A TRANSFERENCE METHOD IN QUANTUM PROBABILITY

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Contents

Introduction

The notion of independent random variables is central in probability theory and has many applications in analysis. Independence is also a fundamental concept in quantum probability, where it can occur in many different forms. In terms of norm estimates for sums of independent variables, free probability often plays the role of the best of all worlds. This is particularly true for applications in the theory of operator spaces. We refer to the so-called Grothendieck's program for operator spaces [7, 33, 44] and also to the noncommutative L_p embedding theory [11, 14, 15] due to the authors. On the other hand, other notions of independence weaker than freeness are often enough in the context of noncommutative Khintchine or Rosenthal type inequalities [12, 21, 24]. A first motivation for this paper was to remove a singularity at $p = 1$ for the classical Rosenthal's inequality [39] and its noncommutative form [20, 21], which is also related to the recent work by Haagerup and Musat [6] on a direct proof of Khintchine inequalities for the generators of the CAR algebras. This easily follows from our main result in this paper, a general transference method which allows us to compare the norm of sums of independent copies with the norm of sums of freely independent copies.

Let us illustrate our transference method. If $\mathcal{M}$ is a von Neumann algebra, let $\mathcal{M}^{\otimes_n}$ be the n-fold tensor product and $\pi_{tens}^k : \mathcal{M} \to \mathcal{M}^{\otimes_n}$ the canonical k-th coordinate homomorphism. It is standard to extend $\pi_{tens}^k : L_p(\mathcal{M}) \to L_p(\mathcal{M}^{\otimes_n})$ for $1 \le p \le \infty$, see e.g. [20]. Similarly, we have k-th coordinate homomorphisms $\pi_{free}^k : \mathcal{M} \to (\mathcal{M}, \phi)^{*_n}$ for free products. Our first result implies that

$$(\tau_p) \qquad \frac{1}{cp} \Big\| \sum_{k=1}^n \pi_{tens}^k(x) \Big\|_p \le \Big\| \sum_{k=1}^n \pi_{free}^k(x) \Big\|_p \le cp \Big\| \sum_{k=1}^n \pi_{tens}^k(x) \Big\|_p$$

for $x \in L_p(\mathcal{M})$ and $n \ge 1$, with uniformly bounded constants when p is close to 1.

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In the following we will write $a \sim_c b$ when $\frac{1}{c} \leq a/b \leq c$. We also need an operator-valued version of (τ_p) for further applications to the theory of operator spaces. This requires the notion of independence over a given subalgebra. Let us formally introduce the notion of 'independent copies' that we will work with. Given a noncommutative probability space $(\mathcal{A}, \varphi)$ equipped with a normal faithful state φ, a von Neumann subalgebra of $\mathcal{A}$ is called conditioned if it is invariant under the action of the modular group σ_t^φ. By Takesaki [40], this holds if and only if there is a φ-invariant normal faithful conditional expectation. Let $\mathcal{N}$ be conditioned in $\mathcal{A}$ with faithful conditional expectation $\mathsf{E}_\mathcal{N} : \mathcal{A} \to \mathcal{N}$. Let us consider two von Neumann subalgebras $\mathcal{M}_1, \mathcal{M}_2$ of $\mathcal{A}$ satisfying $\mathcal{N} \subset \mathcal{M}_1 \cap \mathcal{M}_2$. Then, $\mathcal{M}_1$ and $\mathcal{M}_2$ are called *independent over* $\mathcal{N}$ if

$$\mathsf{E}_\mathcal{N}(a_1 a_2) = \mathsf{E}_\mathcal{N}(a_1)\mathsf{E}_\mathcal{N}(a_2)$$

holds for all $a_1 \in \mathcal{M}_1$ and $a_2 \in \mathcal{M}_2$. Now, if $(\mathcal{M}_k)_{k\geq 1}$ are conditioned subalgebras of $\mathcal{A}$ with $\mathcal{N} \subset \mathcal{M}_k \subset \mathcal{A}$, we shall say that the system $(\mathcal{M}_k)_{k\geq 1}$ is *increasingly independent* if

a) $\big\langle \mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_{k-1} \big\rangle$ and $\mathcal{M}_k$ are independent over $\mathcal{N}$.

We also need a technical notion. Given a von Neumann algebra $\mathcal{M}$ containing $\mathcal{N}$ and a family of $*$-isomorphisms $\pi_k : \mathcal{M} \to \mathcal{M}_k$ with $\mathcal{N} \subset \mathcal{M}_k \subset \mathcal{A}$, we say that $(\mathcal{M}_k)_{k\geq 1}$ is a system of *top-subsymmetric copies of* $\mathcal{M}$ *over* $\mathcal{N}$ if

b) $\pi_{k|_\mathcal{N}} = id$ and

$$\mathsf{E}_\mathcal{N}\big(\pi_{f(1)}(x_1) \cdots \pi_{f(m)}(x_m)\big) = \mathsf{E}_\mathcal{N}\big(\pi_{g(1)}(x_1) \cdots \pi_{g(m)}(x_m)\big)$$

holds for all $f, g : \{1, 2, \ldots, m\} \to \mathbb{N}$ satisfying

- $f_{|_{\{1,2,\ldots,m\}\setminus A}} = g_{|_{\{1,2,\ldots,m\}\setminus A}}$,
- $|A| \leq 2$ and $A = \big\{k \mid f(k) = \max f\big\} = \big\{k \mid g(k) = \max g\big\}$.

Of course, when no subalgebra $\mathcal{N}$ is specified, we shall work with $\mathcal{N} = \langle \mathbf{1}_\mathcal{A} \rangle$ and $\mathsf{E}_\mathcal{N} = \varphi$. Using the assumptions of conditioned subalgebras allows us to provide L_p generalizations of the conditional expectation $\mathsf{E}_\mathcal{N}$ and the isomorphisms π_k, see [20] for details. Intuitively speaking, condition b) means that we are allowed to exchange the top element in the range of f by the top element in the range of g, but only if the top element does not occur more than once or twice. Top-subsymmetry is exactly the technical assumption which makes the argument in [12] work. Nevertheless, as in [12], we can consider two alternative stronger conditions:

b2) *Subsymmetry*: $g(k) = \varphi \circ f(k)$ for any strictly increasing $\varphi : \mathbb{N} \to \mathbb{N}$.

b3) *Symmetry*: $g(k) = \sigma \circ f(k)$ for any permutation σ of the positive integers.

It is clear that the implications below hold

$$\text{Symmetry} \Rightarrow \text{Subsymmetry} \Rightarrow \text{Top-subsymmetry}.$$

Example 1. Tensor product copies. Let

$$\Big(\mathcal{A}_{tens}, \mathcal{M}, \mathcal{M}_{tens}^k, \mathcal{N}; \mathsf{E}_\mathcal{N}\Big) = \Big(\mathcal{N} \bar{\otimes} \mathcal{R}^{\otimes_n}, \mathcal{N} \bar{\otimes} \mathcal{R}, \pi_{tens}^k(\mathcal{M}), \mathcal{N}; id \otimes \varphi_\mathcal{R}^{\otimes_n}\Big)$$

where $n \in \mathbb{N} \cup \{\infty\}$ and the homomorphisms π_{tens}^k are given by

$$\pi_{tens}^k : n \otimes x \in \mathcal{M} \mapsto n \otimes \mathbf{1}_\mathcal{R} \otimes \ldots \otimes \mathbf{1}_\mathcal{R} \otimes \underbrace{x}_{k-\text{th}} \otimes \mathbf{1}_\mathcal{R} \otimes \ldots \otimes \mathbf{1}_\mathcal{R} \in \mathcal{M}_{tens}^k.$$

The $\mathcal{M}_{tens}^k$'s form an independent symmetric system of copies of $\mathcal{M}$ over $\mathcal{N}$.

Example 2. Freely independent copies. Consider

$$\Big(\mathcal{A}_{free}, \mathcal{M}, \mathcal{M}_{free}^k, \mathcal{N}; \mathsf{E}_{\mathcal{N}}\Big) = \Big(*_{\mathcal{N},\,k} \, (\mathcal{M}, \mathcal{E}_{\mathcal{N}}), \mathcal{M}, \pi_{free}^k(\mathcal{M}), \mathcal{N}; \mathsf{E}_{\mathcal{N}}\Big),$$

the reduced $\mathcal{N}$-amalgamated free product of $(\mathcal{M}, \mathcal{E}_{\mathcal{N}})$ with $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$ a normal faithful conditional expectation, see e.g. [18] for details on the construction of reduced amalgamated free product von Neumann algebras. The isomorphism π_{free}^k is the canonical embedding into the k-th component of the free product $\mathcal{A}_{free}$. The $\mathcal{M}_{free}^k$'s form an independent symmetric system of copies of $\mathcal{M}$ over $\mathcal{N}$.

Our notion of noncommutative independent copies is quite general. We refer to [21] for more examples which arise naturally in quantum probability. The first form of our transference principle is the following.

Theorem A. *Let $1 \le p \le 2$ and let $(\mathcal{M}_k)_{k \ge 1}$ be an increasingly independent family of top-subsymmetric copies of $\mathcal{M}$ over $\mathcal{N}$. Then, there exists a positive constant c independent of p and n such that*

$$\mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k \, \pi_k(x) \Big\|_{L_p(\mathcal{A})} \sim_c \mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k \, \pi_{free}^k(x) \Big\|_{L_p(\mathcal{A}_{free})}.$$

If in addition the $\mathcal{M}_k$'s are symmetric, then

$$\Big\| \sum_{k=1}^n \pi_k(x) \Big\|_{L_p(\mathcal{A})} \sim_c \Big\| \sum_{k=1}^n \pi_{free}^k(x) \Big\|_{L_p(\mathcal{A}_{free})}.$$

The second form of transference is stated below.

Theorem B. *Let $1 \le p \le q \le \infty$ and let $(\mathcal{M}_k)_{k \ge 1}$ be an increasingly independent family of top-subsymmetric copies of $\mathcal{M}$ over $\mathcal{N}$. Then, there exists a positive constant c independent of p, q and n such that*

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_p(\mathcal{A}; \ell_q^n)} \sim_c \Big\| \sum_{k=1}^n \pi_{free}^k(x) \otimes \delta_k \Big\|_{L_p(\mathcal{A}_{free}; \ell_q^n)}.$$

We know from [12] that Theorem A holds for $p = 1$. The strategy consists of applying our technique [15, 16] to provide a complete embedding $L_p \to L_1$ which preserves independence. This is done in Section 1 and the rest of the paper will be essentially devoted to the proof of Theorem B, which is similar in nature but requires to adapt all the methods in [12, 15, 16]. As for Theorem A, our strategy is to prove the result in the extremal case $(p, q) = (1, \infty)$ and show that the general statement reduces to it. The extremal case is a consequence of Theorem 3.7, where we characterize the norm in $L_1(\mathcal{A}; \ell_\infty^n(\mathcal{R}))$ of increasingly independent top subsymmetric copies for any finite dimensional von Neumann algebra $\mathcal{R}$. The reduction argument is divided in two cb-embeddings

$$L_p(\mathcal{A}; \ell_q^n) \to L_p(\widehat{\mathcal{A}}; \ell_\infty^{mn}) \to \prod\nolimits_{s, \mathcal{U}} L_1\big(M_s(\widehat{\mathcal{A}})^{\otimes_{\mathrm{k}_s}}; \ell_\infty^{\mathrm{k}_s mn}\big),$$

both preserving independence. Note that this map takes values in an ultraproduct of spaces of the form $L_1(\mathcal{A}'; \ell_\infty^n(\mathcal{R}))$, so that we are in position to apply Theorem 3.7. The second embedding might be of independent interest and will be proved in Theorem 4.4, while the first embedding is the content of Theorem 5.3. In both Theorems A and B, the main new difficulty relies on keeping track of independence in the construction of the embedding.

The drawback is that Theorems A and B only hold for independent copies. This restriction goes back to [12]. Any progress in the non identically distributed case would be very desirable and thus we propose the following problem:

Problem 1. Do the scalar and mixed-norm transference hold for non i.d. variables?

Now we may revisit the singularity of certain constants mentioned above. Given $2 \le p < \infty$, a probability space (Ω, μ) and $f_1, f_2, \ldots \in L_p(\Omega)$ a family of mean-zero independent random variables, Rosenthal's classical inequality gives

$$(\Sigma_p) \qquad \Big(\int_\Omega \big| \sum_{k=1}^n f_k \big|^p d\mu \Big)^{\frac1p} \sim_{c_p} \max \left\{ \Big(\sum_{k=1}^n \|f_k\|_p^p \Big)^{\frac1p}, \Big(\sum_{k=1}^n \|f_k\|_2^2 \Big)^{\frac12} \right\}.$$

As a byproduct, we obtain for $1 \le q \le p < \infty$

$$(\Sigma_{pq}) \qquad \Big(\int_\Omega \big(\sum_{k=1}^n |f_k|^q \big)^{\frac pq} d\mu \Big)^{\frac1p} \sim_{c_{p,q}} \max \left\{ \Big(\sum_{k=1}^n \|f_k\|_p^p \Big)^{\frac1p}, \Big(\sum_{k=1}^n \|f_k\|_q^q \Big)^{\frac1q} \right\}$$

with the f_k's not necessarily mean-zero. Indeed, the case $q = 2$ easily follows from Khintchine and Rosenthal inequalities, while the general case follows from an immediate renormalization argument. Notice that in both cases we end up with the norm of an intersection of Banach spaces, whose dual is the sum of the corresponding dual spaces. This simple observation produces dual inequalities for $1 < p \le 2$ and $1 < p \le q \le \infty$ as follows

$$\Big(\int_\Omega \big| \sum_{k=1}^n f_k \big|^p d\mu \Big)^{\frac1p} \sim_{c_p} \inf_{f_k = \phi_k + \psi_k} \left\{ \Big(\sum_{k=1}^n \|\phi_k\|_p^p \Big)^{\frac1p}, \Big(\sum_{k=1}^n \|\psi_k\|_2^2 \Big)^{\frac12} \right\},$$

$$\Big(\int_\Omega \big(\sum_{k=1}^n |f_k|^q \big)^{\frac pq} d\mu \Big)^{\frac1p} \sim_{c_{p,q}} \inf_{f_k = \phi_k + \psi_k} \left\{ \Big(\sum_{k=1}^n \|\phi_k\|_p^p \Big)^{\frac1p}, \Big(\sum_{k=1}^n \|\psi_k\|_q^q \Big)^{\frac1q} \right\}.$$

It is worth mentioning that the martingale version of Rosenthal inequality [2] was extended to $1 < p \le 2$ from [20] and weak type estimates for $p = 1$ were unknown until [25]. On the other hand, we know from [9] that the best constant c_p in Rosenthal's inequality behaves like $p/\log p$. In particular, we find for (Σ_p) and (Σ_{pq}) a non-removable singularity at ∞ which is carried over to 1 by our duality argument. The problem is to decide whether this singularity is removable —as it happens for the Khintchine inequality— or not, either in the classical or in the quantum setting.

A direct argument to remove it not involving duality seems out of reach by now. Note that precise decompositions $f_k = \phi_k + \psi_k$ have only been studied in [25, 34] for martingales. However, free random variables are fortunately at our disposal and we know from [16, 18] that the free forms of (Σ_p) and (Σ_{pq}) do not have a singularity at ∞ and duality solves in that case our problem. The validity of Khintchine and Rosenthal type inequalities in the extremal case $p = \infty$ is a stamp of free probability, see [26, 38] for related results. Our transference method in Theorems A and B solves our problem for identically distributed variables. It is also worth mentioning that the argument requires freeness even in the classical case with commutative f_k's!

Problem 2. Are the dual forms of (Σ_p) and (Σ_{pq}) singular for non i.d. variables?

Let us comment some further applications of transference. We recall Hiai's construction [8] of the q-deformed analogue of Shlyakhtenko's generalized circular variables. Consider a complex Hilbert space $\mathcal{H}$ equipped with a distinguished unit vector Ω and denote by $\mathcal{F}_q(\mathcal{H})$ the associated q-Fock space. If $q = \pm 1$, we find the well-known Bosonic and Fermionic Fock spaces equipped with the symmetric and antisymmetric structures. When $-1 < q < 1$ we follow [1] and equip it with the q-inner product induced by

$$\big\langle f_1 \otimes \cdots \otimes f_n,\ g_1 \otimes \cdots \otimes g_m \big\rangle_q = \delta_{nm} \sum_{\pi \in \mathcal{S}_n} q^{i(\pi)} \langle f_1,\ g_{\pi(1)} \rangle \cdots \langle f_n,\ g_{\pi(n)} \rangle.$$

Let $\ell_q(e)$ and $\ell_q^*(e)$ stand for the creation and annihilation operators associated to a vector $e \in \mathcal{H}$, see [1] for precise definitions. Assume $\mathcal{H}$ is infinite dimensional and separable, so that we can fix an orthonormal basis $(e_{\pm k})_{k \ge 1}$. Given two sequences $(\lambda_k)_{k \ge 1}$ and $(\mu_k)_{k \ge 1}$ of positive numbers, set

$$gq_k = \lambda_k \ell_q(e_k) + \mu_k \ell_q^*(e_{-k}) \quad \text{and} \quad gq_{k,p} = d_{\phi_q}^{\frac{1}{2p}} gq_k d_{\phi_q}^{\frac{1}{2p}}.$$

The von Neumann algebra generated by the gq_k's in the GNS-construction with respect to the vacuum state $\phi_q(\cdot) = \langle \Omega, \cdot\, \Omega \rangle_q$ will be denoted by Γ_q and represent the q-deformed analogue of the corresponding Araki-Woods factor in the antisymmetric case. Here d_{ϕ_q} denotes the density of ϕ_q.

Corollary A1. *Let $\mathcal{M}$ be a von Neumann algebra and $1 \le p \le 2$. Let us consider a finite sequence $x_1, x_2, \ldots, x_n$ in $L_p(\mathcal{M})$. Then, the following equivalences hold for any $-1 \le q \le 1$ up to a constant c independent of p, q and n*

$$\Big\| \sum_{k=1}^n x_k \otimes gq_{k,p} \Big\|_{L_p(\mathcal{M} \bar{\otimes} \Gamma_q)}$$
$$\sim_c \inf_{x_k = a_k + b_k} \Big\| \Big(\sum\nolimits_k \lambda_k^{\frac{2}{p}} \mu_k^{\frac{2}{p'}} a_k a_k^* \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} + \Big\| \Big(\sum\nolimits_k \lambda_k^{\frac{2}{p'}} \mu_k^{\frac{2}{p}} b_k^* b_k \Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}.$$

The weighted Khintchine type inequalities considered above were already proved in [12, 18, 43]. The novelty of our result relies on the nonsingularity of the constants involved. To be more precise, we explain this point with a series of remarks:

a) In the Fermionic case, Corollary A1 solves a question by Xu in [43]. More concretely, Xu proved the weighted Fermionic Khintchine inequality for $1 < p < \infty$ with singularities at 1 and ∞. The singularity at ∞ is already predicted by the classical Khintchine inequality. However, the first-named author proved in [12] the same inequality for $p = 1$ applying a central limit procedure to a Rosenthal type inequality for independent copies in L_1. This motivated Xu to ask whether the singularity at 1 was removable.

b) On the other hand, Haagerup and Musat recently used in [6] another new argument to prove the weighted Fermionic Khintchine inequality. Their method is not only simpler than the one in [12], but they also managed to improve the constant up to $\sqrt{2}$. Unfortunately, their concrete approach in L_1 seems not to work for $p > 1$ and our proof uses instead the scalar-valued version of our transference method in Theorem A, together with a central limit procedure as in [12].

c) In [18] Corollary A1 was proved with

$$c_{p,q} \le \Big(\frac{2}{\sqrt{1-|q|}} \Big)^{|1-\frac{2}{p}|} \qquad \text{for} \qquad -1 < q < 1.$$

Note that q here has nothing to do with the q in (Σ_{pq})! It was also shown that the same bound applies for the cb-complementation constant $\gamma(p,q)$ of the subspace of $L_p(\Gamma_q)$ generated by the generalized q-gaussians. Corollary A1 now provides a uniform bound for $c_{p,q}$ as far as $1 \le p \le 2$. Moreover, by Corollary A1 and the argument in [18], we have

$$c_{p,q} \sim \gamma(p,q) \quad \text{for} \quad p \ge 2.$$

The following question looks like the next step.

Problem 3. Find accurate estimates for $\gamma(p,q)$ near $(p,q) = (\infty, \pm 1)$.

Our second application has to do with some recent results on operator space L_p embedding theory. Given $1 \le p < q \le 2$ and a von Neumann algebra $\mathcal{M}$, the main result in [15, 16] is the construction of a completely isomorphic embedding of $L_q(\mathcal{M})$ into $L_p(\mathcal{A})$ for some sufficiently large von Neumann algebra $\mathcal{A}$, where both spaces are equipped with their natural operator space structure. We refer to [11, 32, 43] for some prior results. The simplest construction uses free probability techniques and does not produce any singularity in the embedding constants. However, these techniques are not the right ones to show the stability of hyperfiniteness. In other words, whenever $\mathcal{M}$ is hyperfinite we can also take $\mathcal{A}$ to be hyperfinite. Under these conditions, all the known constructions produce a singularity in the constant $\eta(p,q)$ of the cb-embedding $L_q \to L_p$ as $p \to 1$. We solve this by transference.

Corollary A2. *Let $\mathcal{M}$ be hyperfinite and $1 \le p \le q \le 2$. Then, there exists a completely isomorphic embedding of $L_q(\mathcal{M})$ into $L_p(\mathcal{A})$ where both spaces are equipped with their natural operator space structures and satisfy:*

i) *$\mathcal{A}$ is hyperfinite.*
ii) *The constants are independent of p, q.*

So far we have provided applications of Theorem A. Our third application now follows from Theorem B. Pisier proved in [31] that there is no possible cb-embedding of OH into the predual of a semifinite von Neumann algebra. This was generalized by Xu, who proved in [43] that for $1 \le p < q \le 2$ we can not cb-embed C_q or R_q into a semifinite L_p space. In particular, the same applies for

$$S_q = C_q \otimes_h R_q.$$

The following result completes the previous ones by Pisier and Xu.

Corollary B. *If $1 \le p < q \le 2$, there is no cb-embedding of ℓ_q into semifinite L_p.*

In fact, as it was pointed out by Pisier our techniques go a bit further, but that will be explained in the body of the paper. Corollary B, together with the results by Pisier and Xu, justify the relevance of type III von Neumann algebras in operator space L_p embedding theory. The main tools in our argument are:

- The noncommutative form of Rosenthal theorem from [14].
- Xu's nonembedding techniques from [43] à la Grothendieck [7, 33, 44].

- A local cb-embedding
$$x \in S_q^n \mapsto \frac{1}{n^{1/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes \delta_k \in L_p\big(M_{n^2}; \ell_q^{n^2}\big),$$
with constants independent of p, q and n. This result improves a previous one from [13], where the most natural case $p = 1$ was not proved and the constant presented a singularity at 1.

1. A transference method

Given a probability space (Ω, μ), we denote the expectation by $\mathbb{E}(f) = \int f d\mu$. We reserve the symbols $(\varepsilon_k)_{k\geq 1}$ for a sequence of independent Bernoulli random variables, i.e. $\mathrm{Prob}(\varepsilon_k = \pm 1) = \frac{1}{2}$. We shall write $\mathcal{E}_\mathcal{N}$ for $\mathsf{E}_\mathcal{N} \circ \pi_k$. We begin by stating the main result in [12]. The second part is a simple refinement from [16].

Theorem 1.1. *The following inequalities hold for $x \in L_1(\mathcal{M})$:*

i) *If $(\mathcal{M}_k)_{k\geq 1}$ are independent top-subsymmetric over $\mathcal{N}$, then*
$$\begin{aligned} & \mathbb{E}\Big\| \sum_{k=1}^n \varepsilon_k \pi_k(x) \Big\|_{L_1(\mathcal{A})} \\ \sim \ & \inf_{x=a+b+c} n\|a\|_{L_1(\mathcal{M})} + \sqrt{n}\big\|\mathcal{E}_\mathcal{N}(bb^*)^{\frac{1}{2}}\big\|_{L_1(\mathcal{N})} + \sqrt{n}\big\|\mathcal{E}_\mathcal{N}(c^*c)^{\frac{1}{2}}\big\|_{L_1(\mathcal{N})}. \end{aligned}$$

ii) *If moreover, $\mathcal{E}_\mathcal{N}(x) = 0$ and $(\mathcal{M}_k)_{k\geq 1}$ are symmetric over $\mathcal{N}$, then*
$$\begin{aligned} & \Big\| \sum_{k=1}^n \pi_k(x) \Big\|_{L_1(\mathcal{A})} \\ \sim \ & \inf_{x=a+b+c} n\|a\|_{L_1(\mathsf{A})} + \sqrt{n}\big\|\mathcal{E}_\mathcal{N}(bb^*)^{\frac{1}{2}}\big\|_{L_1(\mathcal{N})} + \sqrt{n}\big\|\mathcal{E}_\mathcal{N}(c^*c)^{\frac{1}{2}}\big\|_{L_1(\mathcal{N})}. \end{aligned}$$

In both cases, the relevant constants are independent of $\mathcal{M}$ and $\mathcal{N}$.

We now give some elementary remarks on symmetric tensor products. Given a positive integer m, the symmetric tensor product $\otimes_{sym}^m \mathcal{M}$ is defined as the von Neumann algebra
$$\otimes_{sym}^m \mathcal{M} = \overline{\Big\{ \sum\nolimits_k x_k^{\otimes_m} \mid x_k \in \mathcal{M} \Big\}}^{\mathrm{wot}} \subset \mathcal{M}^{\bar{\otimes}_m}.$$
Let us write $\mathcal{S}_m$ to denote the symmetric group on $\{1, 2, \ldots, m\}$. It is easily seen that the symmetric tensor product $\otimes_{sym}^m \mathcal{M}$ is exactly the fix point algebra of the conditional expectation $\mathcal{E}_{sym}(x_1 \otimes \cdots \otimes x_m) = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(m)}$.

Lemma 1.2. *Let $(\mathcal{M}_k)_{k\geq 1}$ be independent top-subsymmetric copies over $\mathcal{N}$:*

i) *The $\otimes_{sym}^m \mathcal{M}_k$'s are independent top-subsymmetric copies over $\otimes_{sym}^m \mathcal{N}$.*

ii) *Consider the j-th coordinate homomorphisms $\pi_{tens}^j(x) = \mathbf{1} \otimes \ldots \otimes x \otimes \ldots \otimes \mathbf{1}$ into the corresponding m-fold tensor product. Then, given a von Neumann algebra $\mathcal{R}$ and $x \in L_1(\mathcal{M}\bar{\otimes}\mathcal{R})$, the following estimate holds with constants independent of m, n*
$$\mathbb{E}\Big\| \sum_{j=1}^m \pi_{tens}^j\big(\sum_{i=1}^n \varepsilon_i \big[\pi_i \otimes id\big](x) \big) \Big\|_1 \sim_c \mathbb{E}\Big\| \sum_{j=1}^m \pi_{tens}^j\big(\sum_{i=1}^n \varepsilon_i \big[\pi_{free}^i \otimes id\big](x) \big) \Big\|_1.$$

iii) *Moreover, if $\mathcal{E}_{\mathcal{N}} \otimes id(x) = 0$ and the $\mathcal{M}_k$'s are symmetric*

$$\Big\| \sum_{j=1}^m \pi_{tens}^j \big(\sum_{i=1}^n \big[\pi_i \otimes id\big](x)\big) \Big\|_1 \sim_c \Big\| \sum_{j=1}^m \pi_{tens}^j \big(\sum_{i=1}^n \big[\pi_{free}^i \otimes id\big](x)\big) \Big\|_1.$$

Proof. The first assertion trivially follows from the fact that

$$\mathcal{E}_{\otimes_{sym}^m \mathcal{N}}(x \otimes x \otimes \cdots \otimes x) = \mathcal{E}_{\mathcal{N}}(x) \otimes \mathcal{E}_{\mathcal{N}}(x) \otimes \cdots \otimes \mathcal{E}_{\mathcal{N}}(x) \, .$$

To prove ii), we set

$$\begin{aligned} \mathbb{M} &= \otimes_{sym}^m \big(\mathcal{M} \otimes \mathcal{R}\big), \\ \mathbb{M}_k &= \otimes_{sym}^m \big(\mathcal{M}_k \otimes \mathcal{R}\big). \end{aligned}$$

Let $\widehat{\pi}_k : \mathbb{M} \to \mathbb{M}_k$ given by $\widehat{\pi}_k(x^{\otimes_m}) = \big[\pi_k \otimes id\big](x)^{\otimes_m}$. Similarly, consider

$$\widehat{\pi}_{free}^k : x^{\otimes_m} \in \mathbb{M} \mapsto \big[\pi_{free}^k \otimes id\big](x)^{\otimes_m} \in \mathbb{M}_{free}^k = \otimes_{sym}^m \big(\mathcal{M}_{free}^k \otimes \mathcal{R}\big).$$

Let us note that

- $\sum_{j=1}^m \pi_{tens}^j(x) \in L_1(\mathbb{M})$.
- According to i), if we set
$$\mathbb{N} = \otimes_{sym}^m \big(\mathcal{N} \otimes \mathcal{R}\big),$$
$(\mathbb{M}_k)_k$ and $(\mathbb{M}_{free}^k)_k$ are independent top-subsymmetric copies over $\mathbb{N}$.
- Moreover, if we further assume that the $\mathcal{M}_k$'s are symmetric, it also trivially follows that both the $\mathbb{M}_k$'s and the $\mathbb{M}_{free}^k$'s are independent symmetric copies of $\mathbb{M}$ over $\mathbb{N}$.

Under these conditions, we obtain from Theorem 1.1 i)

$$\begin{aligned} \mathbb{E} \Big\| \sum_{j=1}^m \pi_{tens}^j \big(\sum_{i=1}^n \varepsilon_i \big[\pi_i \otimes id\big](x)\big) \Big\|_1 &= \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i \widehat{\pi}_i \big(\sum_{j=1}^m \pi_{tens}^j(x)\big) \Big\|_1 \\ &\sim \mathbb{E} \Big\| \sum_{i=1}^n \varepsilon_i \widehat{\pi}_{free}^i \big(\sum_{j=1}^m \pi_{tens}^j(x)\big) \Big\|_1 \\ &= \mathbb{E} \Big\| \sum_{j=1}^m \pi_{tens}^j \big(\sum_{i=1}^n \varepsilon_i \big[\pi_{free}^i \otimes id\big](x)\big) \Big\|_1. \end{aligned}$$

For independent symmetric mean-zero copies we use Theorem 1.1 ii) instead. □

Proof of Theorem A. Given a complex Hilbert space $\mathcal{H}$, let $\mathcal{F}_{-1}(\mathcal{H})$ denote its antisymmetric Fock space. Write $c(e)$ and $a(e)$ for the creation and annihilation operators associated to $e \in \mathcal{H}$. Given an orthonormal basis $(e_{\pm k})_{k \ge 1}$ of $\mathcal{H}$ and a family $(\mu_k)_{k \ge 1}$ of positive numbers, the sequence $f_k = c(e_k) + \mu_k a(e_{-k})$ satisfies the canonical anticommutation relations and we take $\mathcal{R}$ to be the von Neumann algebra generated by the f_k's. Let $\phi_{\mathcal{R}}$ be the quasi-free state on $\mathcal{R}$ determined by the vacuum. With this construction, $\mathcal{R}$ is the Araki-Woods factor arising from the GNS construction applied to the CAR algebra with respect to $\phi_{\mathcal{R}}$. In fact, using a conditional expectation, we may replace the μ_k's by a sequence $(\mu_k')_{k \ge 1}$ such that for every rational $0 < \lambda < 1$ there are infinitely many $\mu_k' = \lambda/(1+\lambda)$. According to Araki and Woods, we then obtain the hyperfinite III_1 factor $\mathcal{R}$.

Consider the amplification $\widehat{\mathcal{A}} = \mathcal{A}\bar{\otimes}\mathcal{B}(\ell_2)$ and assume for simplicity that $\mathcal{A}$ is σ-finite. A normal strictly semifinite faithful weight $\psi_{\widehat{\mathcal{A}}}$ is determined by a sequence (a net in the general case) of pairs (ψ_n, q_n) such that

- The q_n's are increasing projections in $\widehat{\mathcal{A}}$ with $\mathrm{SOT}-\lim_n q_n = 1$.
- The ψ_n's are normal positive functionals on $\widehat{\mathcal{A}}$ with $\mathrm{supp}\,\psi_n = q_n$.
- The (ψ_n, q_n)'s satisfy the compatibility condition $\psi_{n+1}(q_n x q_n) = \psi_n(x)$.

Let us write k_n for the nondecreasing numbers $\psi_n(q_n) \in (0,\infty)$. We refer to Propositions 8.10, 8.19 and 8.22 of [16] for the fact that there is a normal strictly semifinite faithful weight $\psi_{\widehat{\mathcal{A}}}$ with $\mathrm{k}_n \in \mathbb{N}$ and $\rho_n \in L_1(\mathcal{B}(\ell_2)\bar{\otimes}\mathcal{R})$ such that

$$\|a\|_p \sim_c \lim_{n\to\infty} \big\|a \otimes \rho_n\big\|_{\mathcal{K}^1_{rc_1}(\psi_n \otimes \phi_{\mathcal{R}})}$$

holds for every $a \in L_p(\mathcal{A})$ up to an absolute constant c. The coefficients ρ_n are universal, i.e. do not depend on $\mathcal{A}$. The specific form of $\psi_{\widehat{\mathcal{A}}}$ depends on the spectrum of the operator $f_{|\partial_0} \mapsto f_{|\partial_1}$ which takes the left boundary value of an analytic function f on the strip $0 < \mathrm{Re} z < 1$ to its right boundary value, see Chapter 8 of [16] for further details. Writing $\psi_{\mathcal{R},n} = \psi_n \otimes \phi_{\mathcal{R}}$, we know that $\psi_{\mathcal{R},n} = \mathrm{k}_n \phi_{\mathcal{R},n}$ for some non-faithful state $\phi_{\mathcal{R},n}$ with density $d_{\mathcal{R},n}$, so that $\phi_{\mathcal{R},n}(x) = \mathrm{tr}(d_{\mathcal{R},n} x)$ and the norm on the right takes the form

$$\|z\|_{\mathcal{K}^1_{rc_1}(\psi_n \otimes \phi_{\mathcal{R}})} = \inf_{z = z_1 + d_{\mathcal{R},n}^{\frac12} z_r + z_c d_{\mathcal{R},n}^{\frac12}} \mathrm{k}_n \, \|z_1\|_1 + \mathrm{k}_n^{\frac12} \, \|z_r\|_2 + \mathrm{k}_n^{\frac12} \, \|z_c\|_2.$$

On the other hand, we have $\rho_n = \xi_n \otimes \gamma$ where γ is a mean-zero element of $L_1(\mathcal{R}, \phi_{\mathcal{R}})$ and $\xi_n = \sum_{i,j\le n} e_{ij} \in M_n$. Indeed, see [16, Page 136 and proof of Proposition 8.10] for the claim on γ and the proofs of [16, Lemma 8.21 and Proposition 8.22] to get an idea of how to derive the form of ξ_n, see also [17] for more on this. Taking this into account Theorem 1.1 ii) applies for $\mathcal{N} = \langle \mathbf{1}_{\mathcal{A}} \rangle$ and yields

$$\begin{aligned} (1.1) \quad \mathbb{E}\Big\| \sum\nolimits_k \varepsilon_k \, \pi_k(x) \Big\|_p \quad &\sim \quad \lim_{n\to\infty} \mathbb{E}\Big\| \big(\sum\nolimits_k \varepsilon_k \, \pi_k(x) \big) \otimes \rho_n \Big\|_{\mathcal{K}^1_{rc_1}(\psi_n \otimes \phi_{\mathcal{R}})} \\ &\sim \quad \lim_{n\to\infty} \mathbb{E}\Big\| \sum_{j=1}^n \pi_{tens}^j \big(\sum\nolimits_k \varepsilon_k \, \big[\pi_k \otimes id\big](x \otimes \rho_n) \big) \Big\|_1. \end{aligned}$$

Indeed, in the last step we use

- The mean-zero condition
$$\phi_{\mathcal{R},n}\Big(\sum\nolimits_k \varepsilon_k \, \big[\pi_k \otimes id\big](x \otimes \rho_n) \Big) = 0.$$
- The π_{tens}^j's from Lemma 1.2 provide symmetric independent copies.

The mean-zero condition follows from the fact that γ is mean-zero in $\mathcal{R}$. Now according to Lemma 1.2 ii), we may replace π_k by π_{free}^k at the right hand side of (1.1). This implies the first assertion calculating backwards. For the second assertion, we first note that the argument above is also valid without random signs whenever the $\mathcal{M}_k$'s are symmetric and $\mathcal{E}_{\mathcal{N}}(x) = 0$. In other words, we have

$$(1.2) \qquad \Big\| \sum\nolimits_k \pi_k\big(x - \mathcal{E}_{\mathcal{N}}(x)\big) \Big\|_p \sim_c \Big\| \sum\nolimits_k \pi_{free}^k \big(x - \mathcal{E}_{\mathcal{N}}(x)\big) \Big\|_p.$$

Indeed, we just need to argue as we did above and use Lemma 1.2 iii) for mean-zero symmetric copies. In the general case, we may assume by approximation that only finitely many π_k's are considered. Then we have

$$\begin{aligned}\Big\| \sum_{k=1}^n \pi_k(x) \Big\|_p & \le \Big\| \sum_{k=1}^n \pi_k\big(x - \mathcal{E}_\mathcal{N}(x)\big) \Big\|_p + \Big\| \sum_{k=1}^n \pi_k\big(\mathcal{E}_\mathcal{N}(x)\big) \Big\|_p \\ & = \Big\| \sum_{k=1}^n \pi_k\big(x - \mathcal{E}_\mathcal{N}(x)\big) \Big\|_p + \Big\| \sum_{k=1}^n \pi_{free}^k\big(\mathcal{E}_\mathcal{N}(x)\big) \Big\|_p \\ & \sim_c \Big\| \sum_{k=1}^n \pi_{free}^k\big(x - \mathcal{E}_\mathcal{N}(x)\big) \Big\|_p + \Big\| \sum_{k=1}^n \pi_{free}^k\big(\mathcal{E}_\mathcal{N}(x)\big) \Big\|_p.\end{aligned}$$

We have used (1.2) and the invariances

$$\pi_{k|_\mathcal{N}} = \pi_{free|_\mathcal{N}}^k = id.$$

Moreover, using $\mathcal{E}_\mathcal{N} = \mathsf{E}_\mathcal{N} \circ \pi_{free}^k$ we find $\pi_{free}^k \circ \mathcal{E}_\mathcal{N} = \mathsf{E}_\mathcal{N} \circ \pi_{free}^k$ and

$$\Big\| \sum_{k=1}^n \pi_k(x) \Big\|_p \lesssim_c \Big\| \sum_{k=1}^n \pi_{free}^k(x) \Big\|_p + 2 \Big\| \mathsf{E}_\mathcal{N}\big(\sum_{k=1}^n \pi_{free}^k(x) \big) \Big\|_p \le 3 \Big\| \sum_{k=1}^n \pi_{free}^k(x) \Big\|_p.$$

The reverse inequality follows similarly. □

Remark 1.3. If we allow the constant to be dependent on p, Theorem A holds for $1 < p < \infty$ and not only for copies. Indeed, this follows directly from the noncommutative Rosenthal inequality [21]. As in the classical case, we don't have a uniform constant c_p as $p \to \infty$. On the other hand, we know from Claus Köstler [22] that the symmetry assumption in the second assertion of Theorem A can be replaced by the weaker notion of subsymmetric copies defined in the Introduction.

Remark 1.4. Let M be a finite von Neumann algebra equipped with a faithful normal trace τ and let $u_1, u_2, \ldots, u_n \in \mathsf{M}$ be unitaries. We claim that the inequality below holds for $1 \le p \le 2$, independent top-subsymmetric copies and constants independent of p

$$\begin{aligned}(1.3) \quad & \mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k \pi_k(x) \otimes u_k \Big\|_{L_p(\mathcal{A} \bar{\otimes} \mathsf{M})} \\ & \sim_c \inf_{x = a+b+c} n^{\frac1p} \|a\|_{L_p(\mathcal{M})} + \sqrt{n} \big\| \mathcal{E}_\mathcal{N}(bb^*)^{\frac12} \big\|_{L_p(\mathcal{N})} + \sqrt{n} \big\| \mathcal{E}_\mathcal{N}(c^*c)^{\frac12} \big\|_{L_p(\mathcal{N})}.\end{aligned}$$

Indeed, the case $p = 1$ is a more general version of Theorem 1.1 that was already considered in [12]. Moreover, it can be easily checked that our arguments in Lemma 1.2 and Theorem A are stable under taking tensors with arbitrary unitaries, as far as we work with this refinement of Theorem 1.1. This gives

$$\mathbb{E} \Big\| \sum\nolimits_k \varepsilon_k \pi_k(x) \otimes u_k \Big\|_{L_p(\mathcal{A} \bar{\otimes} \mathsf{M})} \sim_c \mathbb{E} \Big\| \sum\nolimits_k \varepsilon_k \pi_{free}^k(x) \otimes u_k \Big\|_{L_p(\mathcal{A} \bar{\otimes} \mathsf{M})}.$$

If we now combine it with the free Rosenthal inequality from [18]

$$\begin{aligned} & \mathbb{E} \Big\| \sum_{k=1}^n \varepsilon_k \pi_{free}^k(x_k) \Big\|_{L_p(\mathcal{A})} \\ & \sim_c \inf_{x_k = a_k + b_k + c_k} \Big(\sum_{k=1}^n \|a_k\|_p^p \Big)^{\frac1p} + \Big\| \big(\sum_{k=1}^n \mathcal{E}_\mathcal{N}(b_k b_k^*) \big)^{\frac12} \Big\|_p + \Big\| \big(\sum_{k=1}^n \mathcal{E}_\mathcal{N}(c_k^* c_k) \big)^{\frac12} \Big\|_p,\end{aligned}$$

we easily end up with (1.3). This is a form of the noncommutative Rosenthal inequality [21] for independent top-subsymmetric copies with no singularity at $p = 1$ and will be instrumental in the following paragraph.

Proof of Corollary A1. Inequality (1.3) implies the assertion by a central limit procedure which allows us to pass from the three terms in Rosenthal inequality to the two terms in the assertion. The argument for the particular case $p = 1$ was given in Section 8 of [12]. The case $1 < p \leq 2$ just requires simple modifications that we leave to the reader. □

Proof of Corollary A2. The construction in [16] is sketched as follows

$$L_q(\mathcal{M}) \xrightarrow{(\alpha)} \Big(\mathcal{H}^r_{2p',2}(\mathcal{M},\theta) \otimes_{\mathcal{M},h} \mathcal{H}^c_{2p',2}(\mathcal{M},\theta)\Big)_* \xrightarrow{(\beta)} \mathcal{K}^p_{\mathcal{rc}_p}(\phi \otimes \psi \otimes \xi) \xrightarrow{(\gamma)} L_p(\mathcal{A}).$$

We now review the cb-embeddings (α), (β) and (γ) in some detail to identify where the singularity appears. It will happen once in (β) and once in (γ). However, in both cases our version (1.3) of the noncommutative Rosenthal inequality for independent copies —no need of unitaries this time— will allow us to remove the singularity. Namely, the embedding (α) generalizes the so-called 'Pisier's exercise' to embed the Schatten class $S_q = C_q \otimes_h R_q$ into an operator space of the form

$$\big(C_p \oplus \mathrm{OH}\big)/graph(\Lambda_1)^\perp \otimes_h \big(R_p \oplus \mathrm{OH}\big)/graph(\Lambda_2)^\perp,$$

see [15, 43] for further details. When working with general von Neumann algebras this requires to encode complex interpolation in terms of certain spaces of analytic functions. This follows from the factorization

$$L_q(\mathcal{M}) = L^r_{2q}(\mathcal{M}) \otimes_{\mathcal{M},h} L^c_{2q}(\mathcal{M}),$$

Proposition 8.19 in [16] and duality. It turns out that the constants are independent of p, q at this step. Moreover, a more convenient way to write the space between (α) and (β) is by means of a 4-term sum $\mathcal{K}_{p,2}(\psi,\xi)$, with ψ and ξ normal strictly semifinite faithful weights on $\mathcal{M}$ and $\mathcal{B}(\ell_2)$ respectively, see Proposition 2.22 of [16] for further details. The space $\mathcal{K}_{p,2}(\psi \otimes \xi)$ arises from a direct limit

$$\mathcal{K}_{p,2}(\psi \otimes \xi) = \overline{\bigcup_{n\geq 1} \mathcal{K}_{p,2}(\psi_n \otimes \xi_n)},$$

where ψ_n, ξ_n are the restrictions of ψ, ϕ to certain increasing sequences of finite projections. If $\psi_n \otimes \xi_n = \mathrm{k}_n \phi_n$ with ϕ_n a state supported by the projection q_n, it turns out that

$$\mathcal{K}_{p,2}(\psi_n \otimes \xi_n) = \sum_{u,v \in \{2p,4\}} \mathrm{k}_n^{\frac{1}{u}+\frac{1}{v}} L_{(u,v)}(q_n(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))q_n).$$

The definition of asymmetric $L_{(u,v)}$ spaces is recalled in Section 2 below. On the other hand, the space $\mathcal{K}^p_{\mathcal{rc}_p}(\phi \otimes \psi \otimes \xi)$ is also the direct limit of an increasing family of 3-term sums

$$\mathcal{K}^p_{\mathcal{rc}_p}(\psi \otimes \phi \otimes \xi) = \overline{\bigcup_{n\geq 1} \mathcal{K}^p_{\mathcal{rc}_p}(\psi_n \otimes \phi \otimes \xi_n)},$$

where ϕ is the quasi-free state $\phi_{\mathcal{R}}$ in the proof of Theorem A above. The complete embedding $\mathcal{K}_{p,2}(\psi_n \otimes \xi_n) \to \mathcal{K}^p_{\mathcal{rc}_p}(\psi_n \otimes \phi \otimes \xi_n)$ holds up to constants independent on n, see Proposition 8.10 in [16]. The main ingredient in the argument is the noncommutative Rosenthal inequality for independent copies in L_p. Equipped with (1.3), we may now provide a universal constant valid for any $1 \leq p \leq 2$. Thus the embedding (β) holds up to constants independent of p. Let us consider the embedding (γ). Its construction is given in [16, Theorem 8.11]. The idea is to construct cb-embeddings of $\mathcal{K}_{rc_p}^p(\psi_n \otimes \phi \otimes \xi_n)$ for each $n \ge 1$, so that we can take direct limits and hyperfiniteness is preserved. The main tool is a noncommutative Poisson process, an algebraic construction that has nothing to do with constants. Therefore the problem reduces to cb-embed $\mathcal{K}_{rc_p}^p(\psi_n \otimes \phi \otimes \xi_n)$ into $L_p(\mathcal{A}_n)$. A quick look at [16, Theorem 8.11] shows that the only point where the singularity appears is once more from the use of noncommutative Rosenthal inequality for independent copies. Hence, Theorem A applies and produces (1.3), implying the assertion. □

Remark 1.5. According to [15, 16], the assertion in Corollary A2 also holds with $L_q(\mathcal{M})$ replaced by any operator space of the form $\mathrm{X}_1 \otimes_h \mathrm{X}_2$, with X_1 a quotient of a subspace of $R \oplus \mathrm{OH}$ and X_2 a quotient of a subspace of $C \oplus \mathrm{OH}$.

2. Mixed-norms of free variables

In this section we recall several results from [16] for the convenience of the reader. The main result is a variation of the free Rosenthal inequality from [18] which will be instrumental in the course of our argument. The correct formulation involves certain noncommutative function spaces.

2.1. **Conditional L_p spaces.** Inspired by Pisier's theory [29] of noncommutative vector-valued L_p spaces, several noncommutative function spaces have been recently introduced in quantum probability. The first insight came from some of Pisier's fundamental equalities which we briefly review. Let $\mathcal{N}_1, \mathcal{N}_2$ be hyperfinite von Neumann algebras. Given $1 \le p, q \le \infty$, we define $1/r = |1/p - 1/q|$. If $p \le q$, the norm of $x \in L_p(\mathcal{N}_1; L_q(\mathcal{N}_2))$ is given by

$$\inf \Big\{ \|\alpha\|_{L_{2r}(\mathcal{N}_1)} \|y\|_{L_q(\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)} \|\beta\|_{L_{2r}(\mathcal{N}_1)} \, \big| \, x = \alpha y \beta \Big\}.$$

If $p \ge q$, the norm of $x \in L_p(\mathcal{N}_1; L_q(\mathcal{N}_2))$ is given by

$$\sup \Big\{ \|\alpha x \beta\|_{L_q(\mathcal{N}_1 \bar{\otimes} \mathcal{N}_2)} \, \big| \, \alpha, \beta \in \mathsf{B}_{L_{2r}(\mathcal{N}_1)} \Big\}.$$

The hyperfiniteness is an essential assumption in [29]. However, when dealing with mixed $L_p(L_q)$ norms, Pisier's identities remain true for general von Neumann algebras, see [10, 21]. On the other hand, given any von Neumann algebra $\mathcal{M}$, the *row* and *column* subspaces of L_p are defined as follows

$$R_p^n(L_p(\mathcal{M})) = \Big\{ \sum_{k=1}^n x_k \otimes e_{1k} \, \big| \, x_k \in L_p(\mathcal{M}) \Big\} \subset L_p\big(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)\big),$$

$$C_p^n(L_p(\mathcal{M})) = \Big\{ \sum_{k=1}^n x_k \otimes e_{k1} \, \big| \, x_k \in L_p(\mathcal{M}) \Big\} \subset L_p\big(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)\big),$$

where (e_{ij}) denotes the unit vector basis of $\mathcal{B}(\ell_2)$. These spaces are crucial in the noncommutative Khintchine/Rosenthal type inequalities and in noncommutative martingale inequalities, where the row and column spaces are traditionally denoted by $L_p(\mathcal{M};\ell_2^r)$ and $L_p(\mathcal{M};\ell_2^c)$. The norm in these spaces is given by

$$\Big\| \sum_{k=1}^n x_k \otimes e_{1k} \Big\|_{R_p^n(L_p(\mathcal{M}))} = \Big\| \Big(\sum_{k=1}^n x_k x_k^* \Big)^{\frac12} \Big\|_{L_p(\mathcal{M})},$$
$$\Big\| \sum_{k=1}^n x_k \otimes e_{k1} \Big\|_{C_p^n(L_p(\mathcal{M}))} = \Big\| \Big(\sum_{k=1}^n x_k^* x_k \Big)^{\frac12} \Big\|_{L_p(\mathcal{M})}.$$

In what follows we write $R_p^n(L_p(\mathcal{M})) = L_p(\mathcal{M};R_p^n)$ and $C_p^n(L_p(\mathcal{M})) = L_p(\mathcal{M};C_p^n)$.

Now, let us assume that $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and that there exists a normal faithful conditional expectation $\mathcal{E}_\mathcal{N} : \mathcal{M} \to \mathcal{N}$. Then we may define L_p norms of the *conditional square functions*

$$\Big(\sum_{k=1}^n \mathcal{E}_\mathcal{N}(x_k x_k^*) \Big)^{\frac12} \quad \text{and} \quad \Big(\sum_{k=1}^n \mathcal{E}_\mathcal{N}(x_k^* x_k) \Big)^{\frac12}.$$

These norms must be properly defined for $1 \le p \le 2$, see [10] or [16, Chapter 1]. The resulting spaces coincide with the row/column spaces above if $\mathcal{N}$ is $\mathcal{M}$ itself. When $n = 1$ we recover the spaces $L_p^r(\mathcal{M}, \mathsf{E})$ and $L_p^c(\mathcal{M}, \mathsf{E})$ from [10].

We have already introduced $L_p(L_q)$ spaces, row and column subspaces of L_p and some variations associated to a given conditional expectation. All the norms considered so far fit into more general noncommutative function spaces —for not necessarily hyperfinite von Neumann algebras— which we now define. Consider the solid K in $\mathbb{R}^3$ determined by

$$\mathsf{K} = \Big\{ (1/u, 1/v, 1/q) \, \big| \, 2 \le u, v \le \infty, \ 1 \le q \le \infty, \ 1/u + 1/q + 1/v \le 1 \Big\}.$$

Let $\mathcal{N}$ be a conditioned subalgebra of $\mathcal{M}$ and take $1/p = 1/u + 1/q + 1/v$ for some $(1/u, 1/v, 1/q) \in \mathsf{K}$. Then we define the *amalgamated L_p space* associated to the indices (u, q, v) as the subspace $L_u(\mathcal{N})L_q(\mathcal{M})L_v(\mathcal{N})$ of $L_p(\mathcal{M})$ equipped with the norm

$$\inf \Big\{ \|\alpha\|_{L_u(\mathcal{N})} \|y\|_{L_q(\mathcal{M})} \|\beta\|_{L_v(\mathcal{N})} \, \big| \, x = \alpha y \beta \Big\},$$

where the infimum runs over all possible factorizations $x = \alpha y \beta$ with (α, y, β) belonging to $L_u(\mathcal{N}) \times L_q(\mathcal{M}) \times L_v(\mathcal{N})$. Let us now fix $(1/u, 1/v, 1/p) \in \mathsf{K}$ and take $1/s = 1/u + 1/p + 1/v$. Then we define the *conditional L_p space* associated to the indices (u, v) as the completion $L_u^{-1}(\mathcal{N})L_s(\mathcal{M})L_v^{-1}(\mathcal{N})$ of $L_p(\mathcal{M})$ with respect to the norm

$$\sup \Big\{ \|\alpha x \beta\|_{L_s(\mathcal{M})} \, \big| \, \|\alpha\|_{L_u(\mathcal{N})}, \|\beta\|_{L_v(\mathcal{N})} \le 1 \Big\}.$$

Both, amalgamated and conditional L_p spaces, where introduced in [16] and we refer to that paper for a more detailed exposition. It should also be noticed that our terminology $L_u^{-1}(\mathcal{N})L_s(\mathcal{M})L_v^{-1}(\mathcal{N})$ for conditional L_p spaces is different from the one used in [16]. Now we collect the complex interpolation and duality properties of amalgamated and conditional L_p spaces from [16]. Our interpolation identities generalize some previous results by Pisier [28] and recently by Xu [42].

Let K_0 denote the interior of K. Then we have:

a) $L_u(\mathcal{N})L_q(\mathcal{M})L_v(\mathcal{N})$ is a Banach space.

b) $L_{u_\theta}(\mathcal{N})L_{q_\theta}(\mathcal{M})L_{v_\theta}(\mathcal{N})$ is isometrically isomorphic to

$$\Big[L_{u_0}(\mathcal{N})L_{q_0}(\mathcal{M})L_{v_0}(\mathcal{N}), L_{u_1}(\mathcal{N})L_{q_1}(\mathcal{M})L_{v_1}(\mathcal{N}) \Big]_\theta,$$

with $(\frac{1}{u_\theta}, \frac{1}{q_\theta}, \frac{1}{v_\theta}) = (\frac{1-\theta}{u_0} + \frac{\theta}{u_1}, \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \frac{1-\theta}{v_0} + \frac{\theta}{v_1})$.

c) If $(1/u, 1/v, 1/q) \in \mathsf{K}_0$ and $1 - 1/p = 1/u + 1/q + 1/v$

$$\big(L_u(\mathcal{N}) L_q(\mathcal{M}) L_v(\mathcal{N})\big)^* = L_u^{-1}(\mathcal{N}) L_{q'}(\mathcal{M}) L_v^{-1}(\mathcal{N}),$$

$$\big(L_u^{-1}(\mathcal{N}) L_{q'}(\mathcal{M}) L_v^{-1}(\mathcal{N})\big)^* = L_u(\mathcal{N}) L_q(\mathcal{M}) L_v(\mathcal{N}),$$

with respect to the antilinear duality bracket $\langle x, y \rangle = \mathrm{tr}(x^* y)$. A natural way to read the first identity (the second one is its dual) is to say that the dual of the amalgamated $L_{p'}$ space associated to (u, v) is the conditional L_p space associated to (u, v), since

$$1/p' = 1/u + 1/q + 1/v \quad \text{and} \quad 1/p = 1/q' - 1/u - 1/v.$$

We refer the reader to Part I of [16] for some refinements of these results.

2.2. **A variant of free Rosenthal's inequality.** In this paragraph we formulate the free analogue of inequality (Σ_{pq}) in the Introduction and its dual. To be precise, we shall work for convenience with i.d. variables. In that case, it is easily checked that (Σ_{pq}) provides a natural way to realize the space

$$\mathcal{J}_{p,q}^n(\Omega) = n^{\frac{1}{p}} L_p(\Omega) \cap n^{\frac{1}{q}} L_q(\Omega)$$

as an isomorph of a subspace of $L_p(\Omega; \ell_q^n)$. Quite surprisingly, replacing in (Σ_{pq}) independent variables by matrices of independent variables requires to intersect four spaces using the so-called *asymmetric* L_p spaces. This phenomenon was discovered for the first time in [13] and is partly motivated by the isometry $L_p = L_{2p} L_{2p}$ meaning that the p-norm of f is the infimum of $\|g\|_{2p} \|h\|_{2p}$ over all factorizations $f = gh$. Namely, if L_{2p}^r and L_{2p}^c denote the row and column quantizations of L_{2p} determined by definition (2.2) below, the operator space analogue of this isometry is given by the complete isometry $L_p = L_{2p}^r L_{2p}^c$, see below for further details. This leads us to redefine $\mathcal{J}_{p,q}^n$ as

$$\mathcal{J}_{p,q}^n = \Big(n^{\frac{1}{2p}} L_{2p}^r \cap n^{\frac{1}{2q}} L_{2q}^r\Big)\Big(n^{\frac{1}{2p}} L_{2p}^c \cap n^{\frac{1}{2q}} L_{2q}^c\Big).$$

According to [16], we find

$$\mathcal{J}_{p,q}^n = n^{\frac{1}{p}} L_{2p}^r L_{2p}^c \cap n^{\frac{1}{2p}+\frac{1}{2q}} L_{2p}^r L_{2q}^c \cap n^{\frac{1}{2q}+\frac{1}{2p}} L_{2q}^r L_{2p}^c \cap n^{\frac{1}{q}} L_{2q}^r L_{2q}^c. \tag{2.1}$$

These spaces will be rigorously defined below. Our only aim here is to motivate the forthcoming results and definitions. Let us now see how the space in (2.1) generalizes our first definition of $\mathcal{J}_{p,q}^n(\Omega)$. On the Banach space level we have the isometries $L_{2p}^r L_{2q}^c = L_s = L_{2q}^r L_{2p}^c$ with $1/s = 1/2p + 1/2q$. Moreover, again by Hölder inequality it is clear that

$$n^{\frac{1}{s}} \|f\|_s \le \max\Big\{n^{\frac{1}{p}} \|f\|_p, n^{\frac{1}{q}} \|f\|_q\Big\}$$

and the two cross terms in (2.1) disappear. However, in the category of operator spaces the four terms have a significant contribution. The operator space/free version of (Σ_{pq}) is the main result in [16], and goes further than its commutative counterpart. More precisely, in contrast with the classical case, we find a right formulation for $(\Sigma_{\infty q})$. Indeed, as for Khintchine and Rosenthal inequalities, the limit case as $p \to \infty$ holds when replacing independence by freeness.

Now we give detailed definitions and results. Let us write $L_2^r(\mathcal{M})$ and $L_2^c(\mathcal{M})$ for the row/column quantizations of $L_2(\mathcal{M})$ and let $2 \le q \le \infty$. Then, the row/column structures on $L_q(\mathcal{M})$ are defined as follows

$$\begin{array}{rcl} L_q^r(\mathcal{M}) & = & \big[\mathcal{M}, L_2^r(\mathcal{M})\big]_{\frac{2}{q}}, \\ L_q^c(\mathcal{M}) & = & \big[\mathcal{M}, L_2^c(\mathcal{M})\big]_{\frac{2}{q}}. \end{array} \tag{2.2}$$

In fact, a rigorous definition should take Kosaki's embeddings into account as done in [16, Identity (1.3)], but we shall ignore such formalities. Now, if $2 \le u, v \le \infty$ and $1/p = 1/u + 1/v$ for some $1 \le p \le \infty$, we define the *asymmetric L_p space* associated to the pair (u,v) as the $\mathcal{M}$-amalgamated Haagerup tensor product

$$L_{(u,v)}(\mathcal{M}) = L_u^r(\mathcal{M}) \otimes_{\mathcal{M},h} L_v^c(\mathcal{M}). \tag{2.3}$$

That is, we consider the quotient of $L_u^r(\mathcal{M}) \otimes_h L_v^c(\mathcal{M})$ by the closed subspace $\mathcal{I}$ generated by the differences $x_1\gamma \otimes x_2 - x_1 \otimes \gamma x_2$ with $\gamma \in \mathcal{M}$. By a well known factorization argument the norm of an element x in $L_{(u,v)}(\mathcal{M})$ is given by $\|x\|_{(u,v)} = \inf_{x=\alpha\beta} \|\alpha\|_{L_u(\mathcal{M})} \|\beta\|_{L_v(\mathcal{M})}$, see Lemma 1.9 in [16].

- If $\mathcal{M} = M_m$, the space in (2.3) reduces to $S_{(u,v)}^m = C_{u/2}^m \otimes_h R_{v/2}^m$.
- We have a cb-isometry $L_p(\mathcal{M}) = L_{(2p,2p)}(\mathcal{M})$, see [16, Remark 7.5].

Let $1 \le q \le p \le \infty$. According to the discussion which led to (2.1), we know how the general aspect of $\mathcal{J}_{p,q}^n(\mathcal{M})$ should be. Now, equipped with asymmetric L_p spaces we obtain a factorization of noncommutative L_p spaces in the right way:

$$\mathcal{J}_{p,q}^n(\mathcal{M}) = \bigcap_{u,v \in \{2p,2q\}} n^{\frac{1}{u}+\frac{1}{v}} \, L_{(u,v)}(\mathcal{M}).$$

If we take

$$\mathcal{M}_m = M_m(\mathcal{M}) \quad \text{and} \quad \mathsf{E}_m = id_{M_m} \otimes \varphi : \mathcal{M}_m \to M_m$$

for $m \ge 1$ and define

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p} \quad \text{and} \quad \frac{1}{\gamma(u,v)} = \frac{1}{u} + \frac{1}{p} + \frac{1}{v},$$

we have an isometry

$$S_p^m\big(\mathcal{J}_{p,q}^n(\mathcal{M})\big) = \bigcap_{u,v \in \{2r,\infty\}} n^{\frac{1}{\gamma(u,v)}} \, L_u^{-1}(M_m) L_{\gamma(u,v)}(\mathcal{M}_m) L_v^{-1}(M_m). \tag{2.4}$$

The proof can be found in [16]. Let $\mathcal{N}$ be a conditioned subalgebra of $\mathcal{M}$ with corresponding conditional expectation $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$. According to (2.4), we define the $\mathcal{J}$-spaces

$$\mathcal{J}_{p,q}^n(\mathcal{M}, \mathcal{E}_{\mathcal{N}}) = \bigcap_{u,v \in \{2r,\infty\}} n^{\frac{1}{\gamma(u,v)}} \, L_u^{-1}(\mathcal{N}) L_{\gamma(u,v)}(\mathcal{M}) L_v^{-1}(\mathcal{N}).$$

The isometry (2.4) shows us the way to follow. The philosophy is that complete boundedness arises from amalgamation, see [15, 16]. Indeed, instead of working with the o.s.s. of the spaces $\mathcal{J}_{p,q}^n(\mathcal{M})$, it suffices to argue with the Banach space structure of the more general spaces $\mathcal{J}_{p,q}^n(\mathcal{M}, \mathcal{E}_{\mathcal{N}})$. In this spirit, for $1 \le q \le p \le \infty$ we set $1/r = 1/q - 1/p$ and introduce the spaces

$$\mathcal{R}_{2p,q}^n(\mathcal{M}, \mathcal{E}_{\mathcal{N}}) \quad = \quad n^{\frac{1}{2p}} \, L_{2p}(\mathcal{M}) \cap n^{\frac{1}{2q}} \, L_{2r}^{-1}(\mathcal{N}) L_{2q}(\mathcal{M}) L_\infty^{-1}(\mathcal{N}),$$

$$\mathcal{C}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \;=\; n^{\frac{1}{2p}} L_{2p}(\mathcal{M}) \cap n^{\frac{1}{2q}} L_\infty^{-1}(\mathcal{N}) L_{2q}(\mathcal{M}) L_{2r}^{-1}(\mathcal{N}).$$

Remark 2.1. The notion of $\mathcal{M}$-amalgamated Haagerup tensor product $\mathrm{X}_1 \otimes_{\mathcal{M},h} \mathrm{X}_2$ extends naturally to any pair $(\mathrm{X}_1, \mathrm{X}_2)$ of operator spaces such that X_1 contains $\mathcal{M}$ as a right ideal and X_2 does it as a left ideal. We shall write $\mathrm{X}_1 \otimes_\mathcal{M} \mathrm{X}_2$ to denote the underlying Banach space structure of $\mathrm{X}_1 \otimes_{\mathcal{M},h} \mathrm{X}_2$. According to the definition of the Haagerup tensor product and recalling the isometric embeddings $\mathrm{X}_j \subset \mathcal{B}(\mathcal{H}_j)$, we have

$$\|x\|_{\mathrm{X}_1 \otimes_\mathcal{M} \mathrm{X}_2} = \inf \Big\{ \Big\| \big(\sum\nolimits_k x_{1k} x_{1k}^* \big)^{1/2} \Big\|_{\mathcal{B}(\mathcal{H}_1)} \Big\| \big(\sum\nolimits_k x_{2k}^* x_{2k} \big)^{1/2} \Big\|_{\mathcal{B}(\mathcal{H}_2)} \Big\},$$

where the infimum runs over all possible decompositions of $x + \mathcal{I}$ into a finite sum

$$x = \sum\nolimits_k x_{1k} \otimes x_{2k} + \mathcal{I}.$$

This uses the o.s.s. of X_j since row/column square functions live in $\mathcal{B}(\mathcal{H}_j)$ but not necessarily in X_j. Therefore, if X_1 is stable under row square functions (finite sums) and X_2 is stable under column square functions, no o.s.s. on the X_j's is needed to define $\mathrm{X}_1 \otimes_\mathcal{M} \mathrm{X}_2$. In particular, we may consider the Banach space

$$\mathcal{R}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \otimes_\mathcal{M} \mathcal{C}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}).$$

In what follows, we might abuse of the terminology by writing $\mathrm{X}_1 \otimes_{\mathcal{M},h} \mathrm{X}_2$ for the space $\mathrm{X}_1 \otimes_\mathcal{M} \mathrm{X}_2$. If that happens, it will be clear what space do we mean from the context. The reader can find some of the basic properties of this construction in [16, Chapters 6 and 7].

Remark 2.2. The cb-isometry

$$\mathrm{X}_1 \otimes_{\mathcal{M},h} \mathrm{X}_2 \;=\; \big(\mathrm{X}_1 \otimes_h R_m\big) \otimes_{M_m(\mathcal{M}),h} \big(C_m \otimes_h \mathrm{X}_2\big)$$

reflects the behavior of row/column operator spaces with respect to amalgamated tensor products and it is a key property that we will be using along the paper. Indeed, it suffices to understand that

$$\dim R_m \otimes_{M_m,h} C_m = 1,$$

which follows from the fact that $e_{1i} \otimes e_{j1} \sim \delta_{ij} e_{11} \otimes e_{11}$ where $\sim$ refers to the equivalence relation imposed by the quotient of amalgamation. These equivalences can be easily justified by the reader.

The isomorphisms below are the key results in [16]:

a) If $1 \le q \le p \le \infty$, we have

$$\mathcal{J}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \simeq \mathcal{R}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \otimes_\mathcal{M} \mathcal{C}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}).$$

b) If $1 \le p \le \infty$ and $1/q = 1 - \theta + \theta/p$, we have

$$\mathcal{J}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \simeq \big[\mathcal{J}_{p,1}^n(\mathcal{M},\mathcal{E}_\mathcal{N}), \mathcal{J}_{p,p}^n(\mathcal{M},\mathcal{E}_\mathcal{N})\big]_\theta.$$

Moreover, in all cases the involved relevant constants are independent of n.

Remark 2.3. It is worth mentioning that the constants for

$$\mathcal{J}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \simeq \big[\mathcal{J}_{p,1}^n(\mathcal{M},\mathcal{E}_\mathcal{N}), \mathcal{J}_{p,p}^n(\mathcal{M},\mathcal{E}_\mathcal{N})\big]_\theta$$

obtained in [16, Theorem 7.2] have a singularity when $(p,q) \sim (\infty,1)$. To be more explicit, for q small and p large we obtain a constant $c_{p,q} \sim (p-q)/(pq+q-p)$ and the same singularity appears in Theorem 2.4 below. However, the assertion in such theorem holds for the extremal values $(p,q) = (\infty,1)$ and the singularity seems to be removable. It appears as a byproduct of the noncommutative Burkholder inequality, which is used in the argument, see [16, Remark 7.11]. Fortunately, our construction of operator space L_p embeddings in [15, 16] only uses Theorem 2.4 for $q=2$ and no singularity occurs in that case.

In what follows, we shall also need to work with the predual space of $\mathcal{J}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N})$ which is defined as follows. Given $1 \le p \le q \le \infty$, let us consider the index $\frac{1}{r} = \frac{1}{p} - \frac{1}{q}$ and the coefficient $\rho(u,v)$ determined by $\frac{1}{\rho(u,v)} = \frac{1}{p} - \frac{1}{u} - \frac{1}{v}$. Then, we define the space

$$\mathcal{K}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) = \sum_{u,v \in \{2r,\infty\}} n^{\frac{1}{\rho(u,v)}} L_u(\mathcal{N}) L_{\rho(u,v)}(\mathcal{M}) L_v(\mathcal{N}).$$

The antilinear bracket $\langle x,y\rangle_n = n\,\mathrm{tr}(xy^*)$ gives $\mathcal{K}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N})^* = \mathcal{J}_{p',q'}^n(\mathcal{M},\mathcal{E}_\mathcal{N})$, see [16, Remark 7.4]. As for the $\mathcal{J}$-spaces, we shall write $\mathcal{K}_{p,q}^n(\mathcal{M})$ to denote the space $\mathcal{K}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N})$ with $(\mathcal{N},\mathcal{E}_\mathcal{N}) = (\langle \mathbf{1}_\mathcal{M}\rangle, \varphi)$. Having defined the noncommutative forms of our $\mathcal{J}$ and $\mathcal{K}$ spaces, we can now state the free analog of (Σ_{pq}) and its dual from [16]. It is fortunate that in the free case a more general statement holds not requiring copies.

Theorem 2.4. *If $1 \le p \le q \le \infty$, the maps*

$$\sum_{k=1}^n x_k \otimes \delta_k \in \mathcal{K}_{p,q}^1\Big(\ell_\infty^n(\mathcal{M}), \frac{1}{n}\sum_{k=1}^n \mathcal{E}_\mathcal{N}\Big) \mapsto \sum_{k=1}^n \pi_{free}^k(x_k) \otimes \delta_k \in L_p(\mathcal{A}_{free};\ell_q^n)$$
$$\sum_{k=1}^n x_k \otimes \delta_k \in \mathcal{J}_{p',q'}^1\Big(\ell_\infty^n(\mathcal{M}), \frac{1}{n}\sum_{k=1}^n \mathcal{E}_\mathcal{N}\Big) \mapsto \sum_{k=1}^n \pi_{free}^k(x_k) \otimes \delta_k \in L_{p'}(\mathcal{A}_{free};\ell_{q'}^n)$$

are isomorphisms with complemented range and constants independent of n. In particular, considering the restriction to the diagonal subspaces $x_1 = x_2 = \ldots = x_n$ we obtain isomorphisms

$$x \in \mathcal{K}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \mapsto \sum_{k=1}^n \pi_{free}^k(x) \otimes \delta_k \in L_p(\mathcal{A}_{free};\ell_q^n),$$
$$x \in \mathcal{J}_{p',q'}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \mapsto \sum_{k=1}^n \pi_{free}^k(x) \otimes \delta_k \in L_p(\mathcal{A}_{free};\ell_q^n),$$

with complemented range and constants independent of n. Moreover, replacing $(\mathcal{M},\mathcal{N},\mathcal{E}_\mathcal{N})$ by $(M_m(\mathcal{M}), M_m, id_{M_m} \otimes \varphi)$, we obtain complete isomorphisms with completely complemented ranges and constants independent of n

$$x \in \mathcal{K}_{p,q}^n(\mathcal{M}) \mapsto \sum_{k=1}^n \pi_{free}^k(x) \otimes \delta_k \in L_p(\mathcal{R}_{free};\ell_q^n),$$
$$x \in \mathcal{J}_{p',q'}^n(\mathcal{M}) \mapsto \sum_{k=1}^n \pi_{free}^k(x) \otimes \delta_k \in L_p(\mathcal{R}_{free};\ell_q^n).$$

$\mathcal{R}_{free}$ stands for the non-amalgamated free product $(\mathcal{M},\varphi) * (\mathcal{M},\varphi) * \cdots * (\mathcal{M},\varphi)$.

Sketch of the proof. It clearly suffices to prove the first assertion. Since the $\mathcal{J}_{pq}^n$-spaces form an interpolation scale, as observed in Remark 2.3, the same holds for $\mathcal{K}_{pq}^n$ by duality. On the other hand, the spaces $L_p(\mathcal{A}_{free};\ell_q^n)$ are particular examples of amalgamated or conditional L_p spaces (according to the value of p and q) and hence also form an interpolation scale. This is well-known for hyperfinite algebras through Pisier's work, but note that the free product are in general not hyperfinite. It suffices to show the extremal cases. The case $p = q$ is trivial, while the argument for the map $\mathcal{J}_{p',1}^1 \to L_{p'}(\ell_1^n)$ is essentially contained in [13]. Indeed, using

$$\Big\| \sum_{k=1}^n x_k \otimes \delta_k \Big\|_{L_p(\mathcal{A}_{free};\ell_1^n)} = \inf_{x_k = a_k b_k} \Big\| \big(\sum_{k=1}^n a_k a_k^* \big)^{\frac12} \Big\|_p \Big\| \big(\sum_{k=1}^n b_k^* b_k \big)^{\frac12} \Big\|_p,$$

see e.g. [10], the norm estimate for $\mathcal{J}_{p',1}^1$ boils down to

$$\Big\| \big(\sum_k \pi_{free}^k(x_k) \pi_{free}^k(x_k)^* \big)^{\frac12} \Big\|_p \lesssim \Big(\sum_{k=1}^n \|x_k\|_{2p}^{2p} \Big)^{\frac{1}{2p}} + \Big\| \big(\sum_{k=1}^n \mathcal{E}_\mathcal{N}(x_k x_k^*) \big)^{\frac12} \Big\|_{2p} \tag{2.5}$$

and its column version. Equation (2.5) follows from the free Rosenthal inequality [18], see [16, Corollary 5.3]. It is important to recall that, in contrast to the free Rosenthal inequality, we do not require the x_k's to be mean-zero in (2.5). The combination of (2.5) with $\mathcal{J}_{p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \simeq \mathcal{R}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N}) \otimes_\mathcal{M} \mathcal{C}_{2p,q}^n(\mathcal{M},\mathcal{E}_\mathcal{N})$ gives rise to the boundedness of $\mathcal{J}_{p',1}^1 \to L_{p'}(\ell_1^n)$. By interpolation, we deduce that $\mathcal{J}_{p',q'}^1 \to L_{p'}(\ell_{q'}^n)$ is bounded. By duality, see the argument in [13], the boundedness of the inverse and the fact that the range is complemented follows from the boundedness of the map $\mathcal{K}_{pq}^1 \to L_p(\ell_q^n)$. This is even easier and follows from algebraic considerations that can be found in [13, Proposition 3.5]. The proof is complete. □

Remark 2.5. Several remarks are in order:

i) The first result of this kind appeared in [13], where tensor independence played the role of freeness and amalgamation was not considered. According to the classical theory, we found in this case a non-removable singularity as $p \to \infty$. Then, the duality argument produces a singularity as $p \to 1$. As a byproduct of our methods, we shall show in this paper that this singularity is removable. A key point for it is to observe that, in the free setting considered in Theorem 2.4, we only find a (apparently removable) singularity when $(p,q) \to (1,\infty)$ simultaneously.

ii) The variables $\pi_{free}^k(x_k)$ are replaced by $\pi_{free}^k(x_k, -x_k)$ in the formulation of Theorem 2.4 given in [16]. This was done to create mean-zero random variables, a necessary condition for the free Rosenthal inequality [18]. In (2.5) mean-zero random variables are not required and this simplifies our embedding.

iii) The simpler formulation in [13] allowed us to take values in an arbitrary operator space X. In the framework of Theorem 2.4, this requires some additional insight that will be analyzed in a forthcoming paper.

3. Sums of independent copies in $L_1(\ell_\infty)$

Let $(\mathcal{M}_k)_{k\ge1}$ be an increasingly independent family of top-subsymmetric copies of the von Neumann algebra $\mathcal{M}$ over $\mathcal{N}$. Let us also consider a σ-finite von Neumann algebra $\mathcal{R}$ equipped with a normal faithful state ϕ. Given $x \in L_1(\mathcal{M}) \otimes \mathcal{R}$, we want to study sums of independent copies in $L_1(\mathcal{A};\ell_\infty(\mathcal{R}))$. According to Section 2, we know that

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} = \inf_{\pi_k(x) = \alpha y_k \beta} \|\alpha\|_{L_2(\mathcal{A})} \Big(\sup_{1\le k\le n} \|y_k\|_{\mathcal{A}\bar\otimes\mathcal{R}} \Big) \|\beta\|_{L_2(\mathcal{A})}.$$

On the other hand, we have the following formula for any $z \in \mathcal{M}\bar{\otimes}L_1(\mathcal{R})$

$$\|z\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} = \inf \Big\{ \Big\| \Big(\sum_k \mathsf{E}_\mathcal{M}(a_k a_k^*) \Big)^{\frac12} \Big\|_\mathcal{M} \Big\| \Big(\sum_k \mathsf{E}_\mathcal{M}(b_k^* b_k) \Big)^{\frac12} \Big\|_\mathcal{M} \Big\} \tag{3.1}$$

where the infimum runs over all decompositions of z into a finite sum $\sum_k a_k b_k$ and $\mathsf{E}_\mathcal{M} : \mathcal{M}\bar{\otimes}\mathcal{R} \to \mathcal{M}$ is the conditional expectation $\mathsf{E}_\mathcal{M} = id \otimes \phi$. Indeed, recall that the term on the right is the norm of z in

$$L_\infty^r(\mathcal{M}\bar{\otimes}\mathcal{R}; \mathsf{E}_\mathcal{M}) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{R}} L_\infty^c(\mathcal{M}\bar{\otimes}\mathcal{R}; \mathsf{E}_\mathcal{M}).$$

Then, since the following identifications clearly hold

$$\begin{aligned} L_\infty^r(\mathcal{M}\bar{\otimes}\mathcal{R}; \mathsf{E}_\mathcal{M}) &= L_2^{-1}(\mathcal{M}) L_2(\mathcal{M}\bar{\otimes}\mathcal{R}) L_\infty^{-1}(\mathcal{M}), \\ L_\infty^c(\mathcal{M}\bar{\otimes}\mathcal{R}; \mathsf{E}_\mathcal{M}) &= L_\infty^{-1}(\mathcal{M}) L_2(\mathcal{M}\bar{\otimes}\mathcal{R}) L_2^{-1}(\mathcal{M}), \end{aligned}$$

we deduce that we are talking about the norm of z in

$$L_2^{-1}(\mathcal{M}) L_2(\mathcal{M}\bar{\otimes}\mathcal{R}) L_\infty^{-1}(\mathcal{M}) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{R}} L_\infty^{-1}(\mathcal{M}) L_2(\mathcal{M}\bar{\otimes}\mathcal{R}) L_2^{-1}(\mathcal{M}).$$

In these terms, it is clear that (3.1) follows from [16, Proposition 6.9] after taking $(\mathcal{M},\mathcal{N})$ there to be our pair $(\mathcal{M}\bar{\otimes}\mathcal{R},\mathcal{M})$, recall one more time that our terminology for conditional L_p spaces is different from the one used in [16]. Given $\lambda > 0$, we also define the spaces

$$\begin{aligned} \mathcal{R}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N}) &= \mathcal{M} \cap \sqrt{\lambda}\, L_2^{-1}(\mathcal{N}) L_2(\mathcal{M}) L_\infty^{-1}(\mathcal{N}), \\ \mathcal{C}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N}) &= \mathcal{M} \cap \sqrt{\lambda}\, L_\infty^{-1}(\mathcal{N}) L_2(\mathcal{M}) L_2^{-1}(\mathcal{N}). \end{aligned}$$

According to [16], the norm on these spaces has the form

$$\begin{aligned} \|\alpha\|_{\mathcal{R}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N})} &= \max \Big\{ \|\alpha\|_\mathcal{M}, \sqrt{\lambda}\, \big\| \mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12} \big\|_\mathcal{N} \Big\}, \\ \|\beta\|_{\mathcal{C}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N})} &= \max \Big\{ \|\beta\|_\mathcal{M}, \sqrt{\lambda}\, \big\| \mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12} \big\|_\mathcal{N} \Big\}. \end{aligned}$$

Of course, when $\lambda = n$ we recover the spaces $\mathcal{R}_{\infty,1}^n(\mathcal{M},\mathcal{E}_\mathcal{N})$ and $\mathcal{C}_{\infty,1}^n(\mathcal{M},\mathcal{E}_\mathcal{N})$ from the previous section. Now we want to consider the corresponding $\mathcal{J}$-space with values in $L_1(\mathcal{R})$

$$\mathcal{J}_{\infty,1}^\lambda\big(\mathcal{M},\mathcal{E}_\mathcal{N}; L_1(\mathcal{R})\big) = \mathcal{R}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N}) \otimes_\mathcal{M} L_\infty\big(\mathcal{M}; L_1(\mathcal{R})\big) \otimes_\mathcal{M} \mathcal{C}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N}).$$

Note that both $\mathcal{R}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N})$ and $\mathcal{C}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N})$ coincide algebraically with $\mathcal{M}$ itself. Thus, the norm in $\mathcal{J}_{\infty,1}^\lambda\big(\mathcal{M},\mathcal{E}_\mathcal{N}; L_1(\mathcal{R})\big)$ of an element z in the dense subspace $\mathcal{M} \otimes L_1(\mathcal{R})$ is given by

$$\|z\|_{\mathcal{J}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N};L_1(\mathcal{R}))} = \inf_{z=\alpha y\beta} \|\alpha\|_{\mathcal{R}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N})} \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} \|\beta\|_{\mathcal{C}_{\infty,1}^\lambda(\mathcal{M},\mathcal{E}_\mathcal{N})}.$$

When no confusion can arise, we shall write $\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))$.

Lemma 3.1. *We have*

$$\begin{aligned} \|z\|_{\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))} \sim \max \Big\{ & \|z\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}, \\ & \sqrt{\lambda} \inf_{z=\alpha y} \big\| \mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12} \big\|_\mathcal{N} \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}, \\ & \sqrt{\lambda} \inf_{z=y\beta} \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} \big\| \mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12} \big\|_\mathcal{N}, \\ & \lambda \inf_{z=\alpha y\beta} \big\| \mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12} \big\|_\mathcal{N} \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} \big\| \mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12} \big\|_\mathcal{N} \Big\}. \end{aligned}$$

Moreover, the relevant constants are independent of λ.

Proof. The argument can be found in Chapter 6 of [16], see Lemma 6.3. $\square$

Once we have introduced the key spaces, we are ready for some preliminary estimates. In what follows, we shall assume that z is an element in $\mathcal{J}_{\infty,1}^{\lambda}(L_1(\mathcal{R}))$ and we shall work with a factorization $z = \alpha y \beta$ with

$$\max \Big\{ \|\alpha\|_{\mathcal{M}}, \sqrt{\lambda} \, \big\| \mathcal{E}_{\mathcal{N}}(\alpha\alpha^*)^{\frac{1}{2}} \big\|_{\mathcal{N}} \Big\} \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} \max \Big\{ \|\beta\|_{\mathcal{M}}, \sqrt{\lambda} \, \big\| \mathcal{E}_{\mathcal{N}}(\beta^*\beta)^{\frac{1}{2}} \big\|_{\mathcal{N}} \Big\}$$

being $\le \|z\|_{\mathcal{J}_{\infty,1}^{\lambda}(L_1(\mathcal{R}))} + \varepsilon$ for small ε. Let $a = \sqrt{\mathbf{1} - \alpha\alpha^*}, b = \sqrt{\mathbf{1} - \beta^*\beta}$ and

$$A_k = \pi_1(a)\pi_2(a) \cdots \pi_{k-1}(a) \quad , \quad B_k = \pi_{k-1}(b) \cdots \pi_2(b)\pi_1(b).$$

Lemma 3.2. *We have*

$$\Big\| \sum_{k=1}^n A_k \pi_k(z) B_k \otimes \delta_k \Big\|_{L_\infty(\mathcal{A};\ell_1^n(L_1(\mathcal{R})))} \le \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}.$$

Proof. We claim that

$$\begin{aligned} & \Big\| \sum_{k=1}^n \alpha_k y_k \beta_k \otimes \delta_k \Big\|_{L_\infty(\mathcal{A};\ell_1^n(L_1(\mathcal{R})))} \\ & \qquad \le \quad \Big\| \big(\sum_{k=1}^n \alpha_k \alpha_k^* \big)^{\frac{1}{2}} \Big\|_{\mathcal{A}} \Big(\sup_{1 \le k \le n} \|y_k\|_{L_\infty(\mathcal{A};L_1(\mathcal{R}))} \Big) \Big\| \big(\sum_{k=1}^n \beta_k^* \beta_k \big)^{\frac{1}{2}} \Big\|_{\mathcal{A}}. \end{aligned}$$

Indeed, if $(\alpha, \beta) = \big(\sum_k \alpha_k \alpha_k^*, \sum_k \beta_k^* \beta_k \big)$, we find $v_k, w_k \in \mathcal{A}$ with

- $\alpha_k = \alpha^{\frac{1}{2}} v_k$ and $\beta_k = w_k \beta^{\frac{1}{2}}$,
- $\max \Big\{ \| \sum_k v_k v_k^* \|_{\mathcal{A}}, \| \sum_k w_k^* w_k \|_{\mathcal{A}} \Big\} \le 1.$

Since $L_\infty(\mathcal{A};\ell_1^n(L_1(\mathcal{R})))$ is an $\mathcal{A}$-bimodule, it suffices to show

$$\Big\| \sum_{k=1}^n v_k y_k w_k \otimes \delta_k \Big\|_{L_\infty(\mathcal{A};\ell_1^n(L_1(\mathcal{R})))} \le \sup_{1 \le k \le n} \|y_k\|_{L_\infty(\mathcal{A};L_1(\mathcal{R}))}.$$

Factorize $y_k = \sum_s a_{ks} b_{ks}$ in such a way that

$$\max \Big\{ \Big\| \sum_s \mathsf{E}_{\mathcal{A}}(a_{ks} a_{ks}^*) \Big\|_{\mathcal{A}}, \Big\| \sum_s \mathsf{E}_{\mathcal{A}}(b_{ks}^* b_{ks}) \Big\|_{\mathcal{A}} \Big\} \le \|y_k\|_{L_\infty(\mathcal{A};L_1(\mathcal{R}))} + \varepsilon.$$

In particular, we may factorize $v_k y_k w_k = \sum_s v_k a_{ks} b_{ks} w_k$ to deduce the estimate

$$\begin{aligned} & \Big\| \sum_{k=1}^n v_k y_k w_k \otimes \delta_k \Big\|_{L_\infty(\mathcal{A};\ell_1^n(L_1(\mathcal{R})))} \\ & \qquad \le \quad \Big\| \sum_{k,s} \mathsf{E}_{\mathcal{A}} \big(v_k a_{ks} a_{ks}^* v_k^* \big) \Big\|_{\mathcal{A}}^{\frac{1}{2}} \Big\| \sum_{k,s} \mathsf{E}_{\mathcal{A}} \big(w_k^* b_{ks}^* b_{ks} w_k \big) \Big\|_{\mathcal{A}}^{\frac{1}{2}} \\ & \qquad \le \quad \Big\| \sum_k v_k v_k^* \Big\|_{\mathcal{A}}^{\frac{1}{2}} \Big(\sup_{1 \le k \le n} \|y_k\|_{L_\infty(\mathcal{A};L_1(\mathcal{R}))} + \varepsilon \Big) \Big\| \sum_k w_k^* w_k \Big\|_{\mathcal{A}}^{\frac{1}{2}} \\ & \qquad \le \quad \sup_{1 \le k \le n} \|y_k\|_{L_\infty(\mathcal{A};L_1(\mathcal{R}))} + \varepsilon. \end{aligned}$$

This proves our claim if we let $\varepsilon \to 0^+$. Applying it for

$$(\alpha_k, y_k, \beta_k) = \big(A_k \pi_k(\alpha), \pi_k(y), \pi_k(\beta) B_k \big),$$

gives rise to

$$\Big\| \sum_{k=1}^n A_k \pi_k(z) B_k \otimes \delta_k \Big\|_{L_\infty(\mathcal{A};\ell_1^n(L_1(\mathcal{R})))}$$
$$\le \quad \Big\| \big(\sum_{k=1}^n A_k \pi_k(\alpha\alpha^*) A_k^* \big)^{\frac12} \Big\|_{\mathcal{A}} \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} \Big\| \big(\sum_{k=1}^n B_k^* \pi_k(\beta^*\beta) B_k \big)^{\frac12} \Big\|_{\mathcal{A}}.$$

Therefore, the assertion follows from

$$\Big\| \sum\nolimits_k A_k \pi_k(\alpha\alpha^*) A_k^* \Big\|_{\mathcal{A}} \le 1 \quad \text{and} \quad \Big\| \sum_k B_k^* \pi_k(\beta^*\beta) B_k \Big\|_{\mathcal{A}} \le 1.$$

Indeed, these estimates are implicit in [12, Lemma 7.6]. The proof is complete. □

The next lemma requires a further property of the space $L_\infty(\mathcal{M}; L_1(\mathcal{R}))$ used in the definition of the amalgamated tensor product $\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))$. Indeed, we require that for a conditioned subalgebra $\mathcal{N}$ of $\mathcal{M}$ and a normal $*$-homomorphism $\rho : \mathcal{M} \to \mathcal{N}\bar{\otimes}\mathcal{B}(\ell_2)$, we have

$$\big\| \rho \otimes id : L_\infty\big(\mathcal{M}; L_1(\mathcal{R})\big) \to L_\infty\big(\mathcal{N}\bar{\otimes}\mathcal{B}(\ell_2); L_1(\mathcal{R})\big) \big\| \le 1. \tag{3.2}$$

To prove it we use an isometric inclusion

$$L_\infty(\mathcal{M}; L_1(\mathcal{R})) \subset \mathsf{NDEC}(\mathcal{R}^{op}, \mathcal{M}). \tag{3.3}$$

In other words, by slicing $T_x(r) = (id \otimes r)(x)$, we can view $T_x : \mathcal{R}^{op} \to \mathcal{M}$ as a normal decomposable map. The norm in the space of decomposable maps (see [19] for details) is given by

$$\|T_x\|_{dec} = \inf \left\{ \Big\| \begin{pmatrix} S_1 & T_x \\ T_x^* & S_2 \end{pmatrix} \Big\|_{M_2(\mathcal{R}) \to M_2(\mathcal{M})} \text{s.t.} \begin{pmatrix} S_1 & T_x \\ T_x^* & S_2 \end{pmatrix} \text{completely positive} \right\}.$$

To prove (3.3) let us take $a, b \in L_2(\mathcal{M})$ and $M_{ab}(y) = ayb$. For $x \in \mathcal{M} \otimes L_1(\mathcal{R})$ we see that $M_{ab}T_x \in \mathsf{NDEC}(\mathcal{R}^{op}, L_1(\mathcal{M}))$ and is of finite rank. However, every completely positive map $T' : \mathcal{R}^{op} \to \mathcal{M}_*^{op}$ defines an element in $(\mathcal{R}^{op} \otimes_{\max} \mathcal{M}^{op})^*$. Given a finite tensor $z = \sum_j r_j \otimes m_j$ of norm less than one, we may lift it to an element $\hat{z}$ of norm less than one in the unit ball of $\mathcal{R}^{op} \otimes_{\max} \mathcal{M}^{op}$. This implies

$$|\langle \hat{z}, M_{ab}T_x \rangle| \le \|T_x\|_{dec} \|a\|_2 \|b\|_2 \, .$$

Since $M_{ab}T_x$ is of finite rank we know that

$$|\langle \hat{z}, M_{ab}T_x \rangle| = |\langle z, axb \rangle| \, .$$

Then [16, Proposition 6.9] implies that

$$\|x\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} = \sup_{\|a\|_{L_2(\mathcal{M})}, \, \|b\|_{L_2(\mathcal{M})} \le 1} \|axb\|_1 \le \|T_x\|_{dec}.$$

According to (3.1), the converse follows easily by factoring $x = x_1 x_2$ and using that $x_1 x_1^*$ and $x_2^* x_2$ correspond to completely positive maps. Then (3.2) follows from the fact that $\|\rho T_x\|_{dec} \le \|T_x\|_{dec}$.

In the following result, we shall use the conditional expectations $\mathsf{E}_k : \mathcal{A} \to \mathcal{M}_k$.

Lemma 3.3. *If $\delta \le 1/e$ and $\lambda = \delta^{-1} n$, we have*

$$\Big\| \pi_n^{-1} \Big[\mathsf{E}_n \Big(\sum_{k=1}^n (\mathbf{1} - A_k) \pi_n(z) (\mathbf{1} - B_k) \Big) \Big] \Big\|_{\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))} \le 2ne\delta \, \|z\|_{\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))}.$$

Proof. By the normal version of Kasparov's dilation theorem [27], we may assume $\mathsf{E}_n(x) = e_{11}\rho(x)e_{11}$, where $\rho : \mathcal{A} \to \mathcal{M}_n \bar{\otimes} \mathcal{B}(\ell_2)$ is a normal $*$-homomorphism. Let us factorize $z = \alpha y \beta$ as we did before Lemma 3.2, with $\lambda = \delta^{-1} n$. Then we get

$$\mathsf{E}_n\Big(\sum_{k=1}^n (\mathbf{1} - A_k)\pi_n(z)(\mathbf{1} - B_k)\Big) = \mathsf{RDC},$$

where $\mathsf{R}, \mathsf{D}, \mathsf{C}$ are given by

$$\begin{aligned}
\mathsf{R} &= \sum_{k=1}^n e_{11}\rho\big((\mathbf{1} - A_k)\pi_n(\alpha)\big) \otimes e_{1k},\\
\mathsf{D} &= \sum_{k=1}^n \rho(\pi_n(y)) \otimes e_{kk},\\
\mathsf{C} &= \sum_{k=1}^n \rho\big(\pi_n(\beta)(\mathbf{1} - B_k)\big)e_{11} \otimes e_{k1}.
\end{aligned}$$

It is easily checked that

$$\mathsf{RR}^* = \sum_{k=1}^n \mathsf{E}_n\big((\mathbf{1} - A_k)\pi_n(\alpha\alpha^*)(\mathbf{1} - A_k)^*\big) \otimes e_{11}.$$

According to the proof of [12, Lemma 7.8] we have

$$\big\|\pi_n^{-1}\big((\mathsf{RR}^*)^{\frac12}\big)\big\|_{\mathcal{M}} = \big\|\pi_n^{-1}(\mathsf{RR}^*)\big\|_{\mathcal{M}}^{\frac12} \le \sqrt{2ne\delta}\, \|\alpha\|_{\mathcal{M}}. \tag{3.4}$$

To estimate $\pi_n^{-1}\big((\mathsf{RR}^*)^{\frac12}\big)$ in $\mathcal{R}_{\infty,1}^\lambda(\mathcal{M}, \mathcal{E}_\mathcal{N})$ it remains to control the term

$$\sqrt{\delta^{-1}n}\, \big\|\mathcal{E}_\mathcal{N}\big(\pi_n^{-1}(\mathsf{RR}^*)\big)\big\|_\mathcal{N}^{\frac12} = \sqrt{\delta^{-1}n}\, \Big\|\sum_{k=1}^n \mathsf{E}_\mathcal{N}\big((\mathbf{1} - A_k)\pi_n(\alpha\alpha^*)(\mathbf{1} - A_k)^*\big)\Big\|_\mathcal{N}^{\frac12}.$$

Finally, applying Lemma 7.1 (ii) and Lemma 7.7 (iv) from [12], we obtain

$$\begin{aligned}
\big\|\mathcal{E}_\mathcal{N}\big(\pi_n^{-1}(\mathsf{RR}^*)\big)\big\|_\mathcal{N}^{\frac12} &= \Big\|\sum_{k=1}^n \mathsf{E}_\mathcal{N}\big((\mathbf{1} - A_k)\mathcal{E}_\mathcal{N}(\alpha\alpha^*)(\mathbf{1} - A_k)^*\big)\Big\|_\mathcal{N}^{\frac12} \\
&\le \Big\|\sum_{k=1}^n \mathsf{E}_\mathcal{N}\big((\mathbf{1} - A_k)(\mathbf{1} - A_k)^*\big)\Big\|_\mathcal{N}^{\frac12} \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)\big\|_\mathcal{M}^{\frac12} \le \sqrt{2ne\delta}\, \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)\big\|_\mathcal{M}^{\frac12}.
\end{aligned} \tag{3.5}$$

The combination of (3.4) and (3.5) (as well as a symmetric argument) produces

$$\big\|\pi_n^{-1}\big((\mathsf{RR}^*)^{\frac12}\big)\big\|_{\mathcal{R}_{\infty,1}^\lambda} \le \sqrt{2ne\delta}\, \|\alpha\|_{\mathcal{R}_{\infty,1}^\lambda} \quad , \quad \big\|\pi_n^{-1}\big((\mathsf{C}^*\mathsf{C})^{\frac12}\big)\big\|_{\mathcal{C}_{\infty,1}^\lambda} \le \sqrt{2ne\delta}\, \|\beta\|_{\mathcal{C}_{\infty,1}^\lambda}.$$

Since we have the factorization

$$\pi_n^{-1}\Big[\mathsf{E}_n\Big(\sum_{k=1}^n (\mathbf{1} - A_k)\pi_n(z)(\mathbf{1} - B_k)\Big)\Big] = \pi_n^{-1}\big((\mathsf{RR}^*)^{\frac12}\big)\pi_n^{-1}\big(u\mathsf{D}v\big)\pi_n^{-1}\big((\mathsf{C}^*\mathsf{C})^{\frac12}\big)$$

for certain contractions u, v, and $L_\infty(M_n(\mathcal{M}); L_1(\mathcal{R}))$ is an $\mathcal{M}$-bimodule, we deduce

$$\begin{aligned}
\Big\|\pi_n^{-1}\Big[\mathsf{E}_n\Big(\sum_{k=1}^n (\mathbf{1} - A_k)\pi_n(z)(\mathbf{1} - B_k)\Big)\Big]\Big\|_{\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))} \\
&\le 2ne\delta\, \|\alpha\|_{\mathcal{R}_{\infty,1}^\lambda} \|\beta\|_{\mathcal{C}_{\infty,1}^\lambda} \big\|\pi_n^{-1}(\mathsf{D})\big\|_{L_\infty(M_n(\mathcal{M}); L_1(\mathcal{R}))} \\
&\le 2ne\delta\, \|\alpha\|_{\mathcal{R}_{\infty,1}^\lambda} \|\beta\|_{\mathcal{C}_{\infty,1}^\lambda} \big\|\rho(\pi_n(y))\big\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))}
\end{aligned}$$

$$\begin{aligned} &\leq 2ne\delta\Big(\|\alpha\|_{\mathcal{R}^\lambda_{\infty,1}}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}\|\beta\|_{\mathcal{C}^\lambda_{\infty,1}}\Big) \\ &\leq 2ne\delta\Big(\|z\|_{\mathcal{J}^\lambda_{\infty,1}(L_1(\mathcal{R}))}+\varepsilon\Big). \end{aligned}$$

We have applied inequality (3.2). The assertion follows by letting $\varepsilon \to 0$ above. □

Lemma 3.4. *If $\delta \leq 1/e$ and $\lambda = \delta^{-1}n$, we have*

$$\Big\|\sum_{k=1}^n \pi_k(x)\otimes\delta_k\Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} \geq n(1-4e\delta)\,\|x\|_{\mathcal{J}^\lambda_{\infty,1}(L_1(\mathcal{R}))^*}.$$

Proof. By homogeneity, we will assume that

$$\|x\|_{\mathcal{J}^\lambda_{\infty,1}(L_1(\mathcal{R}))^*} = 1.$$

Let $z \in \mathcal{A}\otimes L_1(\mathcal{R})$ be a norm 1 element of $\mathcal{J}^\lambda_{\infty,1}(L_1(\mathcal{R}))$ such that $|\langle x,z\rangle| = 1-\gamma$ and factorize $z=\alpha y\beta$ with $\|\alpha\|_{\mathcal{R}^\lambda_{\infty,1}} = \|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))} = \|\beta\|_{\mathcal{C}^\lambda_{\infty,1}} \leq 1+\gamma$. First we observe from Lemma 3.2 that

$$\Big|\sum_{k=1}^n \big\langle \pi_k(x), A_k\pi_k(z)B_k\big\rangle\Big| \leq (1+\gamma)\Big\|\sum_{k=1}^n \pi_k(x)\otimes\delta_k\Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))}.$$

Now, to work through the error estimate, we use

$$z - azb = z(\mathbf{1}-b) + (\mathbf{1}-a)z - (\mathbf{1}-a)z(\mathbf{1}-b).$$

Hence

$$\begin{aligned} n(1-\gamma) &\leq \Big|\sum_{k=1}^n \langle\pi_k(x),\pi_k(z)\rangle\Big| \\ &\leq \Big|\sum_{k=1}^n \big\langle\pi_k(x),A_k\pi_k(z)B_k\big\rangle\Big| + \Big|\sum_{k=1}^n \big\langle\pi_k(x),\pi_k(z)-A_k\pi_k(z)B_k\big\rangle\Big| \\ &\leq (1+\gamma)\Big\|\sum_{k=1}^n \pi_k(x)\otimes\delta_k\Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} \\ &+ \Big|\sum_{k=1} \big\langle\pi_k(x),(\mathbf{1}-A_k)\pi_k(z)(\mathbf{1}-B_k)\big\rangle\Big| \\ &+ \Big|\sum_{k=1}^n \big\langle\pi_k(x),(\mathbf{1}-A_k)\pi_k(z)\big\rangle\Big| + \Big|\sum_{k=1} \big\langle\pi_k(x),\pi_k(z)(\mathbf{1}-B_k)\big\rangle\Big|. \end{aligned}$$

By top-subsymmetry and [12, Lemma 7.1], we deduce

$$\begin{aligned} &\Big|\sum_{k=1}^n \big\langle\pi_k(x),(\mathbf{1}-A_k)\,\pi_k(z)\big\rangle\Big| = \Big|\big\langle\pi_n(x),\sum_{k=1}^n(\mathbf{1}-A_k)\,\pi_n(z)\big\rangle\Big| \\ &= \Big|\Big\langle\pi_n(x),\mathsf{E}_n\Big(\sum_{k=1}^n(\mathbf{1}-A_k)\Big)\,\pi_n(z)\Big\rangle\Big| = \Big|\Big\langle x,\mathsf{E}_{\mathcal{N}}\Big(\sum_{k=1}^n(\mathbf{1}-A_k)\Big)z\Big\rangle\Big| \\ &\leq \Big\|\mathsf{E}_{\mathcal{N}}\Big(\sum_{k=1}^n(\mathbf{1}-A_k)\Big)z\Big\|_{\mathcal{J}^\lambda_{\infty,1}(L_1(\mathcal{R}))}\|x\|_{\mathcal{J}^\lambda_{\infty,1}(L_1(\mathcal{R}))^*} \leq \Big\|\mathsf{E}_{\mathcal{N}}\Big(\sum_{k=1}^n(\mathbf{1}-A_k)\Big)\Big\|_{\mathcal{N}}. \end{aligned}$$

Similarly, we find

$$\Big|\sum_{k=1}^n \big\langle \pi_k(x), \pi_k(z)\,(\mathbf{1}-B_k)\big\rangle\Big| \le \Big\|\mathsf{E}_\mathcal{N}\Big(\sum_{k=1}^n(\mathbf{1}-B_k)\Big)\Big\|_\mathcal{N}.$$

We refer to [12, Lemma 7.7 (iii)] for

$$\max\left\{\Big\|\mathsf{E}_\mathcal{N}\Big(\sum_{k=1}^n(\mathbf{1}-A_k)\Big)\Big\|_\mathcal{N}, \Big\|\mathsf{E}_\mathcal{N}\Big(\sum_{k=1}^n(\mathbf{1}-B_k)\Big)\Big\|_\mathcal{N}\right\} \le ne\delta.$$

Here we are using implicitly that we have

$$\max\Big\{\big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N}, \big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N}\Big\} \le \frac{1}{\sqrt{\delta^{-1}n}}.$$

Our argument for the symmetric term uses Lemma 3.3 instead

$$\begin{aligned}\Big|\sum_{k=1}^n \big\langle \pi_k(x), (\mathbf{1}-A_k)\,\pi_k(z)\,(\mathbf{1}-B_k)\big\rangle\Big| \\ = \quad \Big|\Big\langle x, \pi_n^{-1}\Big[\mathsf{E}_n\Big(\sum_{k=1}^n(\mathbf{1}-A_k)\,\pi_n(z)\,(\mathbf{1}-B_k)\Big)\Big]\Big\rangle\Big| \le 2ne\delta\,\|x\|_{\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))^*}.\end{aligned}$$

This yields

$$n(1-\gamma) \le (1+\gamma)\Big\|\sum_{k=1}^n \pi_k(x)\otimes\delta_k\Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} + 4ne\delta.$$

Taking $\gamma \to 0^+$, we deduce the assertion and the proof is complete. □

Remark 3.5. Apart from our references to [12] in Section 1, the estimation of the error terms above is the only place in this paper where top-subsymmetry really takes place.

Let us consider the following norms

$$\begin{aligned}\|x\|_{L_1^r(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} &= \inf_{x=ayb}\|a\|_{L_2(\mathcal{N})}\,\|y\|_{\mathcal{M}\bar\otimes\mathcal{R}}\|b\|_{L_2(\mathcal{M})},\\ \|x\|_{L_1^c(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} &= \inf_{x=ayb}\|a\|_{L_2(\mathcal{M})}\|y\|_{\mathcal{M}\bar\otimes\mathcal{R}}\|b\|_{L_2(\mathcal{N})},\\ \|x\|_{L_1^s(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} &= \inf_{x=ayb}\|a\|_{L_2(\mathcal{N})}\,\|y\|_{\mathcal{M}\bar\otimes\mathcal{R}}\|b\|_{L_2(\mathcal{N})}.\end{aligned}$$

Lemma 3.6. *If $z \in \mathcal{M}\otimes L_1(\mathcal{R})$, we have*

$$\begin{aligned}\|z\|_{L_1^r(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})^*} &= \inf_{z=\alpha y}\big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))},\\ \|z\|_{L_1^c(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})^*} &= \inf_{z=y\beta}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}\big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N},\\ \|z\|_{L_1^s(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})^*} &= \inf_{z=\alpha y\beta}\big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}\big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N}.\end{aligned}$$

Proof. Given $x \in \mathcal{M}\otimes\mathcal{R}$, let $x = a_jy_jb_j$ with

$$\begin{aligned}\|x\|_{L_1^r(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} &\sim \|a_1\|_{L_2(\mathcal{N})}\,\|y_1\|_{\mathcal{M}\bar\otimes\mathcal{R}}\,\|b_1\|_{L_2(\mathcal{M})},\\ \|x\|_{L_1^c(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} &\sim \|a_2\|_{L_2(\mathcal{M})}\|y_2\|_{\mathcal{M}\bar\otimes\mathcal{R}}\,\|b_2\|_{L_2(\mathcal{N})},\\ \|x\|_{L_1^s(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} &\sim \|a_3\|_{L_2(\mathcal{N})}\,\|y_3\|_{\mathcal{M}\bar\otimes\mathcal{R}}\,\|b_3\|_{L_2(\mathcal{N})}.\end{aligned}$$

Here $\sim$ means up to $(1+\delta)$ for an arbitrary $\delta > 0$. Then, we have

$$\begin{aligned} \langle x, z\rangle &= \mathrm{tr}_{\mathcal{M}\bar{\otimes}\mathcal{R}}\big(y_j b_j z^* a_j\big) \le \|y_j\|_{\mathcal{M}\bar{\otimes}\mathcal{R}} \|a_j^* z b_j^*\|_{L_1(\mathcal{M}\bar{\otimes}\mathcal{R})} \\ &\le \|a_j\|_{L_2(\mathsf{A}_j)} \|y_j\|_{\mathcal{M}\bar{\otimes}\mathcal{R}} \|b_j\|_{L_2(\mathsf{B}_j)} \sup_{\substack{\|\alpha_j\|_{L_2(\mathsf{A}_j)}\le 1 \\ \|\beta_j\|_{L_2(\mathsf{B}_j)}\le 1}} \|\alpha_j z \beta_j\|_{L_1(\mathcal{M}\bar{\otimes}\mathcal{R})}, \end{aligned}$$

with respect to anti-linear duality and where

$$(\mathsf{A}_1, \mathsf{B}_1, \mathsf{A}_2, \mathsf{B}_2, \mathsf{A}_3, \mathsf{B}_3) = (\mathcal{N}, \mathcal{M}, \mathcal{M}, \mathcal{N}, \mathcal{N}, \mathcal{N}).$$

According to this, it is easily seen that the closure of $\mathcal{M} \otimes L_1(\mathcal{R})$ with respect to the norm (for each $j = 1, 2, 3$) given by the supremum above embeds isometrically into the dual of $L_1^\bullet(\mathcal{M}, \mathcal{E}_\mathcal{N}; \mathcal{R})$ with $(\bullet, j) = (r, 1), (c, 2), (s, 3)$. Therefore, it suffices to see that

$$\sup_{\alpha_1,\beta_1} \|\alpha_1 z \beta_1\|_{L_1(\mathcal{M}\bar{\otimes}\mathcal{R})} = \inf_{z=\alpha y} \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))}, \tag{3.6}$$

$$\sup_{\alpha_2,\beta_2} \|\alpha_2 z \beta_2\|_{L_1(\mathcal{M}\bar{\otimes}\mathcal{R})} = \inf_{z=y\beta} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N}, \tag{3.7}$$

where $\alpha_1, \beta_2 \in \mathsf{B}_{L_2(\mathcal{N})}$ and $\alpha_2, \beta_1 \in \mathsf{B}_{L_2(\mathcal{M})}$; as well as

$$\begin{aligned} &\sup_{\alpha_3,\beta_3} \|\alpha_3 z \beta_3\|_{L_1(\mathcal{M}\bar{\otimes}\mathcal{R})} \\ &\qquad = \inf_{z=\alpha y\beta} \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N} \end{aligned} \tag{3.8}$$

with $\alpha_3, \beta_3 \in \mathsf{B}_{L_2(\mathcal{N})}$. Since the proof of (3.6) and (3.7) is quite similar to that of (3.8), we shall only give a detailed argument for the last one. Given a factorization $z = \alpha y \beta$ and $\alpha_0, \beta_0 \in L_2(\mathcal{N})$, the upper estimate follows from

$$\begin{aligned} &\|\alpha_0 z \beta_0\|_{L_1(\mathcal{M}\bar{\otimes}\mathcal{R})} \\ &\quad\le \|\alpha_0\alpha\|_{L_2(\mathcal{M})} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \|\beta\beta_0\|_{L_2(\mathcal{M})} \\ &\quad\le \|\alpha_0\|_{L_2(\mathcal{N})} \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N} \|\beta_0\|_{L_2(\mathcal{N})}. \end{aligned}$$

For the lower estimate, we set

$$|||z||| = \inf_{z=\alpha y\beta} \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N}.$$

This expression defines a norm. Indeed, the positive definiteness follows from

$$|||z||| \ge \|z\|_{L_\infty^r(\mathcal{M};\mathcal{E}_\mathcal{N}) \otimes_\mathcal{M} L_\infty(\mathcal{M}; L_1(\mathcal{R})) \otimes_\mathcal{M} L_\infty^c(\mathcal{M};\mathcal{E}_\mathcal{N})},$$

while the triangle inequality can be proved following Pisier's factorization argument in [29, Lemma 3.5]. Given $z_0 \in \mathcal{M} \otimes L_1(\mathcal{R})$, let us consider a norm 1 linear functional

$$\phi_{z_0} : \big(\mathcal{M} \otimes L_1(\mathcal{R}), |||\ |||\big) \to \mathbb{C} \quad \text{such that} \quad \phi_{z_0}(z_0) = |||z_0|||.$$

Note that we have

$$\big|\phi_{z_0}(\alpha y \beta)\big| \le \big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)\big\|_\mathcal{N}^{\frac12} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)\big\|_\mathcal{N}^{\frac12}.$$

In particular, we may apply —as in [16, Theorem 3.16 + Proposition 6.9]— a standard Grothendieck-Pietsch separation argument to find states φ_1 and φ_2 in $\mathcal{N}^*$ with associated densities d_1, d_2 in $L_1(\mathcal{N}^{**})$, so that

$$\begin{aligned} \big|\phi_{z_0}(\alpha y \beta)\big| &\le \varphi_1(\mathcal{E}_\mathcal{N}(\alpha\alpha^*))^{\frac12} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \varphi_2(\mathcal{E}_\mathcal{N}(\beta^*\beta))^{\frac12} \\ &= \|d_1^{\frac12}\alpha\|_{L_2(\mathcal{M}^{**})} \|y\|_{L_\infty(\mathcal{M}; L_1(\mathcal{R}))} \|\beta d_2^{\frac12}\|_{L_2(\mathcal{M}^{**})}. \end{aligned} \tag{3.9}$$

We want to construct a norm one functional $\psi : L_1(\mathcal{M}^{**}\bar{\otimes}\mathcal{R}) \to \mathbb{C}$ with

$$\phi_{z_0}(\alpha y\beta) = \psi\big(d_1^{\frac12}\alpha y\beta d_2^{\frac12}\big).$$

Let $e_j = \operatorname{supp} d_j$ be the support of d_j for $j = 1, 2$. We know that the space $L_1(\mathcal{M}^{**}\bar{\otimes}\mathcal{R}) = L_1(\mathcal{M}^{**})\hat{\otimes}L_1(\mathcal{R})$ is given by the operator space tensor product. Therefore elements of the form

$$\xi = \sum_{i,j,k,l=1}^n a_{ik}b_{lj}\, x_{kl}\otimes y_{ij}$$

with

$$\|a\|_2\big\|(x_{kl})\big\|_{M_n(L_1(\mathcal{M}^{**}))}\big\|(y_{kl})\big\|_{M_n(L_1(\mathcal{R}))}\|b\|_2 \le 1$$

are dense in the unit ball of $L_1(\mathcal{M}^{**})\hat{\otimes}L_1(\mathcal{R})$. Note also that $\eta = (\sum_{kl} a_{ik}x_{kl}b_{lj})_{ij}$ is of norm ≤ 1 in $S_1^n(L_1(\mathcal{M}^{**})) = Dec\,(M_n, L_1(\mathcal{M}^{**}))$ where decomposable refers to linear combination of positive elements. Thus we can find $h_1, h_2 \in L_2(\mathcal{M}^{**})$ and $u : M_n \to \mathcal{M}^{**}$ in the unit ball of $Dec\,(M_n, \mathcal{M}^{**})$ such that

$$\sum\nolimits_{kl} a_{ik}x_{kl}b_{lj} = h_1u(e_{ij})h_2.$$

Recall that $Dec\,(M_n, \mathcal{M}^{**}) = Dec\,(M_n, \mathcal{M})^{**}$ and therefore we can find a net of maps u_s in the unit ball of $Dec\,(M_n, \mathcal{M})$ such that $h_1u(e_{ij})h_2 = \lim_s h_1u_s(e_{ij})h_2$. Passing to convex combinations, we may assume that $u_s(e_{ij})$ converges in the strong and strong* topologies, so that $h_1u_sh_2$ converges to η in norm. If we assume additionally that $\xi = e_1\xi e_2$, we may replace h_1 and h_2 by e_1h_1 and h_2e_2. According to Kaplansky's density theorem and the norm density of $\sqrt{d_1}\mathcal{M}^{**}$ in $e_1L_2(\mathcal{M}^{**})$, we see that $\sqrt{d_1}\mathcal{M}$ is norm dense in $e_1L_2(\mathcal{M}^{**})$. Similarly, $\mathcal{M}\sqrt{d_2}$ is norm dense in $L_2(\mathcal{M}^{**})e_2$. Thus we can find $m_{t_1},\, \tilde{m}_{t_2} \in \mathcal{M}$ such that

$$e_1h_1 = \lim\nolimits_{t_1} d_1^{\frac12}m_{t_1} \quad \text{and} \quad h_2e_2 = \lim\nolimits_{t_2} \tilde{m}_{t_2}d_2^{\frac12}.$$

This shows that

$$\xi = \lim\nolimits_s \lim\nolimits_{t_1,t_2} \sum_{i,j=1}^n d_1^{\frac12}m_{t_1}u_s(e_{ij})\tilde{m}_{t_2}d_2^{\frac12}\otimes y_{ij}.$$

We deduce from (3.9) that for fixed s, t_1, t_2

$$\begin{aligned}&\Big|\phi_{z_0}\big(\sum_{i,j=1}^n m_{t_1}u_s(e_{ij})\tilde{m}_{t_2}\otimes y_{ij}\big)\Big| \\ &\qquad \le \quad \big\|d_1^{\frac12}m_{t_1}\big\|_2\Big\|\sum_{i,j=1}^n u_s(e_{ij})\otimes y_{ij}\Big\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}\big\|\tilde{m}_{t_2}d_2^{\frac12}\big\|_2\end{aligned}$$

Recall that $(y_{ij}) \in M_n(L_1(\mathcal{R}))$ has norm ≤ 1. Since u_s is decomposable we see that $u_s\otimes id : M_n(L_1(\mathcal{R})) \to L_\infty(\mathcal{M}; L_1(\mathcal{R}))$ is a contraction, which is easy to check for completely positive u_s. Thus we get

$$\Big\|\sum_{i,j=1}^n u_s(e_{ij})\otimes y_{ij}\Big\|_{L_\infty(\mathcal{M}\otimes\mathcal{R})} \le \|u_s\|_{dec}\big\|(y_{ij})\big\|_{M_n(L_1(\mathcal{R}))},$$

and therefore

$$\psi(\xi) = \lim\nolimits_s \lim\nolimits_{t_1,t_2}\Big|\psi_{z_0}\big(\sum_{i,j=1}^n m_{t_1}u_s(e_{ij})\tilde{m}_{t_2}\otimes y_{ij}\big)\Big| \le 1.$$

Let us resume what we have proved so far. For fixed $n \in \mathbb{N}$ we have shown that $\psi(\sqrt{d_1} m \sqrt{d_2} \otimes y) = \phi_{z_0}(m \otimes y)$ extends to a continuous functional on the Banach space projective tensor product $e_1 L_1(\mathcal{M}^{**}) e_2 \otimes_\pi L_1(\mathcal{R})$ such that

$$\Big|\psi\big(\sum_{ij=1}^{n} x_{ij} \otimes y_{ij}\big)\Big| \le \big\|(x_{ij})\big\|_{S_1^n(L_1(\mathcal{M}^{**}))} \big\|(y_{ij})\big\|_{M_n(L_1(\mathcal{R}))} .$$

Since left and right multiplications with e_1, e_2 are completely contractive, we may extend ψ to $L_1(\mathcal{M}^{**}) \otimes_\pi L_1(\mathcal{R})$ satisfying the same inequality. This means ψ induces a linear map $T_\psi : L_1(\mathcal{R}) \to L_1(\mathcal{M}^{**})^* = \mathcal{M}^{op**}$ such that

$$\big\| id_{M_n} \otimes T_\psi : M_n(\mathcal{R}) \to M_n(\mathcal{M}^{op**}) \big\| \le 1 .$$

Since this is true for all $n \in \mathbb{N}$ we deduce that T_ψ is completely bounded. According to Effors/Ruan's theorem [4, Theorem 7.2.4] $\mathcal{CB}(L_1(\mathcal{R}), \mathcal{M}^{op**}) = \mathcal{R}^{op} \bar{\otimes} \mathcal{M}^{op**} = (L_1(\mathcal{R} \bar{\otimes} \mathcal{M}^{**}))^*$. Therefore ψ corresponds to a norm one functional on $L_1(\mathcal{R} \bar{\otimes} \mathcal{M}^{**})$ such that

$$\psi\big(d_1^{\frac12} x d_2^{\frac12} \otimes y\big) = \phi_{z_0}(x \otimes y) .$$

Now we have to replace $d_1, d_2 \in L_1(\mathcal{N}^{**})$ using an ultraproduct procedure. We recall from [16, Section 6.2] that we have a completely positive, completely isometric $\mathcal{M}$-bimodule map

$$\rho : L_1(\mathcal{M}^{**}) \to \prod\nolimits_{\mathcal{U}} L_1(\mathcal{M}),$$

such that $\rho^* : (\prod_{\mathcal{U}} L_1(\mathcal{M}))^* \to (\mathcal{M}^{op})^{**}$ is a conditional expectation. Thus

$$\rho \otimes id : L_1(\mathcal{M}^{**}) \hat{\otimes} L_1(\mathcal{R}) \to \prod\nolimits_{\mathcal{U}} L_1(\mathcal{M}) \hat{\otimes} L_1(\mathcal{R}).$$

Therefore we find a norm one functional $\psi' : \prod_{\mathcal{U}} L_1(\mathcal{M}) \hat{\otimes} L_1(\mathcal{R}) \to \mathbb{C}$ such that $\psi' \circ (\rho \otimes id) = \psi$. The map ρ also induces a map $\rho_p : L_p(\mathcal{M}^{**}) \to \prod_{\mathcal{U}} L_p(\mathcal{M})$ which remains a $\mathcal{M}$-bimodule map. In particular, we get

$$\rho\big(d_1^{\frac12} x d_2^{\frac12}\big) = \rho_2\big(d_1^{\frac12}\big)\, x\, \rho_2\big(d_2^{\frac12}\big) \quad \text{for} \quad x \in \mathcal{M}.$$

Let us recall that the inclusion $L_2(\mathcal{N}^{**}) \subset L_2(\mathcal{M}^{**})$ is defined with the help of the conditional expectation $\mathcal{E}_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$, more precisely ${\mathcal{E}_{\mathcal{N}}}^{**}$ which is still a (maybe non-faithful) conditional expectation. We recall from [16, Lemma 6.2 ii)] that $\rho(L_1(\mathcal{N}^{**})) \subset \prod_{\mathcal{U}} L_1(\mathcal{N})$ and hence $\rho_2(\sqrt{d_j}) \in \prod_{\mathcal{U}} L_2(\mathcal{N})$. Therefore we find

$$\begin{aligned} |||z_0||| &= \big|\psi' \circ (\rho \otimes id)\big(d_1^{\frac12} z_0 d_2^{\frac12}\big)\big| \\ &\le \big\|\rho_2\big(d_1^{\frac12}\big) z_0 \rho_2\big(d_2^{\frac12}\big)\big\|_{\prod_{\mathcal{U}} L_1(\mathcal{M} \bar{\otimes} \mathcal{R})} \\ &= \lim\nolimits_{i,\mathcal{U}} \big\|\rho_2\big(d_1^{\frac12}\big)_i z_0 \rho_2\big(d_2^{\frac12}\big)_i\big\|_{L_1(\mathcal{M} \bar{\otimes} \mathcal{R})} \\ &\le \sup_{\|\alpha\|_{L_2(\mathcal{N})}, \|\beta\|_{L_2(\mathcal{N})} \le 1} \big\|\alpha z_0 \beta\big\|_{L_1(\mathcal{M} \bar{\otimes} \mathcal{R})} . \end{aligned}$$

This concludes the proof of (3.8). The argument for (3.6) and (3.7) is similar. ☐

Let us define the space

$$\begin{aligned} \mathcal{K}_{1,\infty}^{\lambda}(\mathcal{M}, \mathcal{E}_{\mathcal{N}}; \mathcal{R}) &= \lambda\, L_1(\mathcal{M}; \mathcal{R}) \\ &+ L_1^s(\mathcal{M}, \mathcal{E}_{\mathcal{N}}; \mathcal{R}) \\ &+ \sqrt{\lambda}\, L_1^r(\mathcal{M}, \mathcal{E}_{\mathcal{N}}; \mathcal{R}) \\ &+ \sqrt{\lambda}\, L_1^c(\mathcal{M}, \mathcal{E}_{\mathcal{N}}; \mathcal{R}), \end{aligned}$$

where $L_1(\mathcal{M};\mathcal{R})$ is a shortened way of writing $L_1(\mathcal{M};L_\infty(\mathcal{R}))$. We shall often write $\mathcal{K}_{1,\infty}^\lambda(\mathcal{R})$. The norm of $x \in \mathcal{K}_{1,\infty}^\lambda(\mathcal{R})$ is given by

$$\inf_{x=\sum_1^4 x_j} \lambda \|x_1\|_{L_1(\mathcal{M};\mathcal{R})} + \sqrt{\lambda}\|x_2\|_{L_1^r(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} + \sqrt{\lambda}\|x_3\|_{L_1^c(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})} + \|x_4\|_{L_1^s(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})}.$$

The following result probably holds in larger generality. However, this requires additional fine tuning on the assumptions. For our purpose, finite-dimensional $\mathcal{R}$'s are enough.

Theorem 3.7. *Let us consider a conditioned subalgebra $\mathcal{N}$ of $\mathcal{M}$ and a finite dimensional von Neumann algebra $\mathcal{R}$. Let $(\mathcal{M}_k)_{k\geq 1}$ be an increasingly independent family of top-subsymmetric copies of $\mathcal{M}$ over $\mathcal{N}$. Then, the following estimate holds up to an absolute constant for any $n \geq 1$*

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} \sim \|x\|_{\mathcal{K}_{1,\infty}^n(\mathcal{R})}.$$

Proof. By the triangle inequality

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} \leq n\|x\|_{L_1(\mathcal{M};\mathcal{R})}. \tag{3.10}$$

If $x = ayb$ with $a \in L_2(\mathcal{N})$ and $b \in L_2(\mathcal{M})$, then

$$\pi_k(x) = a\pi_k(y)\pi_k(b) \quad \text{and} \quad \sum_{k=1}^n \pi_k(x) \otimes \delta_k = a\Big(\sum_{k=1}^n \pi_k(y)u_k \otimes \delta_k\Big)\Big(\sum_{k=1}^n \pi_k(b^*b)\Big)^{\frac12},$$

where the u_k's are contractions in $\mathcal{A}$. This immediately gives

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} \leq \sqrt{n}\,\|a\|_{L_2(\mathcal{N})}\|y\|_{\mathcal{M}\bar{\otimes}\mathcal{R}}\|b\|_{L_2(\mathcal{M})}. \tag{3.11}$$

In fact, the same argument provides the remaining individual estimates

$$\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))} \leq \min\Big\{\sqrt{n}\,\|x\|_{L_1^c(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})}, \|x\|_{L_1^s(\mathcal{M},\mathcal{E}_\mathcal{N};\mathcal{R})}\Big\}. \tag{3.12}$$

The combination of (3.10), (3.11) and (3.12) shows that the upper estimate holds contractively. Let us now prove the lower estimate. Since $\mathcal{R}$ is finite dimensional we may characterize the dual space of $\mathcal{K}_{1,\infty}^\lambda$. Indeed, it follows from Lemma 3.1 and Lemma 3.6 that

$$\begin{aligned}
\|z\|_{\mathcal{J}_{\infty,1}^\lambda} \quad \sim \quad & \max\Big\{\|z\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}, \sqrt{\lambda}\inf_{z=\alpha y}\big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}, \\
& \sqrt{\lambda}\inf_{z=y\beta}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}\big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N}, \\
& \lambda\inf_{z=\alpha y\beta}\big\|\mathcal{E}_\mathcal{N}(\alpha\alpha^*)^{\frac12}\big\|_\mathcal{N}\|y\|_{L_\infty(\mathcal{M};L_1(\mathcal{R}))}\big\|\mathcal{E}_\mathcal{N}(\beta^*\beta)^{\frac12}\big\|_\mathcal{N}\Big\} \\
= \quad & \lambda \sup\Big\{|\mathrm{tr}(x^*z)| \,\Big|\, \|x\|_{\mathcal{K}_{1,\infty}^\lambda} \leq 1\Big\}.
\end{aligned}$$

Since the embedding of $\mathcal{K}_{1,\infty}^\lambda$ in its bidual is isometric, we deduce from Lemma 3.4

$$\|x\|_{\mathcal{K}_{1,\infty}^\lambda(\mathcal{R})} \lesssim \lambda\|x\|_{\mathcal{J}_{\infty,1}^\lambda(L_1(\mathcal{R}))^*} \leq 16\,e\Big\| \sum_{k=1}^n \pi_k(x) \otimes \delta_k \Big\|_{L_1(\mathcal{A};\ell_\infty^n(\mathcal{R}))}.$$

We used $\lambda = n/\delta$ and $\delta = 1/8e$ so that $1 - 4e\delta = \frac{1}{2}$. The proof is complete. $\square$

4. A vector-valued embedding result

Given two von Neumann algebras $\mathcal{M}$ and $\mathcal{R}$ as in the previous section, our aim now is to find a complete embedding of $L_p(\mathcal{M};\mathcal{R})$ for each $1 < p < \infty$ into an ultraproduct of the form

$$\prod\nolimits_{n,\mathcal{U}} L_1\big(\mathcal{A}_n; \ell_\infty^{\mathrm{k}_n}(\mathcal{R})\big).$$

Moreover, for our applications we also need some additional information on how such an embedding is constructed in order to maintain the notion of independent copies. In the following, $X_\mathcal{M}$ will be an operator space containing $\mathcal{M}$ as a two-sided ideal. Then we may define

$$L_{2p}(\mathcal{M}) X_\mathcal{M} L_{2p}(\mathcal{M}) = L_{2p}^r(\mathcal{M}) \otimes_{\mathcal{M},h} X_\mathcal{M} \otimes_{\mathcal{M},h} L_{2p}^c(\mathcal{M}).$$

We will also work with subspaces and quotients of

$$\big(\mathcal{M} \oplus L_2^r(\mathcal{M})\big) \otimes_{\mathcal{M},h} X_\mathcal{M} \otimes_{\mathcal{M},h} \big(\mathcal{M} \oplus L_2^c(\mathcal{M})\big).$$

Our main tool is a standard modification of the so-called Pisier's exercise, see [15, 43] and [30, Exercise 7.9]. In other words, a way to reformulate complex interpolation in this setting. We follow the same approach as in [15, 16]. Indeed, let $\mathcal{S}$ be the strip of complex numbers z with $0 \le \mathrm{Re}(z) \le 1$ and let $\partial_0 \cup \partial_1$ be the partition of its boundary $\partial\mathcal{S}$ with ∂_j the line of z's with $\mathrm{Re}(z) = j$. If $0 < \theta < 1$, let μ_θ be the harmonic measure of the point $z = \theta$. This is a probability measure on the boundary $\partial\mathcal{S}$ (with density given by the Poisson kernel in the strip) that can be written as $\mu_\theta = (1-\theta)\mu_0 + \theta\mu_1$, with μ_j being probability measures supported by ∂_j and such that

$$f(\theta) = \int_{\partial\mathcal{S}} f \, d\mu_\theta \tag{4.1}$$

for any bounded analytic f extended non-tangentially to $\partial\mathcal{S}$. Let

$$\mathcal{S}_\mathcal{M}^r = \Big(L_2^c(\partial_0)\bar{\otimes}\mathcal{M}\Big) \oplus \Big(L_2^r(\partial_1) \otimes_h L_2^r(\mathcal{M})\Big),$$

$$\mathcal{S}_\mathcal{M}^c = \Big(\mathcal{M}\bar{\otimes}L_2^r(\partial_0)\Big) \oplus \Big(L_2^c(\mathcal{M}) \otimes_h L_2^c(\partial_1)\Big).$$

The von Neumann algebra tensor product used above is the weak closure of the minimal tensor product, which in this particular case coincides with the Haagerup tensor product since we have either a column space on the left or a row space on the right. In particular, the only difference is that we are taking the closure in the weak operator topology. The direct sums will be taken Hilbertian. Then, if $\mathcal{M}$ comes equipped with a normal strictly semifinite faithful weight ψ and d_ψ denotes the associated density, we define $\mathsf{H}_\theta^r(\mathcal{M})$ as the subspace of all pairs (f_0, f_1) of functions in $\mathcal{S}_\mathcal{M}^r$ such that for every scalar-valued analytic function $g : \mathcal{S} \to \mathbb{C}$ (extended non-tangentially to the boundary) with $g(\theta) = 0$, we have

$$(1-\theta)\int_{\partial_0} g(z)\, d_\psi^{\frac{1}{2}} f_0(z)\, d\mu_0(z) + \theta \int_{\partial_1} g(z) f_1(z)\, d\mu_1(z) = 0.$$

Similarly, the condition on $\mathsf{H}_\theta^c(\mathcal{M}) \subset \mathcal{S}_\mathcal{M}^c$ is

$$(1-\theta)\int_{\partial_0} g(z) f_0(z)\, d_\psi^{\frac{1}{2}}\, d\mu_0(z) + \theta \int_{\partial_1} g(z) f_1(z)\, d\mu_1(z) = 0.$$

We shall also need to consider the subspaces

$$\begin{aligned}\mathsf{H}_{r,0} &= \Big\{(f_0,f_1)\in \mathsf{H}_\theta^r(\mathcal{M}) \,\Big|\, (1-\theta)\int_{\partial_0} d_\psi^{\frac12} f_0 d\mu_0 + \theta\int_{\partial_1} f_1 d\mu_1 = 0\Big\},\\ \mathsf{H}_{c,0} &= \Big\{(f_0,f_1)\in \mathsf{H}_\theta^c(\mathcal{M}) \,\Big|\, (1-\theta)\int_{\partial_0} f_0 d_\psi^{\frac12} d\mu_0 + \theta\int_{\partial_1} f_1 d\mu_1 = 0\Big\}.\end{aligned}$$

We define the $\mathcal{M}$-bimodules

$$\mathcal{H}_r(\mathcal{M},\theta) = \mathsf{H}_\theta^r(\mathcal{M})/\mathsf{H}_{r,0} \quad \text{and} \quad \mathcal{H}_c(\mathcal{M},\theta) = \mathsf{H}_\theta^c(\mathcal{M})/\mathsf{H}_{c,0}.$$

Remark 4.1. We may think of $\mathcal{H}_r(\mathcal{M},\theta)$ as the space of $\mathcal{M}+L_2^r(\mathcal{M})$-valued analytic functions f on the strip, with $f(\partial_0)\subset\mathcal{M}$ and $f(\partial_1)\subset L_2^r(\mathcal{M})$ quotiented by the equivalence relation $f_1\sim f_2$ iff both take the same value at θ. A similar observation holds for $\mathcal{H}_c(\mathcal{M},\theta)$. It is somewhat encoded in the proof of Proposition 4.3 that indeed

$$\mathcal{H}_r(\mathcal{M},\theta) = L_{2p}^r(\mathcal{M}) \qquad \text{and} \qquad \mathcal{H}_c(\mathcal{M},\theta) = L_{2p}^c(\mathcal{M}).$$

In the following we use the notation

$$\mathcal{H}_\theta(X_\mathcal{M}) = \mathcal{H}_r(\mathcal{M},\theta)\otimes_{\mathcal{M},h} X_\mathcal{M} \otimes_{\mathcal{M},h} \mathcal{H}_c(\mathcal{M},\theta).$$

Lemma 4.2. *Given $1\le p\le\infty$, we have a contractive inclusion*

$$S_{2p'}^m L_{2p}(M_m\otimes\mathcal{M}) M_m(X_\mathcal{M}) L_{2p}(M_m\otimes\mathcal{M}) S_{2p'}^m \subset R_m \otimes_h \mathcal{H}_{\frac1p}(X_\mathcal{M})\otimes_h C_m.$$

Proof. We claim that the inclusion

$$S_{2p'}^m L_{2p}(M_m\otimes\mathcal{M}) \subset R_m\otimes_h \mathcal{H}_r(\mathcal{M},1/p)\otimes_h R_m$$

is contractive. Let $x=ab$ be such that $a\in S_{2p'}^m$ and $b\in L_{2p}(M_m\otimes\mathcal{M})$ are norm 1 elements. Using the fact that $R_m\otimes_h\mathcal{H}_r(\mathcal{M},1/p)\otimes_h R_m$ is a right $M_m(\mathcal{M})$-module we may apply polar decomposition and assume that a and b are positive. Indeed, write $ab=|a^*|u_a b = |a^*||b^*u_a^*|u_{ab}$ and use that

$$\big\||a^*|\big\|_{2p'} = \|a\|_{2p'} \qquad \text{and} \qquad \big\||b^*u_a^*|\big\|_{2p}\le \|b\|_{2p}.$$

Define the analytic function $f: z\in\mathcal{S}\mapsto a^{(1-z)p'}b^{pz}$ with $f(\frac1p)=x$. If we set $f_j = f_{|\partial_j}$, it is clear that

$$\begin{aligned}&\|x\|_{R_m\otimes_h\mathcal{H}_r(\mathcal{M},1/p)\otimes_h R_m}\\ &\quad\le \big\|(f_0,f_1)\big\|_{R_m\otimes_h \mathcal{S}_\mathcal{M}^r\otimes_h R_m}\\ &\quad= \Big((1-\tfrac1p)\|f_0\|^2_{R_m\otimes_h(L_2^c(\partial_0)\bar\otimes\mathcal{M})\otimes_h R_m} + \tfrac1p\|f_1\|^2_{R_m\otimes_h(L_2^r(\partial_1)\otimes_h L_2^r(\mathcal{M}))\otimes_h R_m}\Big)^{\frac12}.\end{aligned}$$

The space $R_m\otimes_h(L_2^c(\partial_0)\bar\otimes\mathcal{M})\otimes_h R_m$ is completely isometric to

$$(R_m\otimes_h R_m)\otimes_{M_m,h}\big(C_m\otimes_h(L_2^c(\partial_0)\bar\otimes\mathcal{M})\otimes_h R_m\big)\otimes_{M_m,h}(C_m\otimes_h R_m),$$

which in turn is isometric to $S_2^m L_\infty(M_m\otimes\mathcal{M};L_2^c(\partial_0))$. On the other hand, given any $z\in\partial_0$ we have that $f(z)=a^{p'}u_z$ with u_z being a unitary in $M_m(\mathcal{M})$ for each $z\in\partial_0$. Hence, we get

$$\|u\|_{L_\infty(M_m\otimes\mathcal{M};L_2^c(\partial_0))} = \Big\|\Big(\int_{\partial_0} u_z^*u_z\, d\mu_0(z)\Big)^{\frac12}\Big\|_{M_m\otimes\mathcal{M}} = 1$$

and $\|f_0\|^2_{R_m\otimes_h(L_2^c(\partial_0)\bar{\otimes}\mathcal{M})\otimes_h R_m} \le \|a\|_{2p'}^{2p'} \le 1$. Moreover, it is easy to check that

$$\|f_1\|^2_{R_m\otimes_h(L_2^r(\partial_1)\otimes_h L_2^r(\mathcal{M}))\otimes_h R_m} = \int_{\partial_1} \|v_z b^p\|^2_{L_2(M_m\otimes\mathcal{M})}\, d\mu_1(z) = \|b\|_{2p}^{2p} \le 1.$$

Putting altogether, we deduce our claim. Similarly,

$$L_{2p}(M_m\otimes N)S_{2p'}^m \subset C_m\otimes_h \mathcal{H}_c(\mathcal{M},1/p)\otimes_h C_m$$

holds contractively. Therefore, the assertion follows from

$$\begin{aligned} & R_m\otimes_h \mathcal{H}_{\frac{1}{p}}(X_\mathcal{M})\otimes_h C_m \\ = \quad & \big(R_m\otimes_h \mathcal{H}_r(\mathcal{M},1/p)\otimes_h R_m\big)\bullet M_m(X_\mathcal{M})\bullet\big(C_m\otimes_h\mathcal{H}_c(\mathcal{M},1/p)\otimes_h C_m\big). \end{aligned}$$

where the symbol $\bullet$ stands for the amalgamated tensor product $\otimes_{M_m(\mathcal{M}),h}$. □

Proposition 4.3. *We have a complete isometry*

$$L_p(\mathcal{M};\mathcal{R}) \to \mathcal{H}_{\frac{1}{p}}(\mathcal{M}\bar{\otimes}\mathcal{R}) \quad \textit{for each} \quad 1<p<\infty.$$

Proof. We have

$$\begin{aligned} S_1^m(L_p(\mathcal{M};\mathcal{R})) &= S_{2p'}^m L_p(M_m\otimes\mathcal{M};\mathcal{R})S_{2p'}^m \\ &= S_{2p'}^m L_{2p}(M_m\otimes\mathcal{M})M_m(\mathcal{M}\bar{\otimes}\mathcal{R})L_{2p}(M_m\otimes\mathcal{M})S_{2p'}^m. \end{aligned}$$

Hence Lemma 4.2 implies that

$$S_1^m(L_p(\mathcal{M};\mathcal{R})) \subset R_m\otimes_h \mathcal{H}_{\frac{1}{p}}(\mathcal{M}\bar{\otimes}\mathcal{R})\otimes_h C_m = S_1^m(\mathcal{H}_{\frac{1}{p}}(\mathcal{M}\bar{\otimes}\mathcal{R}))$$

is contractive for all m. Therefore, the inclusion of $L_p(\mathcal{M};\mathcal{R})$ into $\mathcal{H}_{1/p}(\mathcal{M}\bar{\otimes}\mathcal{R})$ is a complete contraction. To complete the argument, we proceed by duality and analyze the inclusion

$$S_1^m(L_{p'}(\mathcal{M};L_1(\mathcal{R}))) \subset M_m(\mathcal{H}_{\frac{1}{p}}(\mathcal{M}\bar{\otimes}\mathcal{R}))^*. \tag{4.2}$$

As in the proof of Lemma 4.2, we may factorize $x = abscd$ with $a,d\in S_{2p}^m$ and $b,c\in L_{2p'}(M_m\otimes\mathcal{M})$ positive norm 1 elements and s being a not necessarily positive norm 1 element in $L_\infty(M_m(\mathcal{M});L_1(\mathcal{R}))$. Let us now consider a norm 1 element $y\in M_m(\mathcal{H}_{1/p}(\mathcal{M}\bar{\otimes}\mathcal{R}))$. Then we may find an analytic function $\xi=\alpha w\beta$ in the equivalence class determined by y such that

- We have $w\in M_k(\mathcal{M}\bar{\otimes}\mathcal{R})$ for some $k\ge 1$ and
$$\alpha = (\alpha_0,\alpha_1)\in M_{m,k}\big(L_2^c(\partial_0)\bar{\otimes}\mathcal{M}\big)\oplus M_{m,k}\big(L_2^r(\partial_1)\otimes_h L_2^r(\mathcal{M})\big),$$
$$\beta = (\beta_0,\beta_1)\in M_{k,m}\big(\mathcal{M}\bar{\otimes}L_2^r(\partial_0)\big)\oplus M_{k,m}\big(L_2^c(\mathcal{M})\otimes_h L_2^c(\partial_1)\big).$$
- The estimate $\|w\|_{M_k(\mathcal{M}\bar{\otimes}\mathcal{R})}\le 1$ holds and
$$\Big((1-\tfrac{1}{p})\,\|\alpha_0\|^2_{M_{m,k}(L_2^c(\partial_0)\bar{\otimes}\mathcal{M})} + \tfrac{1}{p}\,\|\alpha_1\|^2_{M_{m,k}(L_2^r(\partial_1)\otimes_h L_2^r(\mathcal{M}))}\Big)^{\frac{1}{2}} \le 1,$$
$$\Big((1-\tfrac{1}{p})\,\|\beta_0\|^2_{M_{k,m}(\mathcal{M}\bar{\otimes}L_2^r(\partial_0))} + \tfrac{1}{p}\,\|\beta_1\|^2_{M_{k,m}(L_2^c(\mathcal{M})\otimes_h L_2^c(\partial_1))}\Big)^{\frac{1}{2}} \le 1.$$

By adding zeros if necessary, we assume $m=k$ for simplicity. As in Lemma 4.2, we may define $g(z) = a^{zp}b^{(1-z)p'}sc^{(1-z)p'}d^{zp}$. Note that g is also analytic and hence the identity below holds

$$\langle x,y\rangle = \big\langle g(\tfrac{1}{p}),\xi(\tfrac{1}{p})\big\rangle = \int_{\partial\mathcal{S}} \big\langle g(z),\xi(z)\big\rangle\, d\mu_{\frac{1}{p}}(z).$$

Now we claim that

$$(4.3)\qquad \Big| \int_{\partial_0} \big\langle g(z), \xi(z) \big\rangle \, d\mu_0(z) \Big| \ \le \ \Big\| \Big(\int_{\partial_0} \alpha_0(z)^* \alpha_0(z) \, d\mu_0(z) \Big)^{\frac12} \Big\|_{M_m(\mathcal{M})} \times \Big\| \Big(\int_{\partial_0} \beta_0(z) \, \beta_0(z)^* \, d\mu_0(z) \Big)^{\frac12} \Big\|_{M_m(\mathcal{M})}$$

Indeed, since $g_{|\partial_0}(z) = u_z b^{p'} s c^{p'} v_z$ with u_z, v_z unitaries and b, c, s, w are norm 1

$$\begin{aligned} & \Big| \int_{\partial_0} \big\langle g(z), \xi(z) \big\rangle \, d\mu_0(z) \Big| \\ & = \Big| \int_{\partial_0} \mathrm{tr} \big(w \beta_0(z) v_z^* c^{p'} s^* b^{p'} u_z^* \alpha_0(z) \big) \, d\mu_0(z) \Big| \\ & \le \Lambda \Big\| \int_{\partial_0} \alpha_0(z)^* \alpha_0(z) \, d\mu_0(z) \Big\|_{M_m(\mathcal{M})}^{\frac12} \Big\| \int_{\partial_0} \beta_0(z) \, \beta_0(z)^* \, d\mu_0(z) \Big\|_{M_m(\mathcal{M})}^{\frac12}, \end{aligned}$$

where

$$\Lambda \ = \ \|w\|_{M_m(\mathcal{M} \bar\otimes \mathcal{R})} \sup_{z \in \partial_0} \big\| v_z^* c^{p'} s^* b^{p'} u_z^* \big\|_{L_1(M_m(\mathcal{M} \bar\otimes \mathcal{R}))} \ = \ \Lambda_1 \Lambda_2.$$

We have $\Lambda_1 \le 1$ by hypothesis, while Hölder's inequality gives

$$\Lambda_2 \ \le \ \|c^{p'}\|_{L_2(M_m(\mathcal{M}))} \|s^*\|_{L_\infty(M_m(\mathcal{M}); L_1(\mathcal{R}))} \|b^{p'}\|_{L_2(M_m(\mathcal{M}))} \ \le \ 1.$$

This proves (4.3). Similarly, we have

$$\begin{aligned} \Big| \int_{\partial_1} \big\langle g(z), \xi(z) \big\rangle \, d\mu_1(z) \Big| & = \Big| \int_{\partial_1} \big\langle \tilde u_z s \tilde v_z, a^p \alpha_1(z) w \beta_1(z) d^p \, d\mu_1(z) \big\rangle \Big| \\ & \le \int_{\partial_1} \big\| a^p \alpha_1(z) w \beta_1(z) d^p \big\|_{L_1(M_m(\mathcal{M}); \mathcal{R})} \, d\mu_1(z) \\ & \le \Big(\int_{\partial_1} \|a^p \alpha_1(z)\|_2^2 \, d\mu_1(z) \Big)^{\frac12} \|w\|_{M_m(\mathcal{M} \bar\otimes \mathcal{R})} \Big(\int_{\partial_1} \|\beta_1(z) d^p\|_2^2 \, d\mu_1(z) \Big)^{\frac12} \\ & \le \Big\| \int_{\partial_1} \mathrm{tr}_{\mathcal{M}} \big(\alpha_1(z) \alpha_1(z)^* \big) \, d\mu_1(z) \Big\|_{M_m}^{\frac12} \Big\| \int_{\partial_1} \mathrm{tr}_{\mathcal{M}} \big(\beta_1(z)^* \beta_1(z) \big) \, d\mu_1(z) \Big\|_{M_m}^{\frac12}. \end{aligned}$$

In the last inequality we use that a, d are norm 1 in S_{2p}^m. Summarizing, we get

$$\begin{aligned} \big| \langle x, y \rangle \big| & \le (1 - \tfrac1p) \Big| \int_{\partial_0} \langle g, \xi \rangle \, d\mu_0 \Big| + \tfrac1p \Big| \int_{\partial_1} \langle g, \xi \rangle \, d\mu_1 \Big| \\ & \le (1 - \tfrac1p) \, \|\alpha_0\|_{M_m(L_2^c(\partial_0) \bar\otimes \mathcal{M})} \, \|\beta_0\|_{M_m(\mathcal{M} \bar\otimes L_2^r(\partial_0))} \\ & + \tfrac1p \, \|\alpha_1\|_{M_m(L_2^r(\partial_1) \otimes_h L_2^r(\mathcal{M}))} \, \|\beta_1\|_{M_m(L_2^c(\mathcal{M}) \otimes_h L_2^c(\partial_1))} \\ & \le \Big((1 - \tfrac1p) \, \|\alpha_0\|^2 + \tfrac1p \, \|\alpha_1\|^2 \Big)^{\frac12} \Big((1 - \tfrac1p) \, \|\beta_0\|^2 + \tfrac1p \, \|\beta_1\|^2 \Big)^{\frac12} \ \le \ 1. \end{aligned}$$

Therefore, the inclusion (4.2) is contractive and the assertion follows by duality. □

Theorem 4.4. *Given $1 < p < \infty$ and $\mathcal{M}, \mathcal{R}$ as above assuming in addition that $\mathcal{R}$ is finite dimensional. Then there exist states ϕ_n on M_n, positive integers k_n and elements $\xi_n \in L_1(M_n)$ such that we have a complete embedding*

$$x \in L_p(\mathcal{M}; \mathcal{R}) \mapsto \Big(\sum_{j=1}^{\mathrm{k}_n} \pi_{tens}^j (\xi_n \otimes x) \otimes \delta_j \Big) \in \prod\nolimits_{n, \mathcal{U}} L_1 \big(M_n(\mathcal{M})^{\otimes \mathrm{k}_n}; \ell_\infty^{\mathrm{k}_n}(\mathcal{R}) \big).$$

Proof. When restricted to analytic functions, the operator $\Lambda(f|_{\partial_0}) = f|_{\partial_1}$ is densely defined and injective. In combination with (4.1), this allows us to see the subspace of analytic functions on $\mathcal{S}$ vanishing at $1/p$ as the annihilator of the graph of Λ, conveniently regarded as a space of analytic functions. The reader is referred to [15, Remark 2.2] for further details. Moreover, it is also observed in [15] that we can replace Λ by a strictly positive diagonal operator d_λ^{-1} on ℓ_2 without changing the operator space structure. In other words, we have complete isomorphisms

$$u_r : \big(L_2^c(\partial_0) \oplus L_2^r(\partial_1)\big) \Big/ \big\{f \text{ analytic s.t. } f(\tfrac{1}{p}) = 0\big\} \to \big(C \oplus R\big)/graph(\mathsf{d}_\lambda^{-1})^\perp,$$

$$u_c : \big(L_2^r(\partial_0) \oplus L_2^c(\partial_1)\big) \Big/ \big\{f \text{ analytic s.t. } f(\tfrac{1}{p}) = 0\big\} \to \big(R \oplus C\big)/graph(\mathsf{d}_\lambda^{-1})^\perp.$$

For further reference, we set $\xi_r = u_r(1)$ and $\xi_c = u_c(1)$, where 1 denotes the constant function 1 on the strip. Here $\mathsf{d}_\lambda : \ell_2 \to \ell_2$ is a diagonal operator $\mathsf{d}_\lambda(\delta_k) = \lambda_k e_k$ with $0 < \lambda_k < \infty$ and the fact that we may consider the same operator in both cases is also justified in [15]. The exact same argument mentioned above shows that, if we tensorize with the identity map, we also obtain complete isomorphisms

$$u_r : \mathcal{S}_{\mathcal{M}}^r/\mathsf{H}_{r,0} \to \Big[\big(C \bar{\otimes} \mathcal{M}\big) \oplus \big(R \otimes_h L_2^r(\mathcal{M})\big)\Big] \Big/ graph(\mathsf{d}_\lambda^{-1})^\perp,$$

$$u_c : \mathcal{S}_{\mathcal{M}}^c/\mathsf{H}_{c,0} \to \Big[\big(\mathcal{M} \bar{\otimes} R\big) \oplus \big(L_2^c(\mathcal{M}) \otimes_h C\big)\Big] \Big/ graph(\mathsf{d}_\lambda^{-1})^\perp.$$

Let us define

$$\widetilde{\mathcal{H}}_r(\mathcal{M}, 1/p) = u_r(\mathcal{H}_r(\mathcal{M}, 1/p)) \quad \text{and} \quad \widetilde{\mathcal{H}}_c(\mathcal{M}, 1/p) = u_c(\mathcal{H}_c(\mathcal{M}, 1/p)).$$

As in [15], we observe that

$$\Big[\big(C \bar{\otimes} \mathcal{M}\big) \oplus \big(R \otimes_h L_2^r(\mathcal{M})\big)\Big] \Big/ graph(\mathsf{d}_\lambda^{-1})^\perp,$$

$$\Big[\big(\mathcal{M} \bar{\otimes} R\big) \oplus \big(L_2^c(\mathcal{M}) \otimes_h C\big)\Big] \Big/ graph(\mathsf{d}_\lambda^{-1})^\perp,$$

can also be understood as $\mathcal{K}$-spaces. Indeed, if we use anti-linear duality, we have $\mathsf{d}_\lambda^{-1} : x \in C \mapsto (\mathsf{d}_\lambda^{-1}x)^{\mathrm{t}} \in R$ in the first case and $\mathsf{d}_\lambda^{-1} : x \in R \mapsto (x\mathsf{d}_\lambda^{-1})^{\mathrm{t}} \in C$ in the second one. This means that $graph(\mathsf{d}_\lambda^{-1})^\perp$ is spanned by elements of the form $(-\mathsf{d}_\lambda^{-1}x, x)$ in the first quotient and by $(-x\mathsf{d}_\lambda^{-1}, x)$ in the second one. In conclusion this allows us to cb-embed

$$\begin{aligned} \big[u_r \otimes id \otimes u_c\big]\big(\mathcal{H}_{\frac{1}{p}}(\mathcal{M} \bar{\otimes} \mathcal{R})\big) &= \widetilde{\mathcal{H}}_{\frac{1}{p}}(\mathcal{M} \bar{\otimes} \mathcal{R}) \\ &= \widetilde{\mathcal{H}}_r(\mathcal{M}, 1/p) \otimes_{\mathcal{M},h} \big(\mathcal{M} \bar{\otimes} \mathcal{R}\big) \otimes_{\mathcal{M},h} \widetilde{\mathcal{H}}_c(\mathcal{M}, 1/p) \end{aligned}$$

into a four term $\mathcal{K}$-space $\mathcal{K}_\lambda$ with norm given by

$$\|x\|_{\mathcal{K}_\lambda} = \inf_{x = x_1 + \mathsf{d}_\lambda^{-1} x_2 \mathsf{d}_\lambda^{-1} + \mathsf{d}_\lambda^{-1} x_3 + x_4 \mathsf{d}_\lambda^{-1}} \sum_{j=1}^4 \|x_j\|_{\mathsf{E}_j},$$

where the spaces $\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3, \mathsf{E}_4$ are given by

$$\begin{aligned} \mathsf{E}_1 &= \mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{R}, \\ \mathsf{E}_2 &= L_1(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2); \mathcal{R}), \\ \mathsf{E}_3 &= L_2^r(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)) \otimes_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2),h} (\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{R}), \\ \mathsf{E}_4 &= (\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2) \bar{\otimes} \mathcal{R}) \otimes_{\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2),h} L_2^c(\mathcal{M} \bar{\otimes} \mathcal{B}(\ell_2)). \end{aligned}$$

Indeed, all these spaces can be essentially obtained by applying Remark 2.2. For instance, to obtain E_4 we have to show that the given space comes from the choice

$C\bar{\otimes}\mathcal{M}$ and $L_2^c(\mathcal{M}) \otimes_h C$ for the left and right spaces. More concretely, we have a completely isometric embedding of

$$(C\bar{\otimes}\mathcal{M}) \otimes_{\mathcal{M},h} (\mathcal{M}\bar{\otimes}\mathcal{R}) \otimes_{\mathcal{M},h} (L_2^c(\mathcal{M}) \otimes_h C)$$

into E_4. However, according to Remark 2.2 we may embed it cb-isometrically into $(C\bar{\otimes}\mathcal{M} \otimes_h R) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2),h} (C \otimes_h \mathcal{M}\bar{\otimes}\mathcal{R} \otimes_h R) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2),h} (C \otimes_h L_2^c(\mathcal{M}) \otimes_h C)$ which in turn embeds in

$$(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2),h} (\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\bar{\otimes}\mathcal{R}) \otimes_{\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2),h} L_2^c(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)).$$

This completes the argument since the latter space is E_4. On the other hand, since $\mathcal{R}$ is of finite dimension (m say) we know that topologically we may write $\mathcal{K}_\lambda^*$ as follows

$$\begin{array}{rcl}\mathcal{K}_\lambda^* & \simeq & \big((\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))^m\big)^* \cap \big(\mathsf{d}_\lambda^{-1}\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\mathsf{d}_\lambda^{-1}\big)^m \\ & \cap & \big(\mathsf{d}_\lambda^{-1}L_2(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))\big)^m \cap \big(L_2(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\mathsf{d}_\lambda^{-1}\big)^m \\ & = & L_1\big(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\big)^m \cap \big(\mathsf{d}_\lambda^{-1}\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2)\mathsf{d}_\lambda^{-1}\big)^m \\ & \cap & \big(\mathsf{d}_\lambda^{-1}L_2(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))\big)^m \cap \big(L_2(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))\mathsf{d}_\lambda^{-1}\big)^m.\end{array}$$

Here we used the fact that a matrix $[x_{ij}]$ with $x_{ij} \in \mathcal{M} \subset L_1(\mathcal{M})$ belonging to $(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))^{**}$ already belongs to $L_1(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell_2))$. Let $(p_n)_{n\ge1}$ be an increasing sequence of orthogonal projections commuting with d_λ and converging strongly to $\mathbf{1}$. Then we deduce that for $x \in \mathcal{K}_\lambda$ $\lim_{n\to\infty} \big\langle p_n x p_n, y\big\rangle = \langle x, y\rangle$ because we can use convergence in the norm of L_1 or L_2 on at least one side of the bracket. Using a weak*-limit we obtain

$$\|x\|_{\mathcal{K}_\lambda} = \|x\|_{\mathcal{K}_\lambda^{**}} \le \lim_{n,\mathcal{U}} \big\|(1\otimes p_n\otimes 1)x(1\otimes p_n \otimes 1)\big\|_{\mathcal{K}_\lambda} \le \|x\|_{\mathcal{K}_\lambda}$$

for any free ultrafilter $\mathcal{U}$ on the integers. Therefore, allowing to take ultraproducts (as we do) in the final space, it suffices to consider the finite-dimensional case where $\mathcal{B}(\ell_2)$ is replaced by the matrix algebra M_n. Define on M_n

$$\psi_n\Big(\sum_{i,j=1}^n \alpha_{ij}e_{ij}\Big) = \sum_{k=1}^n \lambda_k^2\,\alpha_{kk} \quad \text{and} \quad \phi_n(x) = \frac{\psi_n(x)}{\psi_n(\mathbf{1})}.$$

Since the original d_λ is unbounded, we may assume that $\sum_k \lambda_k^2 > 1$. Then, by approximation we can indeed assume that $\psi_n(\mathbf{1}) = \mathrm{k}_n$ is an integer, see [15] for more details. If d_{ψ_n} and d_{ϕ_n} stand for the corresponding densities, we clearly have

$$\mathsf{d}_\lambda = d_{\psi_n}^{\frac12} = \sqrt{\mathrm{k}_n}\, d_{\phi_n}^{\frac12}.$$

In particular, we may replace $\mathcal{K}_\lambda$ by $\mathcal{K}_n$ with

$$\|x\|_{\mathcal{K}_n} = \inf_{x=\sum_1^4 x_j} \|x_1\|_{\mathsf{E}_{n1}} + \mathrm{k}_n \big\|d_{\phi_n}^{\frac12}x_2 d_{\phi_n}^{\frac12}\big\|_{\mathsf{E}_{n2}} + \sqrt{\mathrm{k}_n}\,\big\|d_{\phi_n}^{\frac12}x_3\big\|_{\mathsf{E}_{n3}} + \sqrt{\mathrm{k}_n}\,\big\|x_4 d_{\phi_n}^{\frac12}\big\|_{\mathsf{E}_{n4}}$$

and where E_{nj} is the result of replacing in E_j the algebra $\mathcal{B}(\ell_2)$ by M_n. We can identify this space in the terminology of Theorem 3.7. Namely, if we fix a positive integer m and set $\mathcal{E}_m = id \otimes \phi_n \otimes \varphi : M_m \otimes M_n \otimes \mathcal{M} \to M_m$, then it is easily checked that we have the following isometric isomorphism

$$S_1^m(\mathcal{K}_n) = \mathcal{K}_{1,\infty}^{\mathrm{k}_n}\big(M_{mn}(\mathcal{M}), \mathcal{E}_m; \mathcal{R}\big).$$

For instance, according to Remark 2.2, $S_1^m(\mathsf{E}_{n3})$ can be written as

$$\Big[R_m \otimes_h L_2^r(M_n(\mathcal{M})) \otimes_h R_m\Big] \otimes_{M_{mn}(\mathcal{M}),h} \Big[C_m \otimes_h M_n(\mathcal{M}\bar{\otimes}\mathcal{R}) \otimes_h R_m\Big] \otimes_{M_m,h} C_{m^2},$$

which in turn is cb-isometric to

$$L_2^r(M_{mn}(\mathcal{M})) \otimes_{M_{mn}(\mathcal{M}),h} (M_{mn}(\mathcal{M}) \bar{\otimes} \mathcal{R}) \otimes_{M_m,h} L_2^c(M_m).$$

In other words, we obtain the space $L_1^c(M_{mn}(\mathcal{M}), \mathcal{E}_m; \mathcal{R})$. The term E_{n4} is handled in the same way while the terms E_{n1} and E_{n2} are even easier. Applying Theorem 3.7 for tensor copies, we have an embedding

$$x \in \mathcal{K}_{1,\infty}^{\mathrm{k}_n}\big(M_{mn}(\mathcal{M}), \mathcal{E}_m; \mathcal{R}\big) \mapsto \sum_{j=1}^{\mathrm{k}_n} \pi_{tens}^j(x) \otimes \delta_j \in L_1\big(\mathcal{A}_{m,n}; \ell_\infty^{\mathrm{k}_n}(\mathcal{R})\big),$$

with $\mathcal{A}_{m,n} = M_m \otimes M_n(\mathcal{M})^{\otimes \mathrm{k}_n}$. In particular, this produces a cb-embedding

$$w_n : x \in \mathcal{K}_n \mapsto \sum_{j=1}^{\mathrm{k}_n} \pi_{tens}^j(x) \otimes \delta_j \in L_1\big(M_n(\mathcal{M})^{\otimes \mathrm{k}_n}; \ell_\infty^{\mathrm{k}_n}(\mathcal{R})\big),$$

with relevant constants independent of n. Finally, the construction of our complete embedding is as follows. We first apply Proposition 4.3 and then we proceed as above. Namely, if $u = u_r \otimes id \otimes u_c$, we have

$$L_p(\mathcal{M}; \mathcal{R}) \stackrel{j}{\longrightarrow} \mathcal{H}_{\frac{1}{p}}(\mathcal{M} \bar{\otimes} \mathcal{R}) \stackrel{u}{\longrightarrow} \widetilde{\mathcal{H}}_{\frac{1}{p}}(\mathcal{M} \bar{\otimes} \mathcal{R}) \stackrel{id}{\longrightarrow} \overline{\bigcup_{n \ge 1} \mathcal{K}_n}$$

and we have constructed a complete embedding

$$\prod\nolimits_{n,\mathcal{U}} w_n : \overline{\bigcup_{n \ge 1} \mathcal{K}_n} \longrightarrow \prod\nolimits_{n,\mathcal{U}} L_1\big(M_n(\mathcal{M})^{\otimes \mathrm{k}_n}; \ell_\infty^{\mathrm{k}_n}(\mathcal{R})\big).$$

Let $q_n : \mathcal{B}(\ell_2) \to M_n$ be the projection into the upper left corner and let us define the element $\xi_n = q_n \xi_r q_n \xi_c q_n \in L_1(M_n)$. Since $u(j(x)) = u(1 \otimes x) = \xi_r \xi_c \otimes x$, the form of the embedding $\prod_{n,\mathcal{U}} w_n \circ u \circ j$ is the one given in the assertion. □

5. Mixed-norm transference and applications

Given a Hilbert space $\mathcal{H}$, we shall write $\mathcal{H}_r$ and $\mathcal{H}_c$ to denote the row and column operator space structures on $\mathcal{H}$. Accordingly, $\mathcal{H}_{r_p}$ and $\mathcal{H}_{c_p}$ stand for the complex interpolation spaces $[\mathcal{H}_r, \mathcal{H}_c]_{1/p}$ and $[\mathcal{H}_c, \mathcal{H}_r]_{1/p}$, respectively. Let us fix $1 \le p \le q \le \infty$ and $n \ge 1$ a positive integer. By Pisier's exercise [30] and some refinements [12, 15, 43], we may construct complete embeddings

$$\alpha_q : C_q^n \to L_2^c(\Omega, \mu_q; \ell_2^n) + L_2^{c_p}(\Omega, \nu_q; \ell_2^n),$$

$$\beta_q : R_q^n \to L_2^r(\Omega, \mu_q; \ell_2^n) + L_2^{r_p}(\Omega, \nu_q; \ell_2^n),$$

for suitable measures μ_q and ν_q on a finite set $\Omega = \{1, 2, \ldots, m\}$ with m depending on n. In fact, an elaborated version of this result was already used in the previous section. A much more concrete approach is available in [17, Lemma 2.2]. Let $\mu_i = \mu_q\{i\}$ and $\nu_i = \nu_q\{i\} = \lambda_i \mu_i$ for some $\lambda_i > 0$. Let us write d_λ for the diagonal operator on ℓ_2^m determined by the λ_i's. That is, $\mathsf{d}_\lambda = \sum_k \lambda_k e_{kk}$. The symbol $+$ above refers as in the previous section to the quotient of the direct sums

$$L_2^c(\Omega, \mu_q; \ell_2^n) \oplus L_2^{c_p}(\Omega, \nu_q; \ell_2^n) \quad \text{and} \quad L_2^r(\Omega, \mu_q; \ell_2^n) \oplus L_2^{r_p}(\Omega, \nu_q; \ell_2^n)$$

by the subspace

$$S = \Big\{ \big(a_{ij}, -\lambda_i^{-\frac{1}{2}} a_{ij}\big) \ \big| \ 1 \le i \le m, \ 1 \le j \le n \Big\}.$$

More concretely, in the first case $a = (a_{ij}) \in L_2^c(\Omega, \mu_q; \ell_2^n)$ is a column with m entries in ℓ_2^n, while in the second case $a = (a_{ij}) \in L_2^r(\Omega, \mu_q; \ell_2^n)$ is a row. In particular, we may write S in each case as follows

$$S_\alpha = \Big\{ \big(a, -\mathsf{d}_\lambda^{-\frac{1}{2}} a\big) \ \big| \ a \in L_2^c(\Omega, \mu_q; \ell_2^n) \Big\} = \Big\{ \big(-\mathsf{d}_\lambda^{\frac{1}{2}} a, a\big) \ \big| \ a \in L_2^c(\Omega, \mu_q; \ell_2^n) \Big\},$$

$$S_\beta = \Big\{ \big(a, -a\mathsf{d}_\lambda^{-\frac{1}{2}}\big) \ \big| \ a \in L_2^r(\Omega, \mu_q; \ell_2^n) \Big\} = \Big\{ \big(-a\mathsf{d}_\lambda^{\frac{1}{2}}, a\big) \ \big| \ a \in L_2^r(\Omega, \mu_q; \ell_2^n) \Big\}.$$

The embedding is of the form

$$\alpha_q(a) = \mathbf{1}_\Omega \otimes a + S_\alpha.$$

The formula for β_q is the same. Let us define

$$S_{\alpha\beta} = S_\alpha \otimes_h \Big(L_2^r(\Omega, \mu_q; \ell_2^n) \oplus L_2^{r_p}(\Omega, \nu_q; \ell_2^n) \Big) + \Big(L_2^c(\Omega, \mu_q; \ell_2^n) \oplus L_2^{c_p}(\Omega, \nu_q; \ell_2^n) \Big) \otimes_h S_\beta.$$

Lemma 5.1. *If $1 \le p \le q \le \infty$, we have a cb-embedding*

$$v_q : \ell_q^n \to \Big(L_2^c(\Omega, \mu_q; \ell_2^n) + L_2^{c_p}(\Omega, \nu_q; \ell_2^n) \Big) \otimes_h \Big(L_2^r(\Omega, \mu_q; \ell_2^n) + L_2^{r_p}(\Omega, \nu_q; \ell_2^n) \Big),$$

$$v_q\big(\xi_1, \xi_2, \ldots, \xi_n\big) = \sum_{k=1}^n \xi_k \, \alpha_q(e_{k1}) \otimes \beta_q(e_{1k}) = \Big[\sum_{k=1}^n \xi_k \, \Big(\sum_{i,j=1}^m e_{ij} \Big) \otimes e_{kk} \Big] + S_{\alpha\beta}.$$

Proof. It follows from our considerations above and $\ell_q^n \subset C_q^n \otimes_h R_q^n$. □

The embedding v_q is special in the sense that its range is contained in the subalgebra $M_m \otimes \ell_\infty^n$, after a suitable change of variables. To explain this we recall that $\mu_q\{i\} = \mu_i$ and $\nu_q\{i\} = \lambda_i \mu_i$. Therefore, the map

$$j : (a_{ij}) \in L_2^c(\Omega, \mu_q; \ell_2^n) \mapsto \big(\sqrt{\mu_i} \, a_{ij} \big) \in C_{mn}$$

is a complete isometry. To respect the sum operation, we have to apply j also on $L_2^{c_p}(\Omega, \nu_q; \ell_2^n)$. If λ stands for the measure on Ω given by $\lambda\{i\} = \lambda_i$, we find another complete isometry $j : L_2^{c_p}(\Omega, \mu_q; \ell_2^n) \to L_2^{c_p}(\Omega, \lambda, \ell_2^n)$. Hence, applying the same argument for the other side and using the terminology

$$L_2^{c_p}(\Omega, \lambda; \ell_2^n) = \mathsf{d}_\lambda^{\frac{1}{2}} C_p^{mn} \quad \text{and} \quad L_2^{r_p}(\Omega, \lambda; \ell_2^n) = R_p^{mn} \mathsf{d}_\lambda^{\frac{1}{2}},$$

we find a complete isometry

$$J : \big(L_2^c(\Omega, \mu_q; \ell_2^n) \oplus L_2^{c_p}(\Omega, \nu_q; \ell_2^n) \big) \otimes_h \big(L_2^r(\Omega, \mu_q, \ell_2^n) \oplus L_2^{r_p}(\Omega, \nu_q; \ell_2^n) \big)$$
$$\to \quad \big(C_{mn} \oplus \mathsf{d}_\lambda^{\frac{1}{2}} C_p^{mn} \big) \otimes_h \big(R_{mn} \oplus R_p^{mn} \mathsf{d}_\lambda^{\frac{1}{2}} \big)$$

with $J = (j, j) \otimes (j, j)$. Passing to quotients, we may replace $\oplus$ by $+$. The key observation here is that algebraically we have $J\left(v_q(\ell_q^n)\right) \subset M_m \otimes \ell_\infty^n$. Indeed, note

$$J\Big[\Big(\sum_{i,j=1}^m e_{ij} \Big) \otimes e_{kk} \Big] = \Big(\sum_{i,j=1}^m \mu_i^{\frac{1}{2}} \mu_j^{\frac{1}{2}} e_{ij} \Big) \otimes \delta_k. \tag{5.1}$$

Before we proceed with our next result, we review the $\mathcal{K}_{p,\infty}^{\mathrm{k}}(M_m, \phi)$ space given by a state ϕ. In this case, given any pair $u, v \in \{2p, \infty\}$, the inclusion map $S_{(u,v)}^m \subset S_p^m$ depends on ϕ. Indeed, we may and will assume that $\phi(x) = \sum_k \phi_k x_{kk}$. Then, the density d_ϕ is indeed a diagonal operator with coefficients ϕ_k and we have Kosaki's embedding

$$x \in S_{(u,v)}^m \mapsto d_\phi^{\frac{1}{2p} - \frac{1}{u}} x \, d_\phi^{\frac{1}{2p} - \frac{1}{v}} \in S_p^m.$$

Therefore, we find

$$\|x\|_{\mathcal{K}^{\mathrm{k}}_{p,\infty}(M_m,\phi)} = \inf\Big\{\mathrm{k}^{\frac1p}\|x^1\|_{S_p^m} + \mathrm{k}^{\frac{1}{2p}}\|x^2\|_{S_{2p}^m} + \mathrm{k}^{\frac{1}{2p}}\|x^3\|_{S_{2p}^m} + \|x^4\|_{M_m}\Big\},$$

where the infimum runs over $x = x^1 + x^2 d_\phi^{\frac{1}{2p}} + d_\phi^{\frac{1}{2p}} x^3 + d_\phi^{\frac{1}{2p}} x^4 d_\phi^{\frac{1}{2p}}$. This gives

$$\|x\|_{\mathcal{K}^{\mathrm{k}}_{p,\infty}(M_m,\phi)} = \inf\Big\{\big\|d_{\mathrm{k}\phi}^{\frac{1}{2p}} x^1 d_{\mathrm{k}\phi}^{\frac{1}{2p}}\big\|_{S_p^m} + \big\|d_{\mathrm{k}\phi}^{\frac{1}{2p}} x^2\big\|_{S_{2p}^m} + \big\|x^3 d_{\mathrm{k}\phi}^{\frac{1}{2p}}\big\|_{S_{2p}^m} + \|x^4\|_{M_m}\Big\},$$

where this time the infimum is taken over $d_\phi^{-\frac{1}{2p}} x\, d_\phi^{-\frac{1}{2p}} = x^1 + x^2 + x^3 + x^4$. A similar calculation applies in the operator-valued setting. In the following result, we shall use the notation

$$\|x\|_{d^\alpha X d^\beta} = \|d^\alpha x\, d^\beta\|_X.$$

Lemma 5.2. *Let us consider a von Neumann algebra $\mathcal{M}$, positive integers m, n, k and a state ϕ on M_m. Let $\mathcal{E}_\mathcal{M} : M_m(\mathcal{M}) \otimes \ell_\infty^{\mathrm{k}n} \to \mathcal{M}$ stand for the conditional expectation*

$$\mathcal{E}_\mathcal{M}\Big(\sum_{i=1}^{\mathrm{k}}\sum_{j=1}^{n} x_{ij}\otimes\delta_{ij}\Big) = \frac{1}{n\mathrm{k}}\sum_{i=1}^{\mathrm{k}}\sum_{j=1}^{n} \phi\otimes id(x_{ij}).$$

Let $\mathcal{R} = M_m(\mathcal{M})\otimes\ell_\infty^n$ and consider the space $X_{\phi,n}^p(\mathcal{M})$ defined by

$$L_p(\mathcal{M};\ell_\infty^n(M_m)) + d_{\mathrm{k}\phi}^{\frac{1}{2p}} L_{2p}(\mathcal{R})L_{2p}(\mathcal{M}) + L_{2p}(\mathcal{M})L_{2p}(\mathcal{R})d_{\mathrm{k}\phi}^{\frac{1}{2p}} + d_{\mathrm{k}\phi}^{\frac{1}{2p}} L_p(\mathcal{R}) d_{\mathrm{k}\phi}^{\frac{1}{2p}}.$$

Then the following identity holds

$$\Big\|\sum_{i=1}^{\mathrm{k}}\sum_{j=1}^{n} d_\phi^{\frac{1}{2p}} x_j\, d_\phi^{\frac{1}{2p}}\otimes\delta_{ij}\Big\|_{\mathcal{K}^{\mathrm{k}n}_{p,\infty}(M_m(\mathcal{M})\otimes\ell_\infty^{\mathrm{k}n},\mathcal{E}_\mathcal{M})} = \Big\|\sum_{j=1}^{n} x_j\otimes\delta_j\Big\|_{X^p_{\phi,n}(\mathcal{M})}.$$

Proof. The subspace of sequences

$$\big(x_1,\ldots,x_1,x_2,\ldots,x_2,\ldots,x_n,\ldots,x_n\big),$$

with every x_j repeated k times is complemented in $\mathcal{K}^{\mathrm{k}n}_{p,\infty}(M_m(\mathcal{M})\otimes\ell_\infty^{\mathrm{k}n},\mathcal{E}_\mathcal{M})$ since it is complemented in the four spaces composing it. Let us assume for simplicity that $\mathcal{M}$ has a normalized trace τ. Our reference state and trace in the construction of the Haagerup L_p spaces are

$$\psi(x_{ij}) = \frac{1}{n\mathrm{k}}\sum_{i=1}^{\mathrm{k}}\sum_{j=1}^{n}\phi\otimes\tau(x_{ij}) \quad \text{and} \quad \mathrm{tr}(x_{ij}) = \frac{1}{n\mathrm{k}}\sum_{i=1}^{\mathrm{k}}\sum_{j=1}^{n}\mathrm{tr}_{M_m}\otimes\tau(x_{ij}).$$

The density is given by $d_\psi = d_\phi\otimes\mathbf{1}_\mathcal{M}$ and letting $\widehat{\mathcal{M}}_{\mathrm{k}mn} = M_m(\mathcal{M})\otimes\ell_\infty^{\mathrm{k}n}$ we have

$$\begin{aligned}
&\mathcal{K}^{\mathrm{k}n}_{p,\infty}\big(\widehat{\mathcal{M}}_{\mathrm{k}mn},\mathcal{E}_\mathcal{M}\big) = (n\mathrm{k})^{\frac1p}\, d_\psi^{\frac{1}{2p}} L_p(\widehat{\mathcal{M}}_{\mathrm{k}mn})\, d_\psi^{\frac{1}{2p}} \\
&+ (n\mathrm{k})^{\frac{1}{2p}}\, d_\psi^{\frac{1}{2p}} L_{2p}(\widehat{\mathcal{M}}_{\mathrm{k}mn})L_{2p}(\mathcal{M}) + (n\mathrm{k})^{\frac{1}{2p}}\, L_{2p}(\mathcal{M})L_{2p}(\widehat{\mathcal{M}}_{\mathrm{k}mn})d_\psi^{\frac{1}{2p}} \\
&+ L_{2p}(\mathcal{M})\, L_\infty(\widehat{\mathcal{M}}_{\mathrm{k}mn})\, L_{2p}(\mathcal{M}) \;=\; Z_1+Z_2+Z_3+Z_4.
\end{aligned}$$

This means that

$$\Big\|\sum_{i=1}^{\mathrm{k}}\sum_{j=1}^{n} d_\phi^{\frac{1}{2p}} x_j\, d_\phi^{\frac{1}{2p}}\otimes\delta_{ij}\Big\|_{\mathcal{K}^{\mathrm{k}n}_{p,\infty}(\widehat{\mathcal{M}}_{\mathrm{k}mn},\mathcal{E}_\mathcal{M})} = \inf_{x_j=x_j^1+x_j^2+x_j^3+x_j^4}\sum_{s=1}^{4}\Big\|\sum\nolimits_{i,j} x_j^s\otimes\delta_{ij}\Big\|_{Z_s}.$$

Let us compute the four norms

$$\begin{aligned}
\Big\| \sum\nolimits_{i,j} x_j \otimes \delta_{ij} \Big\|_{Z_1} &= (n\mathrm{k})^{\frac{1}{p}} \Big(\frac{1}{n\mathrm{k}} \sum\nolimits_{i,j} \big\| d_\phi^{\frac{1}{2p}} x_j d_\phi^{\frac{1}{2p}} \big\|_{L_p(M_m(\mathcal{M}))}^p \Big)^{\frac{1}{p}} \\
&= \mathrm{k}^{\frac{1}{p}} \Big(\sum_{j=1}^n \big\| d_\phi^{\frac{1}{2p}} x_j d_\phi^{\frac{1}{2p}} \big\|_p^p \Big)^{\frac{1}{p}} = \mathrm{k}^{\frac{1}{p}} \Big\| \sum_{j=1}^n x_j \otimes \delta_j \Big\|_{d_\phi^{\frac{1}{2p}} L_p(\mathcal{R}) d_\phi^{\frac{1}{2p}}}, \\
\Big\| \sum\nolimits_{i,j} x_j \otimes \delta_{ij} \Big\|_{Z_2} &= \mathrm{k}^{\frac{1}{2p}} \inf_{d_\phi^{1/2p} x_j = z_j b} \Big(\sum_{j=1}^n \|z_j\|_{L_{2p}(M_m(\mathcal{M}))}^{2p} \Big)^{\frac{1}{2p}} \|b\|_{L_{2p}(\mathcal{M})} \\
&= \mathrm{k}^{\frac{1}{2p}} \Big\| \sum_{j=1}^n x_j \otimes \delta_j \Big\|_{d_\phi^{\frac{1}{2p}} L_{2p}(\mathcal{R}) L_{2p}(\mathcal{M})}, \\
\Big\| \sum\nolimits_{i,j} x_j \otimes \delta_{ij} \Big\|_{Z_4} &= \inf_{x_j = a z_j b} \|a\|_{L_{2p}(\mathcal{M})} \sup_{1 \le j \le n} \|z_j\|_{M_m(\mathcal{M})} \|b\|_{L_{2p}(\mathcal{M})} \\
&= \Big\| \sum_{j=1}^n x_j \otimes \delta_j \Big\|_{L_p(\mathcal{M}; \ell_\infty^n(M_m))}.
\end{aligned}$$

The Z_3-term is calculated as Z_2. The proof is complete. □

Theorem 5.3. *Let $1 \le p \le q \le \infty$ and set $m = |\Omega|$ as above. Then, there exists a state ϕ_m on M_m and a positive integer k_m such that we have a complete embedding*

$$L_p(\mathcal{M}; \ell_q^n) \to L_p\Big(\mathcal{M} \bar{\otimes} \Big[*_{\mathrm{k}_m n} \, (M_m, \phi_m) \Big]; \ell_\infty^{\mathrm{k}_m n} \Big)$$

given by the relation

$$\sum_{j=1}^n x_j \otimes \delta_j \mapsto \sum_{i=1}^{\mathrm{k}_m} \sum_{j=1}^n x_j \otimes \pi_{free}^{ij}(a_m) \otimes \delta_{ij},$$

where $a_m = a_m(p,q) \in M_m$. The relevant constants are independent of m and n.

Proof. By enlarging m if necessary, we may assume that $\sum_i \lambda_i^p = \mathrm{k}_m$ is an integer. Then we define the normalized state $\phi_m(x) = \mathrm{k}_m^{-1} \sum_i \lambda_i^p x_{ii}$. We observe that for arbitrary elements $x_j \in M_m \otimes L_p(\mathcal{M})$, the right hand side of Lemma 5.2 coincides with the norm of diagonal sequences (i.e. $mn \times mn$ matrices which are diagonal on its n-component) in the space

$$L_p\Big(\mathcal{M}; \big(C_{mn} + \mathsf{d}_\lambda^{\frac{1}{2}} C_p^{mn} \big) \otimes_h \big(R_{mn} + R_p^{mn} \mathsf{d}_\lambda^{\frac{1}{2}} \big) \Big).$$

Indeed, we use the properties of the Haagerup tensor product and the fact that projection onto the diagonal $M_n \to \ell_\infty^n$ is completely contractive with respect to all the four interpolation norms. The embedding is given by $u \circ \big(id_{L_p(\mathcal{M})} \otimes (J \circ v_q) \big)$ where u is the first map from Theorem 2.4 for $q = \infty$ and $\mathrm{k}_m n$ instead of n. Note that u is well-defined because of $J(v_q(\ell_q^n)) \subset M_m \otimes \ell_\infty^n$. Moreover, identity (5.1) tells us that

$$a_m = d_\phi^{\frac{1}{2p}} \Big(\sum_{i,j \le \mathrm{k}'_m} \sqrt{\mu_i \mu_j} e_{ij} \Big) d_\phi^{\frac{1}{2p}}$$

with $\mathrm{k}'_m \le \mathrm{k}_m$. The proof is complete. □

Remark 5.4. The constants in Theorem 5.3 are also independent of p, q as far as we do not have $p \to 1$ and $q \to \infty$ simultaneously. The use of Theorem 2.4 produces such singularity, see Remarks 2.3 and 2.5 for further details.

Proof of Theorem B. Let us consider

$$\sum_{k=1}^{n} \pi_k(x) \otimes \delta_k \in L_p(\mathcal{A}; \ell_q^n).$$

According to Theorem 5.3, the following equivalence holds

$$\Big\| \sum_{k=1}^{n} \pi_k(x) \otimes \delta_k \Big\|_{L_p(\mathcal{A};\ell_q^n)} \sim_c \Big\| \sum_{i=1}^{\mathrm{k}_m} \sum_{j=1}^{n} \pi_j(x) \otimes \pi_{free}^{ij}(a_m) \otimes \delta_{ij} \Big\|_{L_p(\mathcal{A}\otimes\mathcal{B};\ell_\infty^{\mathrm{k}_m n})},$$

with $\mathcal{B} = *_{\mathrm{k}_m n}(M_m, \phi_m)$. Applying the complete embedding that we constructed in Theorem 4.4 to the term on the right hand side (note that both Theorems 4.4 and 5.3 provide constants independent of m and n) we obtain a new term in the ultraproduct

$$\prod\nolimits_{s,\mathcal{U}} L_1\Big(\big[M_s \otimes \mathcal{A} \otimes \mathcal{B} \big]^{\otimes_{\mathrm{k}_s}} ; \ell_\infty^{\mathrm{k}_m \mathrm{k}_s n} \Big)$$

of the following form

$$\bigg(\sum_{i=1}^{\mathrm{k}_m} \sum_{j=1}^{n} \sum_{w=1}^{\mathrm{k}_s} \pi_{tens}^{w} \Big[\xi_s \otimes \pi_j(x) \otimes \pi_{free}^{ij}(a_m) \Big] \otimes \delta_{ijw} \bigg)_s$$

for a fixed family of matrices $\xi_s \in M_s$. On the other hand, we have

$$\pi_{free}^{ij} = \pi_{free}^{j} \circ \pi_{free}^{i}$$

by the transitivity of free products, see e.g. Proposition 2.5.5. in [41]. Therefore, if we let $\alpha_j = \pi_j \otimes \pi_{free}^{j}$ and amalgamate over M_s, we may rewrite the term above as follows

$$\bigg(\sum_{i=1}^{\mathrm{k}_m} \sum_{j=1}^{n} \sum_{w=1}^{\mathrm{k}_s} \pi_{tens}^{w} \Big[\alpha_j \big(\xi_s \otimes x \otimes \pi_{free}^{i}(a_m) \big) \Big] \otimes \delta_{ijw} \bigg)_s .$$

Then, arguing as in the proof of Lemma 1.2 (in particular part i), we obtain

$$\bigg(\sum_{j=1}^{n} \widehat{\alpha}_j \Big[\sum_{i=1}^{\mathrm{k}_m} \sum_{w=1}^{\mathrm{k}_s} \pi_{tens}^{w} \big(\xi_s \otimes x \otimes \pi_{free}^{i}(a_m) \big) \otimes \delta_{iw} \Big] \otimes \delta_j \bigg)_s$$

with $\widehat{\alpha}_j$ a tensor amplification of α_j. However, according to Lemma 1.2 i) the $\widehat{\alpha}_j$'s provide an increasingly independent family of top-subsymmetric copies over the symmetric tensor product of M_s. Hence, we are in position to apply Theorem 3.7 with π_j replaced by $\widehat{\alpha}_j$ and $\mathcal{R} = \ell_\infty^{\mathrm{k}_s \mathrm{k}_m}$. Again, the constants are independent of the involved parameters. This gives us a new term which does not depend on the choice of the morphisms $\widehat{\alpha}_j$, so that we may use

$$\widehat{\alpha}_j = \pi_{free}^{j} \otimes \pi_{free}^{j}$$

instead. The assertion is then obtained by calculating backwards. □

Remark 5.5. If we do not require the constant to be (p,q)-independent, Theorem B also holds for $p > q$ by a simple duality argument. The singularity arises in this case from the complementation constant of the subspace of independent copies in $L_p(\mathcal{A}; \ell_q^n)$. As for (Σ_{pq}), this singularity is not removable.

Corollary 5.6. *Let $1 \le p \le q \le \infty$ and let $(\mathcal{M}_k)_{k \ge 1}$ be an increasingly independent family of top-subsymmetric copies of $\mathcal{M}$ over $\mathcal{N}$. Then, we have an isomorphic embedding*

$$x \in \mathcal{K}_{p,q}^n(\mathcal{M}, \mathcal{E}_\mathcal{N}) \mapsto \sum_{k=1}^n \pi_k(x) \otimes \delta_k \in L_p\big(\mathcal{A}; \ell_q^n\big)$$

with complemented range and constants independent of n. In particular, replacing $(\mathcal{M}, \mathcal{N}, \mathcal{E}_\mathcal{N})$ by $(M_m(\mathcal{M}), M_m, id_{M_m} \otimes \varphi)$ so that $\mathcal{A} = M_m(\mathcal{R})$ for some $\mathcal{R}$, we obtain a complete isomorphism with completely complemented range and constants independent of n

$$x \in \mathcal{K}_{p,q}^n(\mathcal{M}) \mapsto \sum_{k=1}^n \pi_k(x) \otimes \delta_k \in L_p\big(\mathcal{R}; \ell_q^n\big).$$

Proof. It follows immediately from Theorem 2.4 and Theorem B. □

Remark 5.7. According to Remarks 2.3 and 2.5 we know that, except for the case $(p,q) \sim (1,\infty)$, the constants in Corollary 5.6 are also independent of p,q. On the other hand, since we are using transference, we need to work with independent copies. In the free case we can also work with non i.d. variables, see Theorem 2.4.

The rest of the paper is devoted to the proof of Corollary B. We begin by stating a refinement of [13, Theorem 4.2] which follows easily from our previous results in this paper. We shall write $L_p(M_n)$ for the Schatten class S_p^n equipped with the normalized trace $\frac{1}{n}\mathrm{tr}_n$.

Lemma 5.8. *Let $1 \le p \le q \le 2$ and a positive integer $n \ge 1$. Then, the following mapping is a complete isomorphism onto a completely complemented subspace with constants independent of p,q and n*

$$\Psi_{pq} : x \in L_q(M_n) \mapsto \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes \delta_k \in L_p(M_{n^{n^2}}; \ell_q^{n^2}).$$

As before, if $1 \le p \le q \le \infty$, the same holds with a singularity when $(p,q) \sim (1,\infty)$.

Proof. According to Theorem 4.2 and Remark 4.3 in [13], the assertion holds for $1 < p \le q \le \infty$ with a constant c_p majorized by $p/p-1$. The fact that it also holds for $p = 1$ now follows from Theorem 3.7 and the argument in [13]. The universality of the constants follows by Corollary 5.6 + Remark 5.7 followed by the original argument [13] again. □

Remark 5.9. The choice $m = n^2$ in $L_p(M_{n^m}; \ell_q^m)$ is optimal, see [13] for details.

In what follows, we will need some preparation on ultraproducts of semifinite von Neumann algebras. Let $(\mathcal{M}_n)$ be a family of semifinite von Neumann algebras with normal semifinite faithful traces (τ_n). We may define $\tau_\mathcal{U}(x_n) = \lim_{n,\mathcal{U}} \tau_n(x_n)$ on the ultraproduct von Neumann algebra

$$\mathcal{M}_\mathcal{U} = \big(\prod\nolimits_{n,\mathcal{U}} L_1(\mathcal{M}_n)\big)^*.$$

Let us set

$$\mathcal{M}_{\mathcal{U},sf} = \overline{\Big\{(q_n x_n q_n)^\bullet \ \Big| \ \lim\nolimits_{n,\mathcal{U}} \tau_n(q_n) < \infty, \, (x_n)^\bullet \in \mathcal{M}_\mathcal{U}\Big\}}^{\mathrm{wot}}.$$

Then it turns out that $\mathcal{M}_{\mathcal{U},sf}$ is a semifinite von Neumann subalgebra of $\mathcal{M}_{\mathcal{U}}$ and $\tau_{\mathcal{U}}((x_n)^\bullet)) = \lim_{n,\mathcal{U}} \tau_n(x_n)$ defines an trace on $\mathcal{M}_{\mathcal{U},sf}$. An appropriate way to check this consist in checking the axioms of a (tracial) Hilbert algebra

$$\mathcal{A} = \Big\{ (x_n)^\bullet \ \Big| \ \lim_{n,\mathcal{U}} \|x_n\| < \infty \text{ and } \lim_{n,\mathcal{U}} \tau(x_n^* x_n) < \infty \Big\},$$

where $(x_n)^\bullet$ corresponds to the equivalence class of a bounded sequence of positive elements in $(\prod_{\mathcal{U}} L_1(\mathcal{M}_n))^*$. Then $\tau_{\mathcal{U}}$ can be extended to a normal semifinite trace on $\mathcal{M}_{\mathcal{U},sf}$ viewed as the closure of $\mathcal{A}$ in the GNS-representation of the Hilbert algebra. We refer to [36] for more on ultraproducts of noncommutative L_p spaces and how they can be identified as a noncommutative L_p space $L_p((\prod_{\mathcal{U}} L_1(\mathcal{M}_n))^*)$. Let $\mu_s(x)$ stand for the generalized s-numbers of x, see [5]. The following will be a key result below.

Lemma 5.10. *Let (x_n) be a bounded sequence in $L_p(\mathcal{M}_n)$ such that*

$$\lim_{\delta \to 0} \lim_{n,\mathcal{U}} \int_0^\delta \mu_s(x_n)^p \, ds = 0 = \lim_{\gamma \to \infty} \lim_{n,\mathcal{U}} \int_\gamma^\infty \mu_s(x_n)^p \, ds.$$

Then we have $(x_n)^\bullet \in L_p(\mathcal{M}_{\mathcal{U},sf})$. Moreover, the converse is also true.

Proof. Given $\varepsilon > 0$, we choose γ, δ such that

$$\max \Big\{ \int_0^\delta \mu_s(x_n)^p \, ds, \int_\gamma^\infty \mu_s(x_n)^p \, ds \Big\} < \varepsilon/2.$$

It is clearly no restriction to assume that the x_n's are positive elements. Let us set $a_n = \mu_\gamma(x_n)$ and $b_n = \mu_\delta(x_n)$. If $q_n = 1_{[a_n,b_n]}(x_n)$, we observe that $\tau(q_n) \le \gamma$ and that $z_n = q_n x_n q_n$ is bounded by b_n. Note that $\delta^p \mu_\delta(x_n) \le \|x_n\|_p^p$ implies that

$$\lim_{n,\mathcal{U}} b_n \le \delta^{-p} \lim_{n,\mathcal{U}} \|x_n\|_p^p$$

is well-defined. Therefore $(z_n)^\bullet \in \mathcal{M}_{\mathcal{U},sf}$. The first assertion then follows from $\|x_n - y_n\|_p^p < \varepsilon$. For the converse we observe that $\mathcal{M}_{\mathcal{U},sf}$ is norm dense in $L_p(\mathcal{M}_{\mathcal{U}})$ and the assertion is trivially true for $(x_n)^\bullet$ in $\mathcal{M}_{\mathcal{U},sf}$. The proof is complete. □

Proof of Corollary B. If $1 \le p < q \le 2$, we shall prove:

a) There is no cb-embedding of $R_q + C_q$ into semifinite L_p.
b) Let $\mathcal{R}_0$ stand for the hyperfinite II_1 factor and assume that there exists a complete embedding of ℓ_q into $L_p(\mathcal{M})$ with $\mathcal{M}$ semifinite. Then, $L_q(\mathcal{R}_0)$ cb-embeds into some semifinite L_p space.

The combination of both results gives rise to the assertion. Indeed, we know from the noncommutative Khintchine inequality [24] that $R_q + C_q$ cb-embeds into $L_q[0,1]$ which also cb-embeds into $L_q(\mathcal{R}_0)$. Therefore, we deduce from a) that there is no cb-embedding of $L_q(\mathcal{R}_0)$ into semifinite L_p. Apply b) to conclude.

Step 1. The proof of a) essentially reproduces Xu's argument in [43]. Assume there exists a complete embedding $j : R_q + C_q \to L_p(\mathcal{M})$ with $\mathcal{M}$ semifinite and equipped with a normal semifinite faithful trace τ. Let

$$j^* : L_{p'}(\mathcal{M}) \to R_q \cap C_q$$

denote the adjoint mapping. Since $R_q \cap C_q$ can be regarded as the diagonal subspace of $R_q \oplus C_q$, we may write $j^* = (\Lambda_1, \Lambda_2)$. Since $\Lambda_1 : L_{p'}(\mathcal{M}) \to R_q$ is completely bounded, we deduce that there exists a positive unit functional $f \in L_{p'/2}(\mathcal{M})^*$ such that

$$\|\Lambda_1(x)\|_{R_q} \le c f(xx^*)^{\frac{1-\theta}{2}} f(x^*x)^{\frac{\theta}{2}} \quad \text{with} \quad 1/q = (1-\theta)/p + \theta/p'.$$

This was proved by Pisier [32] for $p = 1$ and by Xu [44] for $1 < p \le 2$. We also refer to [43, Lemma 5.8] for a precise statement. When $p > 1$, f can be regarded as a positive element in the unit ball of $L_{p/(2-p)}(\mathcal{M})$ while for $p = 1$, f can be taken as a normal state on $\mathcal{M}$ since Λ_1 is normal, see [31] for details. In particular, we deduce

$$\|\Lambda_1(x)\|_{R_q} \le c\,\tau(fxx^*)^{\frac{1-\theta}{2}} \tau(fx^*x)^{\frac{\theta}{2}}.$$

Arguing as in the proof of Theorem 5.6 of [43], we may apply an approximation argument which allows us to assume that $\mathcal{M}$ is finite and $f = \mathbf{1}_{\mathcal{M}}$. In that case our estimate for Λ_1 becomes $\|\Lambda_1(x)\|_{R_q} \le c\,\tau(xx^*)^{1/2}$. Moreover, the same argument for Λ_2 produces

$$\|j^*(x)\|_{R_q \cap C_q} = \max \Big\{ \|\Lambda_1(x)\|_{R_q}, \|\Lambda_2(x)\|_{C_q} \Big\} \le c\,\tau(xx^*)^{\frac{1}{2}}. \tag{5.2}$$

This provides a factorization $j^* = v^* u^*$ with $u^* : L_{p'}(\mathcal{M}) \to L_2(\mathcal{M})$ the natural inclusion map. Arguing (twice) as in [43], we see that u^* becomes a complete contraction when we impose on $L_2(\mathcal{M})$ the o.s.s. of $L_2^{c_{p'}}(\mathcal{M}) \cap L_2^{r_{p'}}(\mathcal{M})$. On the other hand, it follows from (5.2) that v^* is a bounded map between Hilbert spaces so that $v \in S_\infty$. To conclude, we note that $j = uv$ provides a factorization

$$R_q + C_q \xrightarrow{\ v\ } R_p + C_p \xrightarrow{\ u\ } j(R_q + C_q).$$

By a simple modification of [43, Lemma 5.9], we deduce that

$$u \in \mathcal{CB}(R_p + C_p, R_q + C_q) = S_{2pq/|p-q|}.$$

Thus, the identity on $R_q + C_q$ belongs to $S_{2pq/|p-q|}$ which contradicts $1 \le p < q \le 2$.

Step 2. Assume that there exists a cb-embedding j_p of ℓ_q into $L_p(\mathcal{M})$ for some semifinite von Neumann algebra $\mathcal{M}$ equipped with a normal faithful semifinite trace τ. According to Lemma 5.8 and our assumption, we find a cb-embedding

$$\begin{aligned} u_{np} : x \in L_q(M_n) \quad & \mapsto \quad \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes \delta_k \in L_p(M_n^{\otimes_{n^2}}; \ell_q^{n^2}) \\ & \mapsto \quad \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes j_p(\delta_k) \in L_p(M_n^{\otimes_{n^2}} \otimes \mathcal{M}). \end{aligned}$$

Taking ultraproducts, we find a cb-embedding

$$w_p : L_q(\mathcal{R}_0) \to \prod\nolimits_{n,\mathcal{U}} L_p\big(M_n^{\otimes_{n^2}} \otimes \mathcal{M}\big) = L_p(\widehat{\mathcal{M}}_{\mathcal{U}}).$$

On the other hand, by our assumption we may regard ℓ_q as an infinite-dimensional subspace of $L_p(\mathcal{M})$ not containing ℓ_p. According to the noncommutative form [14] of Rosenthal's theorem, given any $p < r < q$ we may find a positive density $d \in L_1(\mathcal{M})$ with $\tau(d) = 1$ and a embedding $j_r : \ell_q \to L_r(\mathcal{M})$ satisfying

$$j_p(x) = d^{\frac{1}{p} - \frac{1}{r}} j_r(x) + j_r(x) d^{\frac{1}{p} - \frac{1}{r}}. \tag{5.3}$$

Let us consider the map

$$
\begin{aligned}
u_{nr} : x \in L_q(M_n) \quad & \overset{u_r^1}{\longmapsto} \quad \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes \delta_k \in L_r(M_n^{\otimes_{n^2}}; \ell_q^{n^2}) \\
& \overset{u_r^2}{\longmapsto} \quad \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes j_r(\delta_k) \in L_r(M_n^{\otimes_{n^2}} \otimes \mathcal{M}).
\end{aligned}
$$

The first half is a complete embedding by Lemma 5.8. The second one is not necessarily bounded since j_r is not necessarily completely bounded. However, it is easily seen that the composition of both is an isomorphic embedding. Namely, we may clearly assume that $x \in L_q(M_n)$ is self-adjoint. In that case,

$$\mathcal{A}_x = \big\langle \pi_{tens}^k(x) \big\rangle_{1 \le k \le n^2}''$$

is a commutative von Neumann algebra. Thus

$$
\begin{aligned}
& \|u_{nr}(x)\|_{L_r(M_{n^{n^2}} \otimes \mathcal{M})} \\
= \quad & \Big\| \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes j_r(\delta_k) \Big\|_{L_r(\mathcal{A}_x \otimes \mathcal{M})} \\
\sim \quad & \Big\| \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_{tens}^k(x) \otimes \delta_k \Big\|_{L_r(\mathcal{A}_x; \ell_q^{n^2})} = \|u_r^1(x)\|_{L_r(\mathcal{A}_x; \ell_q^{n^2})} \sim \|x\|_{L_q(M_{n^{n^2}})}.
\end{aligned}
$$

We may take ultraproducts again and consider

$$w_r : L_q(\mathcal{R}_0) \to \prod\nolimits_{n,\mathcal{U}} L_r\big(M_n^{\otimes_{n^2}} \otimes \mathcal{M}\big) = L_r(\widehat{\mathcal{M}}_{\mathcal{U}}).$$

If $\delta = (\delta_n)^\bullet$ with $\delta_n = \mathbf{1}_{M_{n^{n^2}}} \otimes d$ and according to (5.3), we have

$$w_p(x) = \delta^{\frac{1}{p} - \frac{1}{r}} w_r(x) + w_r(x) \delta^{\frac{1}{p} - \frac{1}{r}}. \tag{5.4}$$

We claim that $w_p(x)$ belongs to $L_p(\widehat{\mathcal{M}}_{\mathcal{U},sf})$, the semifinite part of $\widehat{\mathcal{M}}_{\mathcal{U}}$, for any $x \in L_q(\mathcal{R}_0)$. It suffices to check that the limits of Lemma 5.10 are zero. We do it only for $\delta^{1/p-1/r} w_r(x)$, since the term $w_r(x)\delta^{1/p-1/r}$ is estimated similarly. Since we have $\mu_s(ab) \le \mu_{s/2}(a)\mu_{s/2}(b)$ and $\mu_s(\delta_n) = \mu_s(d)$ for all n, we set $\frac{1}{t} = \frac{1}{p} - \frac{1}{r}$ and obtain

$$
\begin{aligned}
\Big(\int_0^{2\delta} \mu_s\big(\delta_n^{\frac{1}{p} - \frac{1}{r}} u_{nr}(x)\big)^p \, ds \Big)^{\frac{1}{p}} \quad & \le \quad 2^{\frac{1}{p}} \Big(\int_0^\delta \mu_s(d^{\frac{1}{p} - \frac{1}{r}})^t \, ds \Big)^{\frac{1}{t}} \Big(\int_0^\delta \mu_s(u_{nr}(x))^r \, ds \Big)^{\frac{1}{r}} \\
& \le \quad 2^{\frac{1}{p}} \Big(\int_0^\delta \mu_s(d^{\frac{1}{p} - \frac{1}{r}})^t \, ds \Big)^{\frac{1}{t}} \|u_{nr}(x)\|_r \\
& \lesssim \quad 2^{\frac{1}{p}} \Big(\int_0^\delta \mu_s(d^{\frac{1}{p} - \frac{1}{r}})^t \, ds \Big)^{\frac{1}{t}} \|x\|_q.
\end{aligned}
$$

Therefore, we deduce that

$$\lim_{\delta \to 0} \lim\nolimits_{n,\mathcal{U}} \Big(\int_0^\delta \mu_s\big(\delta_n^{\frac{1}{p} - \frac{1}{r}} u_{nr}(x)\big)^p \, ds \Big)^{\frac{1}{p}} = 0.$$

The argument to estimate $\int_\gamma^\infty$ is exactly the same. This completes the proof. $\square$

Remark 5.11. Following a suggestion by G. Pisier let us describe what goes 'wrong' when considering a family $v_n : \ell_q^n \to L_p(\mathcal{M}_n)$ of complete embeddings into a semifinite von Neumann algebras $\mathcal{M}_n$. Note that by local reflexivity such cb-isomorphism exists. Let us consider the following conditions.

i) There exits a cb-embedding of ℓ_q into $L_p(\mathcal{M})$ with $\mathcal{M}$ semifinite.

ii) There exists a sequence $(\mathcal{M}_n)_{n\ge 1}$ of semifinite von Neumann algebras and linear maps $v_n : \ell_q^n \to L_p(\mathcal{M}_n)$ such that $\|v_n\|_{cb}\|v_n^{-1}\|_{cb} \le c$ for all $n \ge 1$ and for every $f \in L_q(0,1)$, the sequence $(f_n)^\bullet$ determined by

$$f_n = v_n\Big(n^{1-\frac{1}{q}} \sum_{k=1}^n \big(\int_{\frac{k-1}{n}}^{\frac{k}{n}} f(x)\,dx\big)\delta_k\Big)$$

belongs to the semifinite part $L_p(\widehat{\mathcal{M}}_{\mathcal{U},sf})$ of $\prod_{n,\mathcal{U}} L_p(\mathcal{M}_n,\tau_n)$.

iii) There exists a sequence $(\mathcal{M}_n)_{n\ge 1}$ of semifinite von Neumann algebras and linear maps $w_n : \ell_q^n \to L_p(\mathcal{M}_n)$ such that $\|w_n\|_{cb}\|w_n^{-1}\|_{cb} \le c$ for all $n \ge 1$ and there exists a sequence of densities $d_n \in L_1(\mathcal{M}_n)$ such that $(d_n)^\bullet$ belongs to the semifinite part of $\prod_{n,\mathcal{U}} L_1(\mathcal{M}_n,\tau_n)$ and

$$w_n(x) = d_n^{\frac{1}{p}-\frac{1}{r}} j_{n,r}(x) + j_{n,r}(x)\, d_n^{\frac{1}{p}-\frac{1}{r}},$$

is Rosenthal's factorization [14] for some contractions $j_{n,r} : \ell_q^n \to L_r(\mathcal{M}_n)$.

iv) There exits a cb-embedding of $L_q(\mathcal{R}_0)$ into $L_p(\mathcal{M})$ with $\mathcal{M}$ semifinite.

We will show that the conditions above are equivalent. Hence, even though a family of complete embeddings $v_n : \ell_q^n \to L_p(\mathcal{M}_n)$ with uniformly controlled constants exists, the uniform integrability condition in ii) or iii) is violated.

Proof. The implication iv) $\Rightarrow$ i) is obvious and we have seen in the proof of Corollary B above that i) $\Rightarrow$ iii), just take the same d_n all the time. The proof of ii) $\Rightarrow$ iii) follows similarly. Indeed, by assumption we obtain a continuous map $v : L_q(0,1) \to L_p(\widehat{\mathcal{M}}_{\mathcal{U},sf})$. We apply the noncommtuative Rosenthal theorem [14] and find $v(f) = d^{\frac{1}{p}-\frac{1}{r}} j(f) + j(f)\, d^{\frac{1}{p}-\frac{1}{r}}$ for a bounded map $j : L_q(0,1) \to L_r(\widehat{\mathcal{M}}_{\mathcal{U},sf})$ and some density $d \in L_1(\widehat{\mathcal{M}}_{\mathcal{U},sf})$. By restricting j to step functions on the intervals $[\frac{k-1}{n}, \frac{k}{n}]$, we have found the complete embeddings w_n from condition iii). For the implication iii) $\Rightarrow$ iv) we apply the argument from our proof of Corollary B. Indeed, it suffices to check that for every self-adjoint $x \in \mathcal{R}_0$, the sequence

$$\begin{aligned} u_n(x) &= \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_k(x) \otimes w_{n^2}(\delta_k) \\ &= \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_k(x) \otimes \big(d_{n^2}^{\frac{1}{p}-\frac{1}{r}} j_{n^2,r}(\delta_k) + j_{n^2,r}(\delta_k)\, d_{n^2}^{\frac{1}{p}-\frac{1}{r}} \big) \end{aligned}$$

belongs to the semifinite part of $\prod_{n,\mathcal{U}} L_p(\mathcal{R}_0^{\otimes_{n^2}} \otimes \mathcal{M}_n)$. Referring to the argument after (5.4), it suffices to note that

$$\Big\| \frac{1}{n^{2/q}} \sum_{k=1}^{n^2} \pi_k(x) \otimes j_{n^2,r}(\delta_k) \Big\|_{L_r(\mathcal{R}_0^{\otimes_{n^2}} \otimes \mathcal{M}_n)}$$

$$\leq \quad \|j_{n^2,r}\| \, \frac{1}{n^{2/q}} \Big(\int_{\mathcal{A}_x} \Big[\sum_{k=1}^{n^2} |\pi_k(x)|^q \Big]^{\frac{r}{q}} d\mu_x \Big)^{\frac{1}{r}} \leq \|j_{n^2,r}\| \|x\|_{\mathcal{R}_0}$$

is uniformly bounded in n for every $x \in \mathcal{R}_0$. The proof is complete. □

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