left\| \left(\sum_{j=1}^{\infty} |H_{\gamma_{g_j}(u)}(f_{g_j})|^2 \right)^{\frac{1}{2}} \right\|_p$$

$$\leq c_p \max \left\{ \left\| \left(\sum_{j=1}^{\infty} |f_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p, \lim_{M \to \infty} M^{\frac{n}{p}} \left\| \left(\sum_{j=1}^{\infty} |\pi_M^{g_j} \gamma_{g_j^{-1}} f_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p \right\},$$

for any sequence g_1, g_2, \ldots in G and any family of functions f_{g_1}, f_{g_2}, \ldots in $L_p(\mathbb{R}^n)$.

Proof. The validity of such an inequality for finite orbits and 1 followsfrom the L_p -boundedness of directional Hilbert transforms. Assume now that $\mathcal{O}_{\gamma}(u)$ is not finite and the inequality in the statement holds for some $p_0 > 2$. According to the proof of Lemma 2.6, there exists an abelian subgroup A such that $\mathcal{O}_{\gamma}(A, u)$ is infinite. In other words, we may assume that G itself is abelian. Decomposing γ into its irreducible parts as above, we get $\mathbb{R}^n = \Delta_{\pm 1} \oplus \Theta_1 \oplus \Theta_2 \oplus \cdots \oplus \Theta_\ell$ with $u = v_{\delta} + \sum_{j} v_{j}$. Pick $1 \leq j_0 \leq \ell$ such that $v_{j_0} \neq 0$ and $|\mathcal{O}_{\gamma}(v_{j_0})| = \infty$, and consider the great circle $\zeta_{j_0} = \Theta_{j_0} \cap \mathbb{S}^{n-1}$. We claim that there exists

- A family $\{g_{km} | m \ge 1, 1 \le k \le 2^{2^m}\}$ in G,
- Rectangles R_{km} and measurable sets E_m in Θ_{i_0} ,

satisfying the following conditions

- a) R_{km} points in the direction of $\pi_{\zeta_{j_0}}(\gamma_{g_{km}}(u))$,
- b) $|E_m| \lesssim \frac{1}{m} \sum_k |R_{km}| \lesssim \frac{1}{m} \sum_k |(3R_{km} \setminus R_{km}) \cap E_m|,$
- c) $R_{jm} \cap R_{km} = \emptyset = \gamma_{g_{jm}^{-1}}(R_{jm}) \cap \gamma_{g_{km}^{-1}}(R_{km})$ for $1 \le j \ne k \le 2^{2^m}$, d) The sets $\{(g_{km}, R_{km})\}_k$ satisfy Lemma 2.5 with $N = 2^{2^m}$ for each $m \ge 1$.

We finish the argument before proving the claim. Fix $m \ge 1$ in what follows and consider an orthonormal basis e_1, e_2, \ldots, e_n so that span $\{e_1, e_2\} = \Theta_{j_0}$. Let us write $\gamma_{g_{km}}(u)_j$ for $\langle \gamma_{g_{km}}(u), e_j \rangle$ and set

$$\lambda = 1 + \max\left\{ \operatorname{length}(R_{km}) \frac{|\gamma_{g_{km}}(u)_j|}{\sqrt{\gamma_{g_{km}}(u)_1^2 + \gamma_{g_{km}}(u)_2^2}} \mid 1 \le k \le 2^{2^m}, \ 3 \le j \le n \right\}.$$

We consider the following prisms in \mathbb{R}^n

$$A_{km} = \left(R_{km} \times [-5\lambda, 5\lambda]^{n-2} \right),$$

$$B_{km} = \left((3R_{km} \setminus R_{km}) \times [-\frac{1}{2}\lambda, \frac{1}{2}\lambda]^{n-2} \right).$$

Taking $f_{g_{km}} = \chi_{A_{km}}$, we have $4|H_{\gamma_{g_{km}}(u)}(f_{g_{km}})| \geq \chi_{B_{km}}$. Indeed, by translational and rotational invariance we may assume that the prism A_{km} is centered at 0 and $\gamma_{g_{km}}(u)_1 = 0$. Now, given $x \in B_{km}$, we have

$$\chi_{A_{km}}(x - t\gamma_{g_{km}}(u)) = \chi_{[-\alpha,\alpha]}(x_1)\chi_{[-\beta,\beta]}(x_2 - t\gamma_{g_{km}}(u)_2) \prod_{j=3}^n \chi_{[-5\lambda,5\lambda]}(x_j - t\gamma_{g_{km}}(u)_j)$$

for some $0 < \alpha \leq \beta = \text{length}(R_{km})$. It is clear that $|x_1| \leq \alpha$ for $x \in B_{km}$. On the other hand, the right hand side vanishes unless $|x_2 - t\gamma_{g_{km}}(u)_2| \leq \beta$. This implies in turn that

$$|t\gamma_{g_{km}}(u)_{2}| \leq \beta + |x_{2}| \leq 4\beta$$

$$\downarrow$$

$$x_{j} - t\gamma_{g_{km}}(u)_{j}| \leq |x_{j}| + |t\gamma_{g_{km}}(u)_{j}| \leq \frac{1}{2}\lambda + \frac{1}{\beta}|t\gamma_{g_{km}}(u)_{2}|\lambda \leq \frac{9}{2}\lambda$$

Hence, $\chi_{A_{km}}(x - t\gamma_{g_{km}}(u)) = \chi_{[-\beta,\beta]}(x_2 - t\gamma_{g_{km}}(u)_2)$ for $x \in B_{km}$ and we obtain

$$H_{\gamma_{g_{km}}(u)}(f_{g_{km}})(x) = H(\chi_{[-\beta,\beta]})(x_2) = \frac{1}{\pi} \log \left| \frac{x_2 + \beta}{x_2 - \beta} \right| \quad \text{for} \quad x \in B_{km}$$

The last logarithm is greater than $\frac{\pi}{4}$ for $\beta \leq |x_2| \leq 3\beta$, which gives the desired estimate $4|H_{\gamma_{g_{km}}(u)}(f_{g_{km}})| \geq \chi_{B_{km}}$. Using this, property b) above, and Hölder's inequality we obtain

$$\begin{split} \lambda^{n-2} \sum_{k=1}^{2^{2^m}} |R_{km}| &\lesssim \quad \lambda^{n-2} \sum_{k=1}^{2^{2^m}} |(3R_{km} \setminus R_{km}) \cap E_m| \\ &= \sum_{k=1}^{2^{2^m}} |B_{km} \cap \left(E_m \times [-\frac{1}{2}\lambda, \frac{1}{2}\lambda]^{n-2}\right)| \\ &\lesssim \quad \int_{E_m} \int_{[-\frac{1}{2}\lambda, \frac{1}{2}\lambda]^{n-2}} \sum_{k=1}^{2^{2^m}} |H_{\gamma_{g_{km}}(u)}(f_{g_{km}})(x)|^2 \, dx \\ &\leq \quad |E_m|^{1-\frac{2}{p_0}} \lambda^{(n-2)(1-\frac{2}{p_0})} \left\| \left(\sum_{k=1}^{2^{2^m}} |H_{\gamma_{g_{km}}(u)}(f_{g_{km}})|^2\right)^{\frac{1}{2}} \right\|_{p_0}^2 \\ &\lesssim \quad \frac{1}{m^{1-\frac{2}{p_0}}} \left(\lambda^{n-2} \sum_{k=1}^{2^{2^m}} |R_{km}|\right)^{1-\frac{2}{p_0}} \left\| \left(\sum_{k=1}^{2^{2^m}} |H_{\gamma_{g_{km}}(u)}(f_{g_{km}})|^2\right)^{\frac{1}{2}} \right\|_{p_0}^2 \end{split}$$

On the other hand, properties c) and d) give

$$\max\left\{ \left\| \left(\sum_{k=1}^{2^{2^{m}}} |f_{g_{km}}|^{2}\right)^{\frac{1}{2}} \right\|_{p_{0}}^{2}, \lim_{M \to \infty} M^{\frac{2n}{p_{0}}} \right\| \left(\sum_{k=1}^{2^{2^{m}}} |\pi_{M}^{g_{km}}\gamma_{g_{km}^{-1}}f_{g_{km}}|^{2}\right)^{\frac{1}{2}} \right\|_{p_{0}}^{2} \right\}$$
$$\lesssim \left(\sum_{k=1}^{2^{2^{m}}} |R_{km} \times [-5\lambda, 5\lambda]^{n-2}|\right)^{\frac{2}{p_{0}}} = (10)^{\frac{2(n-2)}{p_{0}}} \left(\lambda^{n-2}\sum_{k=1}^{2^{2^{m}}} |R_{km}|\right)^{\frac{2}{p_{0}}}$$

with constants independent of m. Combining the two estimates via the inequality in the statement of the lemma, we get a contradiction for m large enough. Therefore it suffices to prove our claim at the beginning of the proof. If $s = s(m) = 2^m$, let us write F_m and $S_{1m}, S_{2m}, \ldots, S_{2^sm}$ for the measurable set E and the pairwise disjoint rectangles $R_1, R_2, \ldots, R_{2^s}$ which appear in [17, Lemma 10.1.1]. Then we find

$$|F_m| \lesssim \frac{1}{m} \sum_k |S_{km}| \lesssim \frac{1}{m} \sum_k |(3S_{km} \setminus S_{km}) \cap F_m|.$$

Of course, these properties remain stable under affine transformations and we may assume that S_{1m} points in the direction $\pi_{\zeta_{j_0}}(u)$. Now, recall that the ζ_{j_0} -shadow of $\mathcal{O}_{\gamma}(u)$ contains $\mathcal{O}_{\gamma}(v_{j_0})$, which in turn is dense in ζ_{j_0} . Therefore, we may pick the group elements g_{km} so that $\pi_{\zeta_{j_0}}(\gamma_{g_{km}}(u))$ points in the direction of S_{km} , or at least as close to it as we want. Note that Fefferman's construction admits small perturbations, so that we can move the rectangles slightly to make them point in the directions we pick. Our final choice of directions will be determined at the end of the proof. If we choose $R'_{km} \sim S_{km}$, it turns out that properties b) and c) hold except possibly for the pairwise disjointness condition

$$\gamma_{g_{jm}^{-1}}(R'_{jm}) \cap \gamma_{g_{km}^{-1}}(R'_{km}) = \emptyset \text{ for } 1 \le j \ne k \le 2^{2^m}.$$

Note however that the given construction is still invariant under translations. In particular, we may take $R'_{km} = \tau(S_{km})$ for a suitable translation τ in Θ_{j_0} . Assume for clarity that $u = e_1$ and take $S_{1m} \sim [0, 3\log(s+2)] \times [0, 2^{-s}]$, so that we roughly have $S_{km} \sim \gamma_{g_{km}}(S_{1m})$. This is essentially the worst possible configuration, since we find a large overlapping for the family $\gamma_{g_{km}^{-1}}(S_{km})$. Taking however $R'_{km} = \tau(S_{km})$ with $\tau(x, y) = (x + L, y)$ for L > 0 large enough, it is easily checked that we get the missing disjointness condition in c). In summary, the rectangles R'_{km} satisfy conditions b) and c) which in turn are stable under small perturbations. It remains to select the g_{km} 's and small perturbations R_{km} so that conditions a) and d) also hold. Since the g_{km} 's are pairwise commuting because G can be taken abelian, it suffices to check that

- $R_{km} \parallel \pi_{\zeta_{j_0}}(\gamma_{g_{km}}(u)),$
- $\sum_{k} |A_{km}| \ge 1$ for all $m \ge 1$, • $\langle \tan \alpha_{jk}^{m} \rangle \ge 2^{s(p_0+1)} \lambda^{n-2} \operatorname{diam}^2 \left(\bigcup_{k=1}^{2^s} \gamma_{g_{km}}^{-1}(R_{km}) \right) = J(p_0, m, \lambda).$

Since $\lambda > 1$ by construction and $|R_{km}| \sim 2^{-s} \log s$, we see that

$$\sum_{k=1}^{2^{s}} |A_{km}| \ge \sum_{k=1}^{2^{s}} |R_{km}| \gtrsim \log s \sim m >> 1.$$

Finally, we select the g_{km} 's so that the first and third conditions above hold. Pick $g_{1m} = e$, so that $R_{1m} = R'_{1m}$ points in the direction of $\pi_{\zeta_{j_0}}(u)$. Then we have to pick g_{2m} so that $\langle \tan \alpha_{12}^m \rangle \geq J(p_0, m, \lambda)$ and $\pi_{\zeta_{j_0}}(\gamma_{g_{2m}}(u))$ is close enough to the direction of S_{2m} , so that R_{2m} will be a small perturbation of $\tau(S_{2m})$ pointing in the direction of $\pi_{\zeta_{j_0}}(\gamma_{2m}(u))$. This is possible since the ζ_{j_0} -shadow of $\mathcal{O}_{\gamma}(u)$ is dense in ζ_{j_0} and $\langle \tan \alpha_{12}^m \rangle \leq J(p_0, m, \lambda)$ holds for finitely many directions. Once g_{1m} and g_{2m} are fixed, pick g_{3m} with $\pi_{\zeta_{j_0}}(\gamma_{g_{3m}}(u))$ close enough to the direction of S_{3m} and

such that $\langle \tan \alpha_{j3}^m \rangle \geq J(p_0, m, \lambda)$ for j = 1, 2. Since the latter inequality fails for finitely many directions, again this is possible by density. Iterating the process we obtain the desired construction and the proof is complete.

Proof of ii) \Rightarrow **iii) in Theorem A.** Since twisted Hilbert transforms are Fourier multipliers on the group von Neumann algebra associated to Γ_{disc} , it is easily checked that $H_u \rtimes_{\gamma} id_{\text{G}}$ is self-adjoint (up to conjugation of the symbol) so that we may assume 2 . Then we combine Lemmas 2.3 and 2.7.

2.5. Twisted de Leeuw's compactification. Let us now prove the implication i) \Rightarrow ii) in Theorem A. We will write H_u and $H_{b,u}$ in this paragraph to distinguish between the *u*-directional Hilbert transform on \mathbb{R}^n and its Bohr compactification respectively. Assume $H_u \rtimes_{\gamma} id_G$ is L_p -bounded for some $1 . By density of trigonometric polynomials, it suffices to prove the <math>L_p$ -boundedness of $H_{b,u} \rtimes_{\gamma} id_G$ for finite sums of the form

$$f_{\mathbf{b}} = \sum_{\xi,g} \widehat{f}(\xi,g) \operatorname{b-exp}_{\xi} \rtimes_{\gamma} \lambda_{\mathbf{G}}(g).$$

Once we have fixed $f_{\rm b}$, we find $(H_{{\rm b},u} \rtimes_{\gamma} id_{\rm G})f_{\rm b} = (\widetilde{H}_{{\rm b},u} \rtimes_{\gamma} id_{\rm G})f_{\rm b}$, where

$$H_{\mathbf{b},u}(\mathbf{b}-\exp_{\xi}) = -i \widetilde{\operatorname{sgn}} \langle u, \xi \rangle \mathbf{b}-\exp_{\xi}$$

and $\widetilde{\operatorname{sgn}}$ is a smoothing of the sign function which coincides with it on the finitely many values $\langle u, \xi \rangle$ with ξ appearing in f_{b} . We will write \widetilde{H}_u for the corresponding smoothing operator in \mathbb{R}^n . Given $\delta > 0$, set $h_{\delta}(x) = (2\pi\delta)^{-n/2} \exp(-|x|^2/2\delta)$ and we claim that

$$\begin{split} \left\| (H_{\mathbf{b},u} \rtimes_{\gamma} id_{\mathbf{G}})f_{\mathbf{b}} \right\|_{L_{p}(\widehat{\Gamma_{\mathrm{disc}}})} &= \left\| (H_{\mathbf{b},u} \rtimes_{\gamma} id_{\mathbf{G}})f_{\mathbf{b}} \right\|_{L_{p}(\widehat{\Gamma_{\mathrm{disc}}})} \\ &= \lim_{\delta \to \infty} \left\| -ih_{\delta}^{\frac{1}{p}} \sum_{\xi,g} \widetilde{\mathrm{sgn}} \langle u, \xi \rangle \widehat{f}(\xi,g) \exp_{\xi} \rtimes_{\gamma} \lambda_{\mathbf{G}}(g) \right\|_{L_{p}(\widehat{\Gamma})} \\ &= \lim_{\delta \to \infty} \left\| (H_{u} \rtimes_{\gamma} id_{\mathbf{G}}) \left(h_{\delta}^{\frac{1}{p}} \sum_{\xi,g} \widehat{f}(\xi,g) \exp_{\xi} \rtimes_{\gamma} \lambda_{\mathbf{G}}(g) \right) \right\|_{L_{p}(\widehat{\Gamma})} \\ &= \lim_{\delta \to \infty} \left\| (H_{u} \rtimes_{\gamma} id_{\mathbf{G}}) \left(h_{\delta}^{\frac{1}{p}} \sum_{\xi,g} \widehat{f}(\xi,g) \exp_{\xi} \rtimes_{\gamma} \lambda_{\mathbf{G}}(g) \right) \right\|_{L_{p}(\widehat{\Gamma})} \\ &\lesssim \lim_{\delta \to \infty} \left\| h_{\delta}^{\frac{1}{p}} \sum_{\xi,g} \widehat{f}(\xi,g) \exp_{\xi} \rtimes_{\gamma} \lambda_{\mathbf{G}}(g) \right\|_{L_{p}(\widehat{\Gamma})} \\ &= \| f_{\mathbf{b}} \|_{L_{p}(\widehat{\Gamma})}. \end{split}$$

Since the inequality follows by hypothesis, the constants are independent of the smoothing. It remains to justify the identities. If $f = \sum_{\xi,g} \hat{f}(\xi,g) \exp_{\xi} \rtimes_{\gamma} \lambda_{G}(g)$ stands for the trigonometric polynomial in $\mathcal{L}(\Gamma)$ with the same Fourier coefficients as $f_{\rm b}$, the second and fifth identities follow from

$$\|f_{\mathbf{b}}\|_{L_{p}(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}})} = \lim_{\delta \to \infty} \|h_{\delta}^{\frac{1}{p}}f\|_{L_{p}(\widehat{\mathbf{\Gamma}})}$$

The proof reduces to p = 1 since $h_{\delta} \rtimes_{\gamma} \mathbf{1}$ is in the center of $\mathcal{L}(\Gamma_{\text{disc}})$, so

$$\|f_{\mathbf{b}}\|_{L_{p}(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}})}^{p} = \left\|\|f_{\mathbf{b}}\|^{p}\right\|_{L_{1}(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}})} = \lim_{\delta \to \infty} \left\|h_{\delta}|f|^{p}\right\|_{L_{1}(\widehat{\mathbf{\Gamma}})} = \lim_{\delta \to \infty} \left\|h_{\delta}^{\frac{1}{p}}f\right\|_{L_{p}(\widehat{\mathbf{\Gamma}})}^{p}.$$

Approximating $|f_{\rm b}|^p$ by a finite sum, it suffices to note that

$$\int_{\mathbb{R}^n_{\text{bohr}}} \text{b-exp}_{\xi}(x) \, d\mu = \delta_{\xi=0} = \lim_{\delta \to \infty} \int_{\mathbb{R}^n} h_{\delta}(x) \exp_{\xi}(x) \, dx,$$

since the cross product with $\lambda_{G}(g)$ only changes both sides by a factor of $\delta_{g=e}$. On the other hand, the third identity in our claim follows from the triangle inequality together with the identity

$$\lim_{\delta \to \infty} \left\| (\widetilde{H}_u \rtimes_{\gamma} id_{\mathcal{G}})(h_{\delta}^{\frac{1}{p}}f) + ih_{\delta}^{\frac{1}{p}} \sum_{\xi,g} \widetilde{\operatorname{sgn}}\langle u,\xi \rangle \widehat{f}(\xi,g) \exp_{\xi} \rtimes_{\gamma} \lambda_{\mathcal{G}}(g) \right\|_{L_p(\widehat{\Gamma})} = 0.$$

Since the sums are finite, we prove that this is true term by term, in which case the cross products with $\lambda_{\rm G}(g)$ are irrelevant. In other words, we just need to show that we have

$$\lim_{\delta \to \infty} \left\| \widetilde{H}_u(h_{\delta}^{\frac{1}{p}} \exp_{\xi}) + ih_{\delta}^{\frac{1}{p}} \widetilde{\operatorname{sgn}} \langle u, \xi \rangle \exp_{\xi} \right\|_p = 0.$$

According to the L_p -boundedness of H_u in \mathbb{R}^n , these expressions are uniformly bounded in δ for fixed 1 . By the three lines lemma, it then suffices to $prove this identity in <math>L_2$ with exponents $(1 - it)/2 + it/q = 1/2 + i\alpha$ with $\alpha \in \mathbb{R}$. By Plancherel theorem

$$\begin{split} &\|\widetilde{H}_{u}(h_{\delta}^{\frac{1}{2}+i\alpha}\exp_{\xi})+ih_{\delta}^{\frac{1}{2}+i\alpha}\widetilde{\operatorname{sgn}}\langle u,\xi\rangle\exp_{\xi}\|_{2}^{2} \\ &= \int_{\mathbb{R}^{n}}\left|\widetilde{\operatorname{sgn}}\langle u,\eta\rangle-\widetilde{\operatorname{sgn}}\langle u,\xi\rangle\right|^{2}\left|\widehat{h_{\delta}^{\frac{1}{2}+i\alpha}}(\eta-\xi)\right|^{2}d\eta \\ &\leq \int_{|\xi-\eta|<\varepsilon}\left|\widetilde{\operatorname{sgn}}\langle u,\eta\rangle-\widetilde{\operatorname{sgn}}\langle u,\xi\rangle\right|^{2}\left|\widehat{h_{\delta}^{\frac{1}{2}+i\alpha}}(\eta-\xi)\right|^{2}d\eta \\ &+ \int_{|\xi-\eta|\geq\varepsilon}\left|\widetilde{\operatorname{sgn}}\langle u,\eta\rangle-\widetilde{\operatorname{sgn}}\langle u,\xi\rangle\right|^{2}\left|\widehat{h_{\delta}^{\frac{1}{2}+i\alpha}}(\eta-\xi)\right|^{2}d\eta = A_{\delta,\varepsilon}+B_{\delta,\varepsilon}. \end{split}$$

Since $h_{\delta}^{\frac{1}{2}+i\alpha}(\xi) = \delta^{\frac{n}{2}}\phi(\sqrt{\delta}\xi)$ for some Schwartz function ϕ , we see that $B_{\delta,\varepsilon} \to 0$ as $\delta \to \infty$ for all $\varepsilon > 0$. On the other hand, sign is uniformly continuous and the integral of h_{δ} is 1, so that $\sup_{\delta>0} A_{\delta,\varepsilon} \to 0$ as $\varepsilon \to 0$. Combining both estimates we obtain the desired 0 limit as $\delta \to \infty$. To justify the fourth identity we argue as for the third, so we may reduce it to show that

$$\lim_{\delta \to \infty} \int_{\mathbb{R}^n} \left| \widetilde{\operatorname{sgn}} \langle u, \eta \rangle - \operatorname{sgn} \langle u, \eta \rangle \right|^2 \widehat{h_{\delta}^{\frac{1}{2} + i\alpha}} (\eta - \xi)^2 \, d\eta \, = \, 0.$$

According to our smoothing, there exists $\varepsilon_0 > 0$ so that $\widetilde{\operatorname{sgn}}\langle u, \eta \rangle = \operatorname{sgn}\langle u, \eta \rangle$ if $|\eta - \xi| < \varepsilon_0$. In particular, the integral above is just defined on $|\eta - \xi| \ge \varepsilon_0$ and the limit again vanishes since sgn and $\widetilde{\operatorname{sgn}}$ are uniformly bounded functions. \Box

2.6. L_p estimates for finite orbits. We now prove the implication iii) \Rightarrow i) in Theorem A. Let $G_u = \{g \in G \mid \gamma_g(u) = u\}$ be the γ -stabilizer of u. Since the index $|G : G_u|$ coincides with $|\mathcal{O}_{\gamma}(u)|$, we have finitely many right cosets G_ug . Let us label them as G_ug_j with $1 \leq j \leq |\mathcal{O}_{\gamma}(u)|$. Thus, we may write any element f in $\mathcal{L}(\Gamma)$ as

$$\begin{split} f &= \int_{\mathbb{R}^n} \sum_{g \in \mathcal{G}} \widehat{f}(\xi, g) \lambda_{\Gamma}(\xi \rtimes_{\gamma} g) d\xi \\ &= \sum_{j=1}^{|\mathcal{O}_{\gamma}(u)|} \int_{\mathbb{R}^n} \sum_{g \in \mathcal{G}_u} \widehat{f}(\xi, gg_j) \lambda_{\Gamma}(\xi \rtimes_{\gamma} gg_j) d\xi \ = \ \sum_{j=1}^{|\mathcal{O}_{\gamma}(u)|} f_j. \end{split}$$

We define

$$F_j = \int_{\mathbb{R}^n} \sum_{g \in \mathcal{G}} \widehat{f}(\xi, gg_j) \lambda_{\Gamma}(\xi \rtimes_{\gamma} g) d\xi$$

and $w_j = \lambda_{\Gamma}(0 \rtimes_{\gamma} g_j)$. If we set $\Gamma_u = \mathbb{R}^n \rtimes_{\gamma} G_u$, let us also write E_u for the conditional expectation $\mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma_u)$. Then, it is clear that we have $f = F_i w_i$ and $f_j = \mathsf{E}_u(F_j)w_j$ for all j, so that $(H_u \rtimes_{\gamma} id_G)(f_j) = (H_u \rtimes_{\gamma} id_{G_u})(\mathsf{E}_u(F_j))w_j$ and $\|\mathsf{E}_u(F_j)\|_p \leq \|F_j\|_p = \|f\|_p$ since w_j is a unitary. This yields

$$\begin{aligned} \left\| (H_u \rtimes_{\gamma} id_{\mathcal{G}})(f) \right\|_{L_p(\widehat{\Gamma})} &\leq \sum_{j=1}^{|\mathcal{O}_{\gamma}(u)|} \left\| (H_u \rtimes_{\gamma} id_{\mathcal{G}})(f_j) \right\|_{L_p(\widehat{\Gamma})} \\ &\leq |\mathcal{O}_{\gamma}(u)| \left\| H_u \rtimes_{\gamma} id_{\mathcal{G}_u} \right\|_{L_p \to L_p} \|f\|_{L_p(\widehat{\Gamma})}. \end{aligned}$$

Thus, it suffices to prove L_p -boundedness of $H_u \rtimes_{\gamma} id_{\mathbf{G}_u}$. In other words, we may assume that u is a fixed point of γ . In that case, it is easy to show that H_u is γ -equivariant and the proof follows easily. For instance, we may proceed by interpolation and duality with our $L_{\infty} \to BMO_u$ estimate below. \square

2.7. Endpoint estimates for finite orbits. The directional Hilbert transform H_u does not have a smooth Calderón-Zygmund kernel and fails to be $L_{\infty} \to BMO$ bounded on \mathbb{R}^n for n > 1, with the usual definition of BMO. Nevertheless, there exists a directional BMO_u space on \mathbb{R}^n satisfying

- a) $H_u: L_{\infty}(\mathbb{R}^n) \to \text{BMO}_u,$ b) $\left[\text{BMO}_u, L_p(\mathbb{R}^n)\right]_{p/q} = L_q(\mathbb{R}^n).$

Namely, the norm in BMO_u is given by

$$\|f\|_{\text{BMO}_{u}} = \sup_{t>0} \left\| \left(S_{u,t} |f|^{2} - |S_{u,t}f|^{2} \right)^{\frac{1}{2}} \right\|_{\infty} \quad \text{with} \quad \widehat{S_{u,t}f}(\xi) = e^{-t|\langle u,\xi\rangle|^{2}} \widehat{f}(\xi).$$

Properties a) and b) arose naturally in a more general setting [22, 23] from geometric group theory and Markov semigroups. In the Euclidean context they might be very well-known. According to Schoenberg's theorem and the fact that $\xi \mapsto |\langle \xi, u \rangle|^2$ is conditionally negative, the semigroup $S_{u,t}$ is Markovian on \mathbb{R}^n . It is however not γ -equivariant and Markovianity is lost in the cross product extension $S_{u,t} \rtimes_{\gamma} id_{\rm G}$, we thank the referee pointing this out. Nevertheless, $S_{u,t}$ is γ -equivariant when we restrict the action γ to the stabilizer G_u and then the crossed products $S_{u,t} \rtimes_{\gamma} id_{G_u}$ yield a Markovian semigroup. In the assumption that $\mathcal{O}_{\gamma}(u)$ is finite, we have seen in the previous paragraph how to write a general element f in $\mathcal{L}(\Gamma)$ as a finite sum

$$f = \sum_{j=1}^{|\mathcal{O}_{\gamma}(u)|} \mathsf{E}_u(fw_j^*) w_j$$

for some unitaries $w_j = \lambda_{\Gamma}(0 \rtimes_{\gamma} g_j)$. This leads to define

$$\|f\|_{\mathrm{BMO}_{u}(\widehat{\mathbf{\Gamma}})} = \max_{1 \le j \le |\mathcal{O}_{\gamma}(u)|} \Big\{ \|\mathsf{E}_{u}(fw_{j}^{*})\|_{\mathrm{BMO}_{u}^{r}(\widehat{\mathbf{\Gamma}})}, \|\mathsf{E}_{u}(fw_{j}^{*})\|_{\mathrm{BMO}_{u}^{c}(\widehat{\mathbf{\Gamma}})} \Big\},\$$

where the row and column BMO norms are given by

$$\|\varphi\|_{\mathrm{BMO}_{u}^{r}(\widehat{\Gamma})} = \sup_{t>0} \left\| \left(\widetilde{S}_{u,t}(\varphi\varphi^{*}) - \widetilde{S}_{u,t}\varphi\widetilde{S}_{u,t}\varphi^{*} \right)^{\frac{1}{2}} \right\|_{\mathcal{L}(\Gamma_{u})},$$

$$\|\varphi\|_{\mathrm{BMO}_{u}^{c}(\widehat{\Gamma})} = \sup_{t>0} \left\| \left(\widetilde{S}_{u,t}(\varphi^{*}\varphi) - \widetilde{S}_{u,t}\varphi^{*}\widetilde{S}_{u,t}\varphi \right)^{\frac{1}{2}} \right\|_{\mathcal{L}(\Gamma_{u})},$$

with $\widetilde{S}_{u,t} = S_{u,t} \rtimes_{\gamma} id_{\mathbf{G}_u}$ and $\Gamma_u = \mathbb{R}^n \rtimes_{\gamma} \mathbf{G}_u$. Since we know this semigroup is Markovian, we may apply Junge/Mei's interpolation theorem to it. We refer to [22] and the references therein for more on noncommutative BMO spaces.

Theorem 2.8. If $\mathcal{O}_{\gamma}(u)$ is finite, then

$$H_u \rtimes_{\gamma} id_{\mathcal{G}} : L_{\infty}(\widehat{\Gamma}) \xrightarrow{cb} \mathrm{BMO}_u(\widehat{\Gamma})$$

The cb means that the map is completely bounded. The same result holds for Γ_{disc} .

Proof. According to our definition of BMO and since

$$\left\| \left(\mathsf{E}_{u}(f_{ij}w_{k}^{*}) \right) \right\|_{M_{n}\otimes\mathcal{L}(\Gamma_{u})} \leq \left\| \left(f_{ij}\right) \right\|_{M_{n}\otimes\mathcal{L}(\Gamma)},$$

it suffices to see that $H_u \rtimes_{\gamma} id_{\mathcal{G}_u}$ takes $L_{\infty}(\widehat{\Gamma}_u)$ to the BMO space associated with the Markov semigroup $S_{u,t} \rtimes_{\gamma} id_{\mathcal{G}_u}$. By the analog of de Leeuw's compactification theorem for $L_{\infty} \to BMO$ boundedness [23, Proof of Theorem 2.4] it suffices to prove the same result for the group $\Gamma_{\text{disc},u} = \mathbb{R}^n_{\text{disc}} \rtimes_{\gamma} \mathcal{G}_u$. On the other hand, since we know that $H_u : L_{\infty}(\mathbb{R}^n) \to BMO_u(\mathbb{R}^n)$ the same holds for H_u on $\mathbb{R}^n_{\text{bohr}}$ —use once more the analogue of de Leeuw's theorem— and the assertion follows from a suitable application of the little Grothendieck inequality, see [23, Lemma 1.2]. \Box

Theorem 2.9. If $\mathcal{O}_{\gamma}(u)$ is finite and G amenable,

$$H_u \rtimes_{\gamma} id_{\mathcal{G}} : L_1(\widehat{\Gamma_{\text{disc}}}) \xrightarrow{cb} L_{1,\infty}(\widehat{\Gamma_{\text{disc}}}).$$

Proof. We may assume that f is a positive trigonometric polynomial. Arguing as in the proof of Theorem 2.8, it suffices to consider the case where u is fixed by γ . This means in particular that $(H_u \rtimes_{\gamma} id_G)f$ is self-adjoint for f positive. Let us consider the *-homomorphism

Since G is amenable, we may construct a Følner averaging sequence:

G = ∪_{j∈J} G_j,
 |G_j| < ∞ and G_{j1} ⊂ G_{j2} for j1 ≤ j2,
 |qG_i \ G_i| = o(|G_i|) for all q ∈ G.

If $p_j = \sum_{g \in G_j} e_{g,g} \in \mathcal{B}(\ell_2(G))$, we claim that following identities hold

$$\begin{split} \left\| (H_u \rtimes_{\gamma} id_{\mathcal{G}})f \right\|_{L_{1,\infty}(\widehat{\Gamma_{\text{disc}}})} \\ &= \sup_{\lambda>0} \lambda \left\| \chi_{(\lambda,\infty)} \Big(\left| H_u \rtimes_{\gamma} id_{\mathcal{G}} \right)f \right| \Big) \right\|_{L_1(\widehat{\Gamma_{\text{disc}}})} \\ &= \sup_{\lambda>0} \lambda \lim_j \frac{1}{|\mathcal{G}_j|} \left\| p_j \rho \Big(\chi_{(\lambda,\infty)} \Big(\left| H_u \rtimes_{\gamma} id_{\mathcal{G}} \right)f \right| \Big) \Big) p_j \right\|_{L_1(\mathbb{R}^n_{\text{bohr}};S_1(\ell_2(\mathcal{G})))} \\ &= \sup_{\lambda>0} \lambda \lim_j \frac{1}{|\mathcal{G}_j|} \left\| p_j \Big(\chi_{(\lambda,\infty)} \Big(\left| \rho \left[(H_u \rtimes_{\gamma} id_{\mathcal{G}})f \right] \right| \Big) \Big) p_j \right\|_{L_1(\mathbb{R}^n_{\text{bohr}};S_1(\ell_2(\mathcal{G})))} \\ &= \sup_{\lambda>0} \lambda \lim_j \frac{1}{|\mathcal{G}_j|} \left\| \chi_{(\lambda,\infty)} \Big(\left| p_j \rho \left[(H_u \rtimes_{\gamma} id_{\mathcal{G}})f \right] p_j \right| \Big) \right\|_{L_1(\mathbb{R}^n_{\text{bohr}};S_1(\ell_2(\mathcal{G})))}. \end{split}$$

Let us first finish the argument assuming this claim. If $f = \sum_{g} f_g \rtimes_{\gamma} \lambda_{\mathcal{G}}(g)$,

$$\begin{aligned} p_{j}\rho\big[(H_{u}\rtimes_{\gamma}id_{\mathcal{G}})f\big]p_{j} \\ &= p_{j}\Big(\sum_{g,h}\gamma_{g^{-1}}(H_{u}(f_{gh^{-1}}))\otimes e_{g,h}\Big)p_{j} \\ &= p_{j}\Big(\sum_{g,h}H_{u}(\gamma_{g^{-1}}(f_{gh^{-1}}))\otimes e_{g,h}\Big)p_{j} \\ &= p_{j}\Big(\big[H_{u}\otimes id_{\mathcal{B}(\ell_{2}(\mathcal{G}))}\big](\rho f)\Big)p_{j} = \big[H_{u}\otimes id_{\mathcal{B}(\ell_{2}(\mathcal{G}))}\big](p_{j}\rho(f)p_{j}). \end{aligned}$$

The second identity uses that u is fixed by γ . Writing Haar integration in $\mathbb{R}^n_{\text{bohr}}$ as a limit of averages on arbitrary large cubes as we did in Paragraph 2.2, a standard Fubini argument shows that the $L_1 \to L_{1,\infty}$ boundedness of $H_u \otimes id_{\mathcal{B}(\ell_2(G))}$ reduces to that of $H \otimes id_{\mathcal{B}(\ell_2(G))}$ where H stands for the one-dimensional Hilbert transform in the Bohr compactification of \mathbb{R} . Such a weak type boundedness was proven by Randrianantoanina in his work [47] on Hilbert transforms associated to maximal subdiagonal algebras. Thus, combining this with our claim we deduce that we have

$$\left\| \left(H_u \rtimes_{\gamma} id_{\mathcal{G}} \right) f \right\|_{L_{1,\infty}(\widehat{\mathbf{\Gamma}_{\text{disc}}})} \lesssim \lim_{j} \frac{1}{|\mathcal{G}_j|} \left\| p_j \rho(f) p_j \right\|_{L_1(\mathbb{R}^n_{\text{bohr}}; S_1(\ell_2(\mathcal{G})))} = \left\| f \right\|_{L_1(\widehat{\mathbf{\Gamma}_{\text{disc}}})}.$$

The last identity above follows as in the second identity of our claim, which we now justify. The first identity is just the definition of the $L_{1,\infty}$ quasi-norm. The second follows from Neuwirth/Ricard's matrix-valued form of Szegö's theorem [39], some details —also needed for the fourth identity— can be found below. The third follows since ρ is a *-homomorphism. Indeed, f is a trigonometric polynomial so that $(H_u \rtimes_{\gamma} id_G)f$ is a bounded operator. Therefore, we may replace $\chi_{(\lambda,\infty)}$ by $\chi_{(\lambda,M)}$ for M large enough and argue by polynomial approximation. The last identity can be proved by following [39, Proof of Theorem 2.1] again, where the idea is to approximate $\chi_{(\lambda,M)}|\cdot|$ by polynomials and estimate

$$\lim_{j} \frac{1}{|\mathbf{G}_{j}|} \left\| p_{j} P\left(\rho \left[(H_{u} \rtimes_{\gamma} id_{\mathbf{G}}) f \right] \right) p_{j} - P\left(p_{j} \rho \left[(H_{u} \rtimes_{\gamma} id_{\mathbf{G}}) f \right] p_{j} \right) \right\|_{1}$$

for each polynomial P. As we have

$$p_j x^k p_j - (p_j x p_j)^k = p_k x^{k-1} (x p_j - p_j x p_j) + (p_j x^{k-1} p_j - (p_j x p_j)^{k-1}) x p_j,$$

an induction argument yields the inequality

$$\left\| p_j x^k p_j - (p_j x p_j)^k \right\|_1 \le (k-1) \|x\|_{\infty}^{k-1} \|x p_j - p_j x p_j\|_1$$

On the other hand, analyzing the trigonometric polynomial f term by term, we are reduced to showing that $|\mathbf{G}_j|^{-1} ||Ap_j - p_j Ap_j||_1 \to 0$ for $A = \rho(\mathrm{b-exp}_{\xi} \rtimes_{\gamma} \lambda_{\mathbf{G}}(g))$ and this follows from the relation

$$\left\|Ap_j - p_j Ap_j\right\|_1 = \left\|\sum_{h \in \mathcal{G}_j \setminus g^{-1}\mathcal{G}_j} b\text{-}\mathrm{exp}_{\gamma_{gh}^{-1}(\xi)} \otimes e_{gh,h}\right\|_1 \le |g\mathcal{G}_j \setminus \mathcal{G}_j|$$

and the fact that $|gG_j \setminus G_j| = o(|G_j|)$, which follows from the amenability of G. \Box

End of the proof of Theorem A. We have proved i) \Leftrightarrow ii) \Leftrightarrow iii) and the endpoint estimates. The remaining equivalence with iv) is now very simple. Indeed, the boundedness for finite orbits follows from the triangle inequality since any block of rows is contractively complemented in the Schatten *p*-class $S_p(G)$. When the orbit is infinite, unboundedness follows by picking $f_{g,h} = \delta_{h=e}f_g$ so that the resulting square function inequality only holds for orbits with no Kakeya shadows.

Remark 2.10. Given the unboundedness for infinite orbits and our $L_{\infty} \to BMO$ estimate for finite orbits —which hold in the category of operator spaces— we see that L_p -boundedness is equivalent to complete L_p -boundedness for twisted Hilbert transforms and 1 , which was not clear*a priori*.

Remark 2.11. Given a Fourier multiplier T_m on \mathbb{R}^n —for example, the directional Hilbert transform in this paper— and a orthogonal representation $\gamma : \mathbf{G} \to O(n)$, we may consider three noncommutative forms of such an operator

- (A) The matrix operator $f_{g,h} \otimes e_{g,h} \mapsto \gamma_{g^{-1}} T_m(f_{g,h}) \otimes e_{g,h}$, (B) The cross product operator $f_g \rtimes_{\gamma} \lambda_{\mathcal{G}}(g) \mapsto T_m(f_g) \rtimes_{\gamma} \lambda_{\mathcal{G}}(g)$,
- (C) The cocycle form of the multiplier in $\mathcal{L}(G)$: $\lambda_G(g) \mapsto m_{b(q)} \lambda_G(g)$.

In terms of L_p -boundedness, [39] gives (A) \Rightarrow (B) for G discrete amenable and [23] gives (B) \Rightarrow (C) for arbitrary discrete G, see Paragraph 3.2 below. One could wonder when the reverse implications hold. When dealing with directional Hilbert transforms, Theorem A shows (A) \Leftrightarrow (B) for any discrete G, while the comment after the statement of Corollary C proves that $(C) \Rightarrow (B)$ fails in general.

2.8. Some examples. Let us illustrate our main result with a few constructions of semidirect products for which either all, none or some of the directions u in the unit sphere of \mathbb{R}^n satisfy our finite-orbit condition. This will show how Theorem A can provide positive and negative answers to the L_p boundedness of twisted Hilbert transforms for $1 as a function of <math>(G, \gamma, n, u)$.

According to Theorem A, all possible twisted Hilbert transforms in $\mathbb{R}^n \rtimes_{\gamma} G$ are L_p -bounded for 1 whenever the dimension <math>n = 1, the group G is finite or the action γ is trivial. This is the case for instance of the infinite dihedral group $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle \alpha, \beta \rangle$, which can be presented as $\mathbb{Z} \rtimes_{\gamma} \mathbb{Z}_2$ with $\gamma_{\pm 1} = \pm i d_{\mathbb{Z}}$ and $\Lambda: \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathbb{Z} \rtimes_{\gamma} \mathbb{Z}_2$ the group homomorphism

$$\Lambda(\alpha) = (0, -1)$$
 and $\Lambda(\beta) = (1, -1)$.

This shows $\mathbb{Z}_2 * \mathbb{Z}_2$ as a subgroup of $\mathbb{R}^n \rtimes_{\gamma} G$ with n = 1 and $G = \mathbb{Z}_2$ finite. In this case, the orbit of any $u \in \mathbb{R}$ has at most two elements and Theorem A shows that all twisted Hilbert transforms in $\mathbb{Z}_2 * \mathbb{Z}_2$ are L_p -bounded for 1 . Otherexamples of the same kind with n > 1 and G not finite are easily found, take for instance \mathbb{Z} acting on \mathbb{R}^2 by rotations R_k of angle $\frac{2\pi}{m}(k \mod m)$. Note also that for n = 2 the cardinality of the orbit $|\mathcal{O}_{\gamma}(u)|$ does not depend on $u \neq 0$, so that either all or none of the twisted Hilbert transforms are L_p -bounded. If (say) \mathbb{Z} acts on \mathbb{R}^2 with γ_1 a rotation R_1 of angle $\theta \in \mathbb{R} \setminus 2\pi \mathbb{Q}$, then all orbits are infinite (periodic) and no twisted Hilbert transform is L_p -bounded for $p \neq 2$. This example was already mentioned at the end of the Introduction. A similar situation appears taking $G = \mathbb{Z} * \mathbb{Z}$ the free group with two generators a_1, a_2 and n = 3. Namely, the orthogonal action we use goes back to the proof of the Banach-Tarski paradox. If $\theta \in \mathbb{R} \setminus 2\pi \mathbb{Q}$, the subgroup of SO(3) generated by

$$A_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

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is isomorphic to \mathbb{F}_2 under the mapping

$$\mathbb{F}_2 \ni \underbrace{a_{k_1}^{n_1} a_{k_2}^{n_2} \cdots a_{k_r}^{n_r}}_{w} \mapsto \underbrace{A_{k_1}^{n_1} A_{k_2}^{n_2} \cdots A_{k_r}^{n_r}}_{\gamma_w} \in SO(3)$$

with $k_1, k_2, \ldots, k_r \in \{1, 2\}, k_j \neq k_{j+1}$ and $n_1, n_2, \ldots, n_r \in \mathbb{Z}$. It is very simple to show that the action $\gamma : \mathbb{F}_2 \curvearrowright \mathbb{R}^3$ has no finite orbits. On the contrary, if we impose that the γ_{a_j} 's $(1 \leq j \leq m)$ belong to a finite subgroup G of O(n), it determines an orthogonal action $\mathbb{F}_m \curvearrowright \mathbb{R}^n$ with finite orbits everywhere. Note that by freeness any choice of γ_{a_j} is admissible. In fact, imposing such a condition for $1 \leq j \leq d$ and setting γ_{a_j} to be irrational rotations on 2-dimensional subspaces of \mathbb{R}^n for $d+1 \leq j \leq m$, we find examples of the form $\mathbb{R}^n \rtimes_{\gamma} \mathbb{F}_m$ for which some (but not all) directions $u \in \mathbb{S}^{n-1}$ lead to L_p -bounded twisted Hilbert transforms. The simplest such example arises with (m, n) = (1, 4) and $\gamma_k = R^k_{\alpha} \oplus R^k_{\beta} \in O(4)$, with rotation angles $(\alpha, \beta) = 2\pi(\frac{1}{4}, e)$ say. Only the directions u = (x, y, 0, 0) lead to L_p -bounded twisted Hilbert transforms.

Other classical groups arise as semidirect products of non-orthogonal actions. Let us consider for instance the discrete Heisenberg group, which is usually described as $\mathbb{H}_n = \mathbb{R} \times \mathbb{Z}^n \times \mathbb{Z}^n$ with group law $(x, \mathbf{a}, \mathbf{b})(x', \mathbf{a}', \mathbf{b}') = (x + x' + \mathbf{B}(\mathbf{a}, \mathbf{b}'), \mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}')$ for some bilinear form **B** on $\mathbb{Z}^n \times \mathbb{Z}^n$. However, it can also be understood as the semidirect product $(\mathbb{R} \times \mathbb{Z}^n) \rtimes_{\gamma} \mathbb{Z}^n$ with action $\gamma_{\mathbf{a}}(x, \mathbf{b}) = (x + \mathbf{B}(\mathbf{a}, \mathbf{b}), \mathbf{b})$. This gives

$$((x, \mathbf{b}), \mathbf{a})((x', \mathbf{b}'), \mathbf{a}') = ((x, \mathbf{b}) + \gamma_{\mathbf{a}}(x', \mathbf{b}'), \mathbf{a} + \mathbf{a}')$$

as expected, so that \mathbb{H}_n arises as a subgroup of $\mathbb{R}^{n+1} \rtimes_{\gamma} \mathbb{Z}^n$ since the action γ naturally extends to $\mathbb{R} \times \mathbb{R}^n$. Note however that $\gamma_{\mathbf{a}} \in \operatorname{GL}(n+1) \setminus O(n+1)$ in general. Another example of this kind is the Poincaré group, which arises as a cross product extension of the Lorentz group. Theorem A does not apply directly to this case, but perhaps modifying our arguments could lead to generalize our result to non-orthogonal actions. We believe this is an interesting problem.

2.9. On dimension free estimates. The directional Hilbert transforms act as 1-dimensional operators in the direction they point to. It is therefore clear that their $L_p \to L_p$ norm do not depend on the dimension of ambient space where they are defined. In analogy, it would be interesting to show dimension free estimates for twisted Hilbert transforms. Such a result for twisted Riesz transforms $R_u \rtimes id_G$ has been recently proved in [24]. A revision of our argument shows a few points where the constants depend on the dimension. Notably our use of (dimension-dependent) Littlewood-Paley estimates is not one of them since we apply them in dimension one. The statement of Lemma 2.3 evidences the constant $M^{\frac{n}{p}}$, but this is just a middle point in our way to Lemma 2.5 which in turn can be made dimension free by picking $\tan \alpha_{jk} \in \mathbb{R} \setminus \mathbb{Q}$. Once this is noted, it follows from the proof of Lemma 2.7 and Paragraph 2.6 that

$$C_1(p)10^{-\frac{n}{p}}\log\log\left(|\mathcal{O}_{\gamma}(u)|\right)^{\left|\frac{1}{2}-\frac{1}{p}\right|} \le \left\|H_u \rtimes_{\gamma} id_{\mathcal{G}}\right\|_{p \to p} \le C_2(p)|\mathcal{O}_{\gamma}(u)|.$$

The constant $10^{-n/p}$ arises from the inequality $4|H_{\gamma_{g_{km}}(u)}(f_{g_{km}})| \geq \chi_{B_{km}}$ in the proof of Lemma 2.7 and unfortunately we do not see how to make that estimate dimension free replacing B_{km} by something bigger. This discussion leaves the problem open for the interested reader. Moreover, it is also relevant to study the

case $n = \infty$ by considering twisted Hilbert transforms $H_u \rtimes_{\gamma} id_{\rm G}$ on the discretized group $\mathbb{R}^{\infty}_{\rm disc} \rtimes_{\gamma} {\rm G}$. Note that the corresponding group von Neumann algebra can be described as $L_{\infty}(\mathbb{R}^{\infty}_{\rm bohr}) \rtimes_{\gamma} {\rm G}$. Here it is crucial to work with the Haar measure on the Bohr compactification of \mathbb{R}^{∞} , the Lebesgue measure is meaningless. In the context of cocycle forms of directional Hilbert transforms (considered in the next section) any result for $n = \infty$ would allow infinite-dimensional cocycles in our results on Fourier L_p -summability, see Remark 3.6 for more details. In addition to the above-mentioned difficulty with the constant $10^{-n/p}$, the case $n = \infty$ would demand a revision of Lemma 2.6.

3. LACUNARITY, COCYCLES, AND CONVERGENCE OF FOURIER SERIES

In this section we analyze the more general frameworks considered in Theorem B and Corollary C. We will also establish some connections between these problems and the L_p -boundedness of directional maximal operators or idempotent Fourier multipliers on \mathbb{R} .

3.1. Lacunary γ -suborbits. Given $n \geq 2$ and a set of directions $\Omega \subset \mathbb{S}^{n-1}$ in the unit sphere, the directional maximal operator M_{Ω} is defined on smooth functions $f : \mathbb{R}^n \to \mathbb{C}$ by

$$M_{\Omega}f(x) = \sup_{w \in \Omega} \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x - t\omega)| dt.$$

The sets Ω in the circle for which M_{Ω} is bounded in \mathbb{R}^2 can now be described with remarkable accuracy. Bateman recently proved in [1] that M_{Ω} is L_q -bounded for some/any $1 < q < \infty$ iff Ω is a finite union of lacunary sets of finite order in the sense of Sjögren/Sjölin [49]. In higher dimensions, the only known results were due to Carbery and Nagel/Stein/Wainger [7, 38]. In a recent paper [42], we obtain more general results and characterize the L_q -boundedness for arbitrary dimensions. In particular, we prove that M_{Ω} is L_q -bounded for any $1 < q < \infty$ provided that Ω is HD-lacunary (see the introduction for the definition). This will be the key ingredient in the following lemma.

Lemma 3.1. If $1 and <math>\Omega$ is HD-lacunary, then

$$\left\|\left(\sum_{\omega\in\Omega}|H_{\omega}f_{\omega}|^{2}\right)^{\frac{1}{2}}\right\|_{p}\lesssim \left\|\left(\sum_{\omega\in\Omega}|f_{\omega}|^{2}\right)^{\frac{1}{2}}\right\|_{p}.$$

Proof. Since H_{ω} is essentially self-dual and the case p = 2 is clear, we may clearly assume that p > 2. Let $\frac{1}{q} = 1 - \frac{2}{p}$, then we select v in the positive part of the unit ball of $L_q(\mathbb{R}^n)$ such that

$$\begin{split} \left\| \left(\sum_{\omega \in \Omega} |H_{\omega} f_{\omega}|^{2} \right)^{\frac{1}{2}} \right\|_{p}^{2} &= \left\| \sum_{\omega \in \Omega} |H_{\omega} f_{\omega}|^{2} \right\|_{\frac{p}{2}} \\ &= \sum_{\omega \in \Omega} \int_{\mathbb{R}^{n}} |H_{\omega} f_{\omega}|^{2}(x) \operatorname{v}(x) dx \\ &= \sum_{\omega \in \Omega} \int_{\mathbb{R}^{n} \ominus \mathbb{R} \omega} \left(\int_{\mathbb{R} \omega} |H(f_{z,\omega})(s)|^{2} \operatorname{v}_{z,\omega}(s) ds \right) dz, \end{split}$$

where $f_{z,\omega}(s) = f_{\omega}(z+s\omega)$ and $v_{z,\omega}(s) = v(z+s\omega)$ for $z \perp \omega$ and H stands for the Hilbert transform on \mathbb{R} . Now we pick some $0 < \delta < 1$ and use the Hardy-Littlewood maximal operator M on \mathbb{R} to get

$$\mathbf{v}_{z,\omega}(s) \le M(\mathbf{v}_{z,\omega}^{\frac{1}{\delta}})^{\delta}(s) = \sup_{r>0} \left(\frac{1}{2r} \int_{-r}^{r} \mathbf{v}^{\frac{1}{\delta}} \left(z + (s-t)\omega\right) dt\right)^{\delta} \le M_{\Omega}(\mathbf{v}^{\frac{1}{\delta}})^{\delta} (z+s\omega).$$

It is well-known [15] that $w_{\delta} = M(v_{z,\omega}^{1/\delta})^{\delta}$ is an A_2 Muckenhoupt weight with A_2 contants depending only on δ . Since the Hilbert transform is bounded on $L_2(\mathbb{R}, \mathbf{w}_{\delta}(s)ds)$ with norm depending —linearly, see [43]— on the A_2 norm of \mathbf{w}_{δ} we conclude

$$\left\|\sum_{\omega\in\Omega}|H_{\omega}f_{\omega}|^{2}\right\|_{\frac{p}{2}}\lesssim\sum_{\omega\in\Omega}\int_{\mathbb{R}^{n}}|f_{\omega}(x)|^{2}M_{\Omega}(\mathbf{v}^{\frac{1}{\delta}})^{\delta}(x)\,dx\leq\|M_{\Omega}\|_{q\delta\to q\delta}^{\delta}\left\|\sum_{\omega\in\Omega}|f_{\omega}|^{2}\right\|_{\frac{p}{2}}.$$

This also follows from [9]. We now use HD-lacunarity and the result from [42]. \Box

Remark 3.2. Given a discrete group G and a length function $\psi : G \to \mathbb{R}_+$, a countable subset $\Delta = \{\delta_j \mid j \ge 1\} \subset G$ will be called ψ -lacunary when the following condition holds

$$\sup_{j\geq 1}\frac{\psi(\delta_{j+1})}{\psi(\delta_j)}\leq \lambda_{\psi}<1.$$

Arguing as in Lemma 2.1, we may construct a sequence of smooth functions h_m on \mathbb{R}_+ fulfilling the hypotheses of Lemma 1.2 for the group Γ_{disc} and the length function $\xi \rtimes_{\gamma} g \mapsto \psi(g)$, so that

$$\|f\|_{L_p(\widehat{\Gamma_{\mathrm{disc}}})} \sim \Big\| \sum_{\delta \in \Delta} f_{\delta} \rtimes_{\gamma} \lambda_{\mathrm{G}}(\delta) \otimes e_{\delta} \Big\|_{L_p(\widehat{\Gamma_{\mathrm{disc}}}; \ell^2_{rc})}$$

for every $f \in L_{\Delta,p}(\widehat{\Gamma_{\text{disc}}})$ $(1 and constants depending only on <math>p, \lambda_{\psi}$.

Proof of Theorem B. Enumerating $\Lambda = \{g_j \mid j \geq 1\}$, we know by hypothesis that there exists some ω in the unit sphere so that $\gamma_{g_j}^{-1}(u) \to \omega$ lacunarly as $j \to \infty$. In particular, if $1 \le j \le M \ll N$ we see that

$$\begin{aligned} |\gamma_{g_{j+1}^{-1}}(u) - \gamma_{g_{N}^{-1}}(u)| &\leq |\gamma_{g_{N}^{-1}}(u) - \omega| + |\gamma_{g_{j+1}^{-1}}(u) - \omega| \\ &\leq (1+\lambda) |\gamma_{g_{N}^{-1}}(u) - \omega| + \lambda |\gamma_{g_{j}^{-1}}(u) - \gamma_{g_{N}^{-1}}(u)|. \end{aligned}$$

In particular, if N = N(M) is large enough we find for $1 \le j \le M$

$$\begin{aligned} \left| \gamma_{g_{j+1}g_{N}^{-1}}(u) - u \right| &= \left| \gamma_{g_{j+1}^{-1}}(u) - \gamma_{g_{N}^{-1}}(u) \right| \\ &\leq \sqrt{\lambda} \left| \gamma_{g_{j}^{-1}}(u) - \gamma_{g_{N}^{-1}}(u) \right| = \sqrt{\lambda} \left| \gamma_{g_{j}g_{N}^{-1}}(u) - u \right|. \end{aligned}$$

For sufficiently large M we approximate the *p*-norm of $f \in L_{\Lambda,p}(\Gamma_{\text{disc}})$

$$\|f\|_{L_p(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}})} \sim \left\|\sum_{j=1}^{\mathrm{M}} f_{g_j} \rtimes_{\gamma} \lambda_{\mathrm{G}}(g_j)\right\|_{L_p(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}})} = \left\|\sum_{j=1}^{\mathrm{M}} f_{g_j} \rtimes_{\gamma} \lambda_{\mathrm{G}}(g_j g_{\mathrm{N}}^{-1})\right\|_{L_p(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}})}.$$

We may now apply Remark 3.2. Indeed, consider

- The set $\Delta_{\mathcal{M}} = \{g_j g_{\mathcal{N}}^{-1} | 1 \le j \le \mathcal{M}\},$ The length function $\psi_{\gamma,u}(g) = |\gamma_g(u) u|^2.$

We refer to Paragraph 1.3 to justify that $\psi_{\gamma,u}$ is a length. According to our estimates above, we see that $\Delta_{\rm M}$ is $\psi_{\gamma,u}$ -lacunary and Remark 3.2 yields the following norm equivalence with $\delta_j = g_j g_{\rm N}^{-1}$

$$\left\|f\right\|_{L_p(\widehat{\Gamma_{\mathrm{disc}}})} \sim \left\|\sum_{\delta \in \Delta_{\mathrm{M}}} f_{\delta g_{\mathrm{N}}} \rtimes_{\gamma} \lambda_{\mathrm{G}}(\delta) \otimes e_{\delta}\right\|_{L_p(\widehat{\Gamma_{\mathrm{disc}}}; \ell_{rc}^2)}$$

Using the same equivalence for $(H_u \rtimes_{\gamma} id_G)f$, we are reduced to proving

$$\left\|\sum_{\delta \in \Delta_{\mathrm{M}}} H_{u}(f_{\delta g_{\mathrm{N}}}) \rtimes_{\gamma} \lambda_{\mathrm{G}}(\delta) \otimes e_{\delta}\right\|_{L_{p}(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}};\ell_{rc}^{2})} \lesssim \left\|\sum_{\delta \in \Delta_{\mathrm{M}}} f_{\delta g_{\mathrm{N}}} \rtimes_{\gamma} \lambda_{\mathrm{G}}(\delta) \otimes e_{\delta}\right\|_{L_{p}(\widehat{\mathbf{\Gamma}_{\mathrm{disc}}};\ell_{rc}^{2})}$$

for 1 and constants independent of M. As explained in Section 1, these norms are sums/intersections of row and column spaces for <math>p smaller/greater than 2. In particular, it suffices to show that this inequality holds for row and column spaces independently. In the row case, the inequality reads as

$$\left\| \left(\sum_{j=1}^{M} |H_u(f_{g_j})|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \left\| \left(\sum_{j=1}^{M} |f_{g_j}|^2 \right)^{\frac{1}{2}} \right\|_p$$

which clearly holds from the L_p -bdness of H_u . In the column case we have

$$\left\| \left(\sum_{j=1}^{\mathcal{M}} \left| H_{\gamma_{g_{\mathcal{N}}g_{j}^{-1}}(u)}(\gamma_{g_{\mathcal{N}}g_{j}^{-1}}f_{g_{j}}) \right|^{2} \right)^{\frac{1}{2}} \right\|_{p} \lesssim \left\| \left(\sum_{j=1}^{\mathcal{M}} |\gamma_{g_{\mathcal{N}}g_{j}^{-1}}f_{g_{j}}|^{2} \right)^{\frac{1}{2}} \right\|_{p}$$

for functions $f_g \in L_p(\mathbb{R}^n_{\text{bohr}})$. Arguing as in Paragraph 2.5, we are reduced to proving such an inequality in the Euclidean space $L_p(\mathbb{R}^n)$. Now we use our second assumption which gives HD-lacunarity for the suborbit $\mathcal{O}_{\gamma}(\Lambda^{-1}, u)$ —and therefore also for the set $\gamma_{g_N} \gamma_{g_j}^{-1}(u)$ — in conjunction with Lemma 3.1 to deduce the validity of such a square function inequality in $L_p(\mathbb{R}^n)$ for 1 with absolute constantsindependent of M. This completes the proof.

Remark 3.3. Theorem B admits several generalizations. Namely, we could work with other length functions ψ for which Λ were ψ -lacunary as long as Λ is a finite covering of the suborbit $\mathcal{O}_{\gamma}(\Lambda, u)$: $\sup_{g \in \Lambda} |\{h \in \Lambda \mid \gamma_h(u) = \gamma_g(u)\}| < \infty$. On the other hand, more general notions of ψ -lacunarity may be considered. It would be interesting to obtain Littlewood-Paley estimates for ψ -lacunary sequence of higher order in the sense of [49], with which one could relax the conditions in Theorem B.

Remark 3.4. We have found in Lemma 2.1 a twisted form on $L_p(\mathbb{R}^n_{\text{bohr}})$ of Meyer's square function inequality. Now we may provide necessary and sufficient conditions for this inequality to hold. Indeed, it follows from the proof of Theorem A that not admitting Kakeya shadows is necessary for the orbits/suborbits considered. On the other hand, being HD-lacunary is sufficient, as we see from Lemma 3.1 and de Leeuw's compactification like in Paragraph 2.5. Simpler arguments —ergodicity and transference $\mathbb{R}^n \leftrightarrow \mathbb{R}^n_{\text{bohr}}$ are not needed in the Euclidean-Lebesguean case—yield the same conclusions for the twisted Meyer's inequality on $L_p(\mathbb{R}^n)$. It is an open problem to decide if there exist sets of directions which do not admit Kakeya shadows and which fail to be HD-lacunary.

3.2. Convergence of Fourier series in the ψ -metric. Corollary C follows from our results above on the discretized algebra $\mathcal{L}(\Gamma_{disc})$, an intertwining identity from [23] and standard Fourier methods.

Proof of Corollary C. If dim $\mathcal{H} = n$ the mapping $\pi_{\psi} : \mathcal{L}(G) \to \mathcal{L}(\mathbb{R}^n_{\text{disc}}) \rtimes_{\gamma} G$ determined by $\lambda_G(g) \mapsto \text{b-exp}_{b(g)} \rtimes_{\gamma} \lambda_G(g)$ is a trace preserving *-homomorphism. The key property is that

$$\pi_{\psi} \circ H_{\psi,u} = (H_u \rtimes_{\gamma} id_{\mathcal{G}}) \circ \pi_{\psi},$$

which can be easily checked. Note also that $\pi_{\psi}(L_{\Lambda,p}(\widehat{\mathbf{G}})) = L_{\Lambda,p}(\widehat{\Gamma_{\text{disc}}})$. This allows us to represent $H_{\psi,u}$ as the restriction of the γ -twisted Hilbert transform to the image of π_{ψ} . In particular, the assertions in a) and b) on the boundedness of $H_{\psi,u}$ follow from the corresponding boundedness of $H_u \rtimes_{\gamma} id_{\mathbf{G}}$ considered in Theorems A and B. On the other hand, the L_p -density of trigonometric polynomials —for which the convergence results hold trivially— allows us to emulate the standard argument in \mathbb{T}^n for L_p convergence of Fourier series. In other words, we must show that

$$f \mapsto \sum_{g \in \mathcal{G}} \chi_{\mathrm{RK}}(b(g)) \widehat{f}(g) \lambda_{\mathcal{G}}(g)$$

defines an L_p -bounded Fourier multiplier with constants independent of R. By the intertwining identity above, it suffices to prove uniform L_p -boundedness for the family $T_{\rm RK} \rtimes_{\gamma} id_{\rm G}$, where $T_{\rm RK}$ is the Fourier multiplier in $L_p(\mathbb{R}^n_{\rm bohr})$ with Fourier symbol $\chi_{\rm RK}$. If we denote the faces of K by $\partial_j K$ $(1 \leq j \leq m)$, this in turn factorizes as a finite product of semispace Fourier multipliers of the form $S_{u_j,v_j} \rtimes_{\gamma} id_{\rm G}$ with $u_j \perp \partial_j K, v_j \in {\rm R} \partial_j K$ and

$$\widehat{S_{u_j,v_j}f}(\xi) = \chi_{\mathbb{R}_+} \langle \xi - v_j, u_j \rangle \widehat{f}(\xi) \quad \Rightarrow \quad S_{u_j,v_j}f = M_{v_j} \circ S_{u_j} \circ M_{-v_j}f$$

where $S_u f = \frac{1}{2}(id + iH_u)$ and $M_v f = \text{b-exp}_v f$. Since the modulations $M_v \rtimes_{\gamma} id_{\text{G}}$ are L_p -isometries, the convergence result in c) follows once again from Theorem A applied to $H_{u_j} \rtimes_{\gamma} id_{\text{G}}$. It remains to justify the necessity in c). Consider the cocycle (**H**, **b**, γ) in Γ_{disc} defined as follows

$$\mathbf{H} = \mathbb{R}^n_{\text{disc}}, \qquad \mathbf{b}(\xi, g) = \xi, \qquad \boldsymbol{\gamma}(\xi, g) = \gamma_g.$$

The associated $\psi(\xi,g) = |\xi|^2$ yields $H_{\psi,u} = H_u \rtimes_{\gamma} id_G$ and finiteness of $\gamma(G)$ gives

$$\lim_{\mathbf{R}\to\infty}\left\|f-\sum_{\boldsymbol{\xi}\in\mathbf{RK}}\sum_{g\in\mathbf{G}}\widehat{f}(\boldsymbol{\xi},g)\lambda_{\Gamma_{\mathrm{disc}}}(\boldsymbol{\xi}\rtimes_{\gamma}g)\right\|_{L_{p}(\widehat{\Gamma_{\mathrm{disc}}})}=0$$

from the same argument above. Now we assume that the limit above vanishes. By a standard application of the uniform boundedness principle, we deduce that $\sup_{R>0} ||T_{RK} \rtimes_{\gamma} id_{G}||_{p \to p} < \infty$. Since we have already seen how translations of the Fourier symbol can be written in terms of conjugation against isometric modulation maps, we also deduce that we must have

$$\sup_{\mathbf{R}>0} \left\| T_{\tau_{\mathbf{R},j}\mathbf{R}\mathbf{K}} \rtimes_{\gamma} i d_{\mathbf{G}} \right\|_{p \to p} < \infty,$$

where $T_{\tau_{\mathrm{R},j}\mathrm{RK}}$ is the Fourier multiplier associated to the symbol $\chi_{\tau_{\mathrm{R},j}\mathrm{RK}}$. If we pick the translations $\tau_{\mathrm{R},j}$ so that $\partial_j \mathrm{K} \cap \partial_j \tau_{\mathrm{R},j} \mathrm{RK} \neq \emptyset$, we see that $\tau_{\mathrm{R},j}\mathrm{RK}$ approximates the semispace determined by the face $\partial_j \mathrm{K}$. Applying Fatou's lemma, we conclude that $H_{u_j} \rtimes_{\gamma} id_{\mathrm{G}}$ must be L_p -bounded for all $1 \leq j \leq m$. The result finally follows from another application of Theorem A. **Remark 3.5.** As with Remark 3.3, we may also consider different lengths for c).

Remark 3.6. Our argument on the sufficiency of $|\mathcal{O}_{\gamma}(u)| < \infty$ for L_p -boundedness of $H_u \rtimes_{\gamma} id_{\mathcal{G}}$ is easily modified to work in $\mathbb{R}^{\infty}_{\text{disc}} \rtimes_{\gamma} \mathcal{G}$. In particular, statement a) and the sufficiency part of c) in Corollary C remain valid in this case.

Remark 3.7. As it was justified in the Introduction, condition a) in Corollary C no longer provides a characterization of the L_p -boundedness of $H_{\psi,u}$ for arbitrary discrete groups. Such a characterization appears to be harder and would yield examples of idempotent Fourier multipliers on group von Neumann algebras for arbitrary discrete groups. A characterization of the L_p boundedness of $H_{\psi,u}$ for G = \mathbb{R} —i.e. idempotent Fourier multipliers determined by restriction from inner cocycles— will appear in [41].

Remark 3.8. As we have observed in the Introduction, for crossed products of the form $\mathbb{Z}^n \rtimes_{\gamma} G$ all possible actions fixing \mathbb{Z}^n must satisfy $N = \sup_{\xi \in \mathbb{R}^n} |\mathcal{O}_{\gamma}(\xi)| < \infty$ and Corollary C applies in full generality. This is however a restrictive scenario and we might wonder about the L_p -boundedness of twisted Hilbert transforms $H_u \rtimes_{\gamma} id_G$ acting on crossed products of the form $L_{\infty}(\mathbb{T}^n) \rtimes_{\gamma} G$. This is a cross product von Neumann algebra with a measure preserving action, which does not necessarily come from an action $G \curvearrowright \mathbb{Z}^n$. Compare for instance the algebras $\ell_{\infty}(\mathbb{Z}_p) \rtimes_{\alpha} \mathbb{Z}_p$ and $\mathcal{L}(\mathbb{Z}_p) \rtimes_{\beta} \mathbb{Z}_p^*$, with $\alpha_j(k) = j + k$ and $\beta_j(k) = jk$. The first one is *-isomorphic to M_n and the cross product does not come from a group action, while the second one is a group von Neumann algebra $L_{\infty}(\mathbb{T}^n) \rtimes_{\gamma} G$ is not necessarily a group von Neumann algebra and relating L_p -boundedness of $H_u \rtimes_{\gamma} id_G$ in this algebra with the direction u seems more involved.

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