SHARP FOURIER TYPE AND COTYPE
WITH RESPECT TO COMPACT SEMISIMPLE LIE GROUPS

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ABSTRACT. Sharp Fourier type and cotype of Lebesgue spaces and Schatten classes with respect to an arbitrary compact semisimple Lie group are investigated. In the process, a local variant of the Hausdorff-Young inequality on such groups is given.

INTRODUCTION

Let $1 \leq p \leq 2$. An operator space E is said to have Fourier type p with respect to the compact group G if the vector-valued Fourier transform extends to a completely bounded map

$$\mathcal{F}_{G,E} : L_E^p(G) \longrightarrow \mathcal{L}_E^{p'}(\widehat{G})$$

where $p' = p/(p-1)$ is the exponent conjugate to p. That is, a vector-valued Hausdorff-Young inequality of exponent p is satisfied. Similarly, if we replace the operator $\mathcal{F}_{G,E}$ by its inverse, we get the notion of Fourier cotype p' of E with respect to G. Following the notation of [8], we define the constants

$$\mathcal{C}_p^1(E,G) = \|\mathcal{F}_{G,E}\|_{cb(L_E^p(G), \mathcal{L}_E^{p'}(\widehat{G}))} \qquad \text{and} \qquad \mathcal{C}_{p'}^2(E,G) = \|\mathcal{F}_{G,E}^{-1}\|_{cb(\mathcal{L}_E^p(\widehat{G}), L_E^{p'}(G))}.$$

The Fourier type and cotype become stronger conditions on the pair (E,G) as the exponents p and p' approach 2. This gives rise to the notions of sharp Fourier type and cotype exponents. The present paper grew out of the project to investigate the sharp Fourier type and cotype of Lebesgue spaces L^p and Schatten classes S^p and it is a natural continuation of [8]. However, as we shall see below, some other results have appeared in the process which are interesting on their own right.

In section 1 we recall that the natural candidates for the sharp Fourier type and cotype of L^p and S^p –where now $1 \leq p \leq \infty$– are $\min(p,p')$ and $\max(p,p')$ respectively. To justify that this guess is right, one would have to show that for $1 \leq p < q \leq 2$

$$\begin{array}{llllll}
(a) & \mathcal{C}_q^1(L^p(\Omega), G) & = & \mathcal{C}_{q'}^2(L^{p'}(\Omega), G) & = & \infty \\
(b) & \mathcal{C}_q^1(L^{p'}(\Omega), G) & = & \mathcal{C}_{q'}^2(L^p(\Omega), G) & = & \infty
\end{array}$$

with the obvious modifications for the Schatten classes. Section 2 is devoted to make some remarks about (a) and (b). First we show that, to have any chance of getting positive answers to these questions, we have to require the group G not

to be finite and the operator spaces L^p and S^p to be infinite-dimensional. Then, under such assumptions, one can easily get the following inequality

$$\mathcal{C}_q^1(L^p(\Omega), G) \geq \limsup_{n \to \infty} \ \mathcal{C}_q^1(l^p(n), G)$$

and the analog for $L^{p'}(\Omega)$. Therefore the growth of $\mathcal{C}_q^1(l^p(n), G)$ and $\mathcal{C}_q^1(l^{p'}(n), G)$ provides a possible way to obtain (a) and (b). In the last part of section 2 we analyze the vector-valued Lebesgue spaces and Schatten classes.

The growth of $\mathcal{C}_q^1(l^p(n), G)$ is investigated in section 3. To be precise, if G stands for a compact semisimple Lie group and $1 \leq p < q \leq 2$, then there exists a constant $0 \leq \mathcal{K}(G, q) \leq 1$ such that $\mathcal{C}_q^1(l^p(n), G) \geq \mathcal{K}(G, q) \ n^{1/p - 1/q}$ for all $n \geq 1$. If one is able to see that $\mathcal{K}(G, q) > 0$, this result gives (a). Moreover, we would obtain optimal growth since $\mathcal{C}_q^1(l^p(n), G) \leq n^{1/p - 1/q}$ for any compact group. We shall see that

$$\mathcal{K}(G, q) = \inf_{n \geq 1} \sup \left\{ \frac{\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|f\|_{L^q(G)}} : \ f \ \text{central}, \ f \in L^q(G), \ \text{supp}(f) \subset \mathcal{U}_n \right\}$$

where $\{\mathcal{U}_n : n \geq 1\}$ is a basis of neighborhoods of $\mathbf{1}$, the identity element of G. The Hausdorff-Young inequality on compact groups provides $\mathcal{K}(G, q) \leq 1$. The interesting point lies in the inequality $\mathcal{K}(G, q) > 0$ which constitutes a local variant of the Hausdorff-Young inequality on G with exponent q.

Sections 4 and 5 are completely devoted to the proof of this local inequality. In the abelian setting, the particular case $G = \mathbb{T}$ was explored by Andersson in [1]. The basic idea is to consider a function $f : \mathbb{T} \to \mathbb{C}$ as a complex-valued function on $\mathbb{R}$ supported in $[-1/2, 1/2)$. Then, by expressing the norm of $\widehat{f}$ on $L^{q'}(\mathbb{R})$ as a Riemann sum, one obtains

$$\frac{\|\widehat{f}\|_{L^{q'}(\mathbb{R})}}{\|f\|_{L^q(\mathbb{R})}} = \lim_{k \to \infty} \frac{\|\widehat{\varphi}_k\|_{L^{q'}(\mathbb{T})}}{\|\varphi_k\|_{L^q(\mathbb{T})}}$$

where $\varphi_k(t) = k^{1/q} f(kt)$. This gives $\mathcal{K}(\mathbb{T}, q) \geq \mathcal{B}_q$ –where $\mathcal{B}_q = \sqrt{q^{1/q}/q'^{1/q'}}$ stands for the constant of Babenko-Beckner, see [2] and [3]– but in fact the equality holds, as it was proved by Sjölin in [15]. We show here that Andersson's argument, suitably modified, is also valid in the context of compact semisimple Lie groups. In section 4 we summarize the main results of the structure and representation theory of compact semisimple Lie groups that will be used in the process. Then we use these algebraic results to get an expression for the Fourier transform of central functions $f : G \to \mathbb{C}$ in terms of the Fourier transform $\mathcal{F}_{\mathbf{T}}$ on the maximal torus $\mathbf{T}$ of G. This will allow us to work over the maximal torus where we know that Andersson obtained a satisfactory result. However, in the non-commutative setting, the degree d_π of an irreducible representation π does not have to be 1. And we shall see that this becomes a further obstacle to be treated in section 5. There we combine some results –as the Weyl dimension formula– concerning the representation theory of compact semisimple Lie groups with classical harmonic analysis to avoid this difficulty.

On the other hand, if we notice that $\mathcal{C}_q^1(l^{p'}(n), G) = \mathcal{C}_{q'}^2(l^p(n), G)$, we can understand the growth of this constant as the dual problem of the growth of $\mathcal{C}_q^1(l^p(n), G)$ in the sense that we substitute the Fourier transform operator $\mathcal{F}_{G, l^p(n)}$ by its inverse. Therefore, since the dual object is no longer a group –as it is when G is abelian– we do not have a Fourier inversion theorem and we should not expect to

reconstruct the proof given in sections 3, 4 and 5 step by step. At the time of this writing, we are not able to solve this problem and so we pose it as follows:

Problem: Let G be any compact semisimple Lie group and let $1 \leq p < q \leq 2$. Does the estimate $\mathcal{C}_q^1(l^{p'}(n), G) \geq \mathcal{K}(G, q) \ n^{1/p-1/q}$ hold for some positive constant $\mathcal{K}(G, q)$ depending only on G and q?

Finally, we point to a non-commutative notion of Rademacher type for operator spaces, see [9]. We think this notion could be helpful in order to study the growth of $\mathcal{C}_q^1(l^{p'}(n), G)$.

1. Statement of the problem

All throughout this paper some basic notions of operator space theory and non-commutative vector-valued discrete L^p spaces will be assumed. The definitions and results about operator spaces that we are using can be found in the book of Effros and Ruan [5], while for the study of our non-commutative L^p spaces the reader is referred to [12], where Pisier analyzes them in detail. In any case all the analytic preliminaries of this paper are summarized in [8], where we study the Fourier type and cotype of an operator space with respect to a compact group. In order to state the problem we want to solve, we begin by recalling the definitions and the main properties of Fourier type and cotype.

Let G be a compact topological group endowed with its Haar measure μ normalized so that $\mu(G) = 1$ and let $\pi \in \widehat{G}$ be an irreducible unitary representation of G of degree d_π. Here the symbol $\widehat{G}$ stands for the dual object of G. Given an operator space E, it was shown in [8] that –by fixing a basis on the representation space of each $\pi \in \widehat{G}$– the Fourier transform operator $\mathcal{F}_{G,E}$ for functions defined on G and with values on E, has the form $f \in L_E^1(G) \longmapsto \big(\widehat{f}(\pi)\big)_{\pi \in \widehat{G}} \in \mathcal{M}_E(\widehat{G})$, where

$$\widehat{f}(\pi) = \int_G f(g)\pi(g)^\star d\mu(g) \qquad \text{and} \qquad \mathcal{M}_E(\widehat{G}) = \prod_{\pi \in \widehat{G}} M_{d_\pi} \otimes E.$$

Here M_n denotes the space of $n \times n$ complex matrices. Let $1 \leq p < \infty$, if $S_n^p(E)$ stands for the vector-valued Schatten class on $M_n \otimes E$, we define the spaces

- $\mathcal{L}_E^p(\widehat{G}) = \Big\{ A \in \mathcal{M}_E(\widehat{G}) : \ \|A\|_{\mathcal{L}_E^p(\widehat{G})} = \Big(\sum_{\pi \in \widehat{G}} d_\pi \|A^\pi\|_{S_{d_\pi}^p(E)}^p \Big)^{1/p} < \infty \Big\}$

- $\mathcal{L}_E^\infty(\widehat{G}) = \Big\{ A \in \mathcal{M}_E(\widehat{G}) : \ \|A\|_{\mathcal{L}_E^\infty(\widehat{G})} = \sup_{\pi \in \widehat{G}} \|A^\pi\|_{S_{d_\pi}^\infty(E)} < \infty \Big\}.$

We write $\mathcal{L}^p(\widehat{G})$ for the case $E = \mathbb{C}$. Finally, let $1 \leq p \leq 2$, then by the Hausdorff-Young inequality on compact groups –see [8] or Kunze's paper [10]– it is not difficult to check that $\mathcal{F}_{G,E}(L^p(G) \otimes E) \subset \mathcal{L}^{p'}(\widehat{G}) \otimes E$ and $\mathcal{F}_{G,E}^{-1}(\mathcal{L}^p(\widehat{G}) \otimes E) \subset L^{p'}(G) \otimes E$. This motivates the following definitions.

Definition 1.1. Let $1 \leq p \leq 2$ and let p' denote its conjugate exponent. The operator space E has *Fourier type* p with respect to the compact group G if the Fourier transform operator

$$\mathcal{F}_{G,E} : L^p(G) \otimes E \to \mathcal{L}^{p'}(\widehat{G}) \otimes E$$

can be extended to a completely bounded operator from $L_E^p(G)$ into $\mathcal{L}_E^{p'}(\widehat{G})$. In that case $\mathcal{C}_p^1(E, G)$ will stand for its *cb* norm.

Definition 1.2. In the same fashion, the operator space E has *Fourier cotype p'* with respect to the compact group G if the inverse

$$\mathcal{F}_{G,E}^{-1} : \mathcal{L}^p(\widehat{G}) \otimes E \to L^{p'}(G) \otimes E$$

can be extended to a completely bounded operator from $\mathcal{L}_E^p(\widehat{G})$ to $L_E^{p'}(G)$. As before we shall denote its *cb* norm by $\mathcal{C}_{p'}^2(E, G)$.

One of the properties proved in [8] is that every operator space has Fourier type 1 and Fourier cotype ∞ with respect to any compact group. In particular, the complex interpolation method for operator spaces –see Pisier's work [11]– provides the following result.

Lemma 1.3. *Let $1 \le p_1 \le p_2 \le 2$ and assume that E has Fourier type p_2 with respect to G, then E has Fourier type p_1 with respect to G. Similarly, Fourier cotype p_2' of E with respect to G implies Fourier cotype p_1' of E with respect to G.*

Therefore, the Fourier type and cotype become stronger conditions on the pair (E, G) as the exponent p –and consequently its conjugate p'– tends to 2. So lemma 1.3 gives rise to the following definition.

Definition 1.4. The *sharp Fourier type and cotype exponents* of an operator space E with respect to the compact group G are defined respectively by

$$p_1(E, G) \quad = \quad \sup\{p \le 2 : E \text{ has Fourier type } p \text{ with respect to } G\}$$
$$p_2(E, G) \quad = \quad \inf\{p' \ge 2 : E \text{ has Fourier cotype } p' \text{ with respect to } G\}.$$

If E has Fourier type $p_1(E, G)$ with respect to G we say that E has *sharp Fourier type $p_1(E, G)$*. The *sharp Fourier cotype* of E is defined analogously.

In order to simplify the statement of the problem we shall need the following lemma –see [8]– which analyzes the Fourier type and cotype of the dual $E^\star$ of an operator space E with respect to a compact group G.

Lemma 1.5. *Let $1 \le p \le 2$ and let p' be the conjugate exponent of p. Then we have the equalities $\mathcal{C}_p^1(E^\star, G) = \mathcal{C}_{p'}^2(E, G)$ and $\mathcal{C}_{p'}^2(E^\star, G) = \mathcal{C}_p^1(E, G)$.*

The problem we want to investigate in this paper is how to find out the sharp Fourier type and cotype of Lebesgue spaces and Schatten classes. Concerning these topics we present here a result given in [8] from which we start out. In what follows $(\Omega, \mathcal{A}, \nu)$ will denote a σ-finite or regular measure space and $S_{\mathbb{N}}^p$ the classical Schatten class over the space of compact operators on l^2.

Theorem 1.6. *Let $1 \le p \le \infty$, then the spaces $L^p(\Omega)$, S_n^p and $S_{\mathbb{N}}^p$ have Fourier type $\min(p, p')$ and Fourier cotype $\max(p, p')$. In fact, the vector-valued Fourier transform –or its inverse– is a complete contraction in each of the cases considered.*

Therefore, if we consider two exponents p and q such that $1 \le p < q \le 2$, we would like to find out conditions on G and Ω under which

(a) $\quad \mathcal{C}_q^1(L^p(\Omega), G) \quad = \quad \mathcal{C}_{q'}^2(L^{p'}(\Omega), G) \quad = \quad \infty$

(b) $\quad \mathcal{C}_q^1(L^{p'}(\Omega), G) \quad = \quad \mathcal{C}_{q'}^2(L^p(\Omega), G) \quad = \quad \infty$

with the obvious modifications for the Schatten classes.

2. Some remarks about the problem

In this section we shall point out some remarks about the problem we have just stated. We begin by showing some necessary conditions that should hold to obtain a positive answer to our question. Second we wonder about sufficient conditions that we shall work with along the rest of this paper. Finally we study what happens if we consider vector-valued L^p spaces –or Bochner-Lebesgue spaces– and vector-valued Schatten classes.

2.1. **Necessary conditions.** The first necessary condition we are talking about is on the compact group G. We have to exclude **finite groups** from our treatment since, as we shall see immediately, every operator space E has sharp Fourier type and cotype 2 with respect to any finite group. Anyway the next result is a bit more accurate.

Proposition 2.1. *Let G be a finite group, then every operator space E satisfies the estimates $\mathcal{C}_p^1(E,G)$, $\mathcal{C}_{p'}^2(E,G) \leq |G|^{1/p'}$ for $1 \leq p \leq 2$.*

Proof. Let us assume that $\mathcal{C}_2^1(E,G) \leq |G|^{1/2}$ for every operator space E, then we have $\mathcal{C}_2^2(E,G) = \mathcal{C}_2^1(E^\star,G) \leq |G|^{1/2}$ by duality. The desired estimates are then obtained by complex interpolation from the equalities $\mathcal{C}_1^1(E,G) = \mathcal{C}_\infty^2(E,G) = 1$ –proved in [8]– and the case $p = 2$. Therefore we focus our attention on the case $p = 2$. It suffices to check that for all $m \geq 1$ and any family of functions $\{f_{ij} : G \to E\}_{1 \leq i,j \leq m}$

$$\Big(\sum_{\pi \in \widehat{G}} d_\pi \big\| \big(\, \widehat{f_{ij}}(\pi) \, \big) \big\|_{S_{d_\pi m}^2(E)}^2 \Big)^{1/2} \leq |G|^{1/2} \, \big\| \big(\, f_{ij} \, \big) \big\|_{S_m^2(L_E^2(G))}.$$

But if $G = \{g_1, g_2, \ldots g_n\}$, then

$$
\begin{aligned}
\big\| \big(\, \widehat{f_{ij}}(\pi) \, \big) \big\|_{S_{d_\pi m}^2(E)} &= \big\| \big(\, \frac{1}{n} \sum_{k=1}^n f_{ij}(g_k) \pi(g_k)^\star \, \big) \big\|_{S_{d_\pi m}^2(E)} \\
&\leq \frac{1}{n} \sum_{k=1}^n \| \pi(g_k)^\star \|_{S_{d_\pi}^2} \big\| \big(\, f_{ij}(g_k) \, \big) \big\|_{S_m^2(E)} \\
&\leq d_\pi^{1/2} \big\| \big(\, f_{ij} \, \big) \big\|_{S_m^2(L_E^2(G))}
\end{aligned}
$$

Therefore we obtain

$$\big\| \big(\, \widehat{f_{ij}} \, \big) \big\|_{S_m^2(\mathcal{L}_E^2(\widehat{G}))} \leq \sqrt{\sum_{\pi \in \widehat{G}} d_\pi^2} \, \big\| \big(\, f_{ij} \, \big) \big\|_{S_m^2(L_E^2(G))}$$

and, since $\sum_{\pi \in \widehat{G}} d_\pi^2 = |G|$ by the Peter-Weyl theorem, we are done. $\square$

Next we show that we can not work with measure spaces $(\Omega, \mathcal{A}, \nu)$ which are a **union of finitely many ν-atoms**. Before that we need to define the *cb* distance between two operator spaces. It is due to Pisier and it constitutes the analog of the Banach-Mazur distance between two Banach spaces in the context of operator space theory. Given two operator spaces E_1 and E_2, we define their *cb* distance by the relation $d_{cb}(E_1, E_2) = \inf\{\|u\|_{cb(E_1,E_2)} \|u^{-1}\|_{cb(E_2,E_1)}\}$ where the infimum runs over all complete isomorphisms $u : E_1 \to E_2$. The following result –also extracted

from [8]– relates the Fourier type and cotype of two operator spaces E_1 and E_2 with their cb distance.

Lemma 2.2. *Let $1 \le p \le 2$ and let E_1, E_2 be operator spaces, then we have the estimates $\mathcal{C}_p^1(E_2, G) \le d_{cb}(E_1, E_2) \; \mathcal{C}_p^1(E_1, G)$ and $\mathcal{C}_{p'}^2(E_2, G) \le d_{cb}(E_1, E_2) \; \mathcal{C}_{p'}^2(E_1, G)$.*

Proposition 2.3. *Let $1 \le p < q \le 2$ and assume that $(\Omega, \mathcal{A}, \nu)$ is a union of finitely many ν-atoms. Then every compact group G satisfies the following estimates*

$$\begin{array}{rcccl}
\mathcal{C}_q^1(L^p(\Omega), G) & = & \mathcal{C}_{q'}^2(L^{p'}(\Omega), G) & \le & \nu(\Omega)^{1/p - 1/q} \\
\mathcal{C}_q^1(L^{p'}(\Omega), G) & = & \mathcal{C}_{q'}^2(L^p(\Omega), G) & \le & \nu(\Omega)^{1/q' - 1/p'}.
\end{array}$$

In fact, since $1/p - 1/q = 1/q' - 1/p'$, we have the same bound for both cb norms.

Proof. Applying lemma 2.2 and the last part of theorem 1.6 we get the estimates $\mathcal{C}_q^1(L^p(\Omega), G) \le d_{cb}(L^p(\Omega), L^q(\Omega))$ and $\mathcal{C}_q^1(L^{p'}(\Omega), G) \le d_{cb}(L^{p'}(\Omega), L^{q'}(\Omega))$. On the other hand it is straightforward to check that, for $1 \le p_1 < p_2 \le \infty$ and such a measure space $(\Omega, \mathcal{A}, \nu)$, we have $d_{cb}(L^{p_1}(\Omega), L^{p_2}(\Omega)) \le \nu(\Omega)^{1/p_1 - 1/p_2}$. $\square$

In other words, we do not allow **finite dimensional** Lebesgue spaces. Since the cb distance between two Schatten classes of the same finite dimension is also finite, the arguments used in proposition 2.3 –theorem 1.6 and lemma 2.2– are also valid to show that the unique Schatten classes with any possibility to make theorem 1.6 sharp are those of infinite dimension.

2.2. **Sufficient conditions.** The Fourier type and cotype of the subspaces of a given operator space E are bounded above by the respective type and cotype of E. The proof of this result is straightforward, see [8].

Lemma 2.4. *Let $1 \le p \le 2$ and let F be a closed subspace of E, then we have the estimates $\mathcal{C}_p^1(F, G) \le \mathcal{C}_p^1(E, G)$ and $\mathcal{C}_{p'}^2(F, G) \le \mathcal{C}_{p'}^2(E, G)$.*

After the conditions above, we shall work in the sequel with infinite compact groups and infinite dimensional Lebesgue spaces and Schatten classes. Since the measure space $(\Omega, \mathcal{A}, \nu)$ is no longer a union of finitely many ν-atoms, we obtain that the n-dimensional space $l^p(n)$ is a closed subspace of $L^p(\Omega)$ for all $n \ge 1$ and any $1 \le p \le \infty$. Moreover, recalling that the subspace of diagonal matrices of S_n^p is completely isomorphic to $l^p(n)$, we deduce that the same happens for the Schatten classes $S_\mathbb{N}^p$. Hence sharpness of theorem 1.6 will be guaranteed if, for $1 \le p < q \le 2$, we have

$$\begin{array}{llclccl}
(a') & \mathcal{C}_q^1(l^p(n), G) & = & \mathcal{C}_{q'}^2(l^{p'}(n), G) & \longrightarrow & \infty & \text{as } n \to \infty \\
(b') & \mathcal{C}_q^1(l^{p'}(n), G) & = & \mathcal{C}_{q'}^2(l^p(n), G) & \longrightarrow & \infty & \text{as } n \to \infty.
\end{array}$$

Therefore our aim from now on will be the study of the growth of the constants $\mathcal{C}_q^1(l^p(n), G)$ and $\mathcal{C}_q^1(l^{p'}(n), G)$. The first remark about these constants that we can already make is that both have a common upper bound

$$\mathcal{C}_q^1(l^p(n), G), \; \mathcal{C}_q^1(l^{p'}(n), G) \le n^{1/p - 1/q}.$$

This is an obvious consequence of proposition 2.3.

2.3. **Vector-valued spaces.** Theorem 1.6 was also studied in [8] for vector-valued spaces, here is the statement of the result obtained.

Theorem 2.5. *Let $1 \leq p \leq \infty$ and let E be an operator space having Fourier type $\min(p, p')$ –respectively Fourier cotype $\max(p, p')$– with respect to G. Then the spaces $L_E^p(\Omega)$, $S_n^p(E)$ and $S_{\mathbb{N}}^p(E)$ have Fourier type $\min(p, p')$ –respectively Fourier cotype $\max(p, p')$– with respect to G.*

Let $1 \leq p \leq \infty$ and $\min(p, p') < q \leq 2$. Let E be as in theorem 2.5, then lemma 2.4 gives the following estimates

$$
\begin{array}{llllll}
\mathcal{C}_q^1(L_E^p(\Omega), G) & \geq & \mathcal{C}_q^1(L^p(\Omega), G) & \qquad \mathcal{C}_q^1(L_E^p(\Omega), G) & \geq & \mathcal{C}_q^1(E, G) \\
\mathcal{C}_{q'}^2(L_E^p(\Omega), G) & \geq & \mathcal{C}_{q'}^2(L^p(\Omega), G) & \qquad \mathcal{C}_{q'}^2(L_E^p(\Omega), G) & \geq & \mathcal{C}_{q'}^2(E, G)
\end{array}
$$

with the obvious modifications for the Schatten classes. Hence we have shown that sharp Fourier type or cotype of $L^p(\Omega)$ –respectively $S_{\mathbb{N}}^p$– provides sharp Fourier type or cotype of $L_E^p(\Omega)$ –respectively $S_{\mathbb{N}}^p(E)$– with respect to G. Also the same conclusion is obtained assuming sharp Fourier type or cotype of E. In particular, the sufficient condition given above also works for vector-valued spaces. Therefore we focus our attention on the growth of the constants $\mathcal{C}_q^1(l^p(n), G)$ and $\mathcal{C}_q^1(l^{p'}(n), G)$.

3. On the growth of $\mathcal{C}_q^1(l^p(n), G)$.

We shall assume in what follows that G is a **compact semisimple Lie group**. Semisimplicity is an essential assumption in the arguments we shall be using. Anyway, for the moment, the only property of such groups that we shall apply is the existence of a maximal torus $\mathbf{T}$ in G. The following result gives, in particular, part (a) in section 1 –with the obvious modifications for Schatten classes– whenever we work with infinite dimensional operator spaces and compact semisimple Lie groups.

Theorem 3.1. *Let $1 \leq p < q \leq 2$ and let G be a compact semisimple Lie group. Then there exists a constant $0 < \mathcal{K}(G, q) \leq 1$ depending on G and q such that for all $n \geq 1$*

$$
\mathcal{K}(G, q) \ n^{1/p - 1/q} \leq \mathcal{C}_q^1(l^p(n), G) \leq n^{1/p - 1/q}.
$$

In particular we observe that the growth of $\mathcal{C}_q^1(l^p(n), G)$ is optimal for compact semisimple Lie groups. The proof of this result starts out applying the existence of a maximal torus $\mathbf{T}$ to consider a countable family $\{g_k : k \geq 1\}$ of pairwise commuting elements of G, just take $g_k \in \mathbf{T}$. For every $n \geq 1$ we take $\mathcal{U}_n$ to be a neighborhood of $\mathbf{1}$ –the identity element of G– satisfying

$$
g_j^{-1} \mathcal{U}_n \cap g_k^{-1} \mathcal{U}_n = \emptyset \qquad \text{for} \ \ 1 \leq j, k \leq n \ \text{ and } \ j \neq k.
$$

We recall here that we can always consider a central function f_n supported in $\mathcal{U}_n$ and belonging to $L^q(G)$, for example take $\mathcal{U}_n$ to be invariant under conjugations –see lemma (5.24) of [6]– and $f_n = 1_{\mathcal{U}_n}$ where $1_{\mathcal{U}}$ stands for the characteristic function of $\mathcal{U}$. Henceforth f_n will be a central function in $L^q(G)$ supported in $\mathcal{U}_n$, to be fixed later. Then we define the function $\Phi_n : G \to \mathbb{C}^n$ by $\Phi_n(g) = (f_n(g_1 g), f_n(g_2 g), \dots f_n(g_n g))$. We obviously have the estimate

$$
\mathcal{C}_q^1(l^p(n), G) \geq \frac{\|\widehat{\Phi}_n\|_{\mathcal{L}_{l^p(n)}^{q'}(\widehat{G})}}{\|\Phi_n\|_{L_{l^p(n)}^q(G)}}.
$$

So it suffices to prove that this quotient is bounded below by $\mathcal{K}(G, q) \ n^{1/p - 1/q}$. The following lemma will be very helpful for that purpose.

Lemma 3.2. *Let $1 \leq p_1, p_2 \leq \infty$, $\pi \in \widehat{G}$ and $n \geq 1$. Consider the matrix-valued vector $A_{\pi,n} = (\pi(g_1), \pi(g_2), \ldots \pi(g_n))$, then*

$$\|A_{\pi,n}\|_{l^{p_1}_{S^{p_2}_{d_\pi}}(n)} = \|A_{\pi,n}\|_{S^{p_2}_{d_\pi}(l^{p_1}(n))} = n^{1/p_1} d_\pi^{1/p_2}.$$

Proof. Since $g_1, g_2, \ldots g_n$ are pairwise commuting, there exists a basis of $\mathbb{C}^{d_\pi}$ of common eigenvectors of $\pi(g_1), \pi(g_2) \ldots \pi(g_n)$. Therefore, in that basis, all these matrices are diagonal

$$\pi(g_k) = \begin{pmatrix} \theta_1^k & & \\ & \ddots & \\ & & \theta_{d_\pi}^k \end{pmatrix}.$$

Moreover, $|\theta_j^k| = 1$ for $1 \leq j \leq d_\pi$ because of the unitarity of $\pi(g_k)$. Hence, applying the complete isometry –see corollary (1.3) of [12]– between $l_E^{p_2}(d_\pi)$ and the subspace of diagonal matrices of $S^{p_2}_{d_\pi}(E)$, we easily obtain the desired equality. $\square$

(i) **The value of $\|\widehat{\Phi}_n\|_{\mathcal{L}^{q'}_{l^p(n)}(\widehat{G})}$.** We begin by recalling that, since f_n is central,

$$\widehat{f_n}(\pi) = \frac{1}{d_\pi} \int_G f_n(g)\overline{\chi_\pi(g)}d\mu(g) \ 1_{d_\pi} = \gamma_{\pi,n} 1_{d_\pi}$$

by Schur's lemma. Here χ_π is the irreducible character associated to π and 1_m denotes the identity matrix of order $m \times m$. On the other hand $f_n(g_k\cdot)$ is the translation by g_k of f_n, therefore

$$\widehat{\Phi}_n(\pi) = \frac{1}{d_\pi} \int_G f_n(g)\overline{\chi_\pi(g)}d\mu(g) \ (\pi(g_1), \pi(g_2), \ldots \pi(g_n)) = \gamma_{\pi,n} A_{\pi,n}.$$

So we get, by lemma 3.2, the following equality

$$\begin{aligned}
\|\widehat{\Phi}_n\|_{\mathcal{L}^{q'}_{l^p(n)}(\widehat{G})} &= \Big(\sum_{\pi \in \widehat{G}} d_\pi |\gamma_{\pi,n}|^{q'} \|A_{\pi,n}\|^{q'}_{S^{q'}_{d_\pi}(l^p(n))} \Big)^{1/q'} \\
&= n^{1/p} \Big(\sum_{\pi \in \widehat{G}} d_\pi^2 |\gamma_{\pi,n}|^{q'} \Big)^{1/q'} = n^{1/p} \ \|\widehat{f_n}\|_{\mathcal{L}^{q'}(\widehat{G})}.
\end{aligned}$$

(ii) **The value of $\|\Phi_n\|_{L^p_{l^p(n)}(G)}$.** We have

$$\begin{aligned}
\|\Phi_n\|_{L^q_{l^p(n)}(G)} &= \Big(\int_G \Big(\sum_{k=1}^n |f_n(g_k g)|^p \Big)^{q/p} d\mu(g) \Big)^{1/q} \\
&= \Big(\sum_{k=1}^n \|f_n(g_k\cdot)\|^q_{L^q(G)} \Big)^{1/q} = n^{1/q} \ \|f_n\|_{L^q(G)}
\end{aligned}$$

since the sets $\{g_k^{-1}\mathcal{U}_n : 1 \leq k \leq n\}$ are pairwise disjoint.

In summary, we have obtained that $\mathcal{C}^1_q(l^p(n), G) \geq \mathcal{K}(G, q, n) \ n^{1/p-1/q}$ where the constant $\mathcal{K}(G, q, n)$ is given by

$$\mathcal{K}(G, q, n) = \frac{\|\widehat{f_n}\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|f_n\|_{L^q(G)}}.$$

If we define $\mathcal{K}(G, q) = \inf_{n \geq 1} \mathcal{K}(G, q, n)$, it is obvious that $\mathcal{K}(G, q) \leq 1$ by the Hausdorff-Young inequality on compact groups. Thus it remains to check that $\mathcal{K}(G, q) > 0$. For that aim, since we have not fixed f_n yet, we need to see that

$$\inf_{n \geq 1} \sup \left\{ \frac{\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|f\|_{L^q(G)}} : \quad f \text{ central}, \quad f \in L^q(G), \quad \text{supp}(f) \subset \mathcal{U}_n \right\} > 0.$$

We shall prove this fact in section 5 where we study the supremum of the Hausdorff-Young quotient for central functions supported in arbitrary small sets. As we shall see immediately, semisimplicity of G will be essential in our proof.

4. A SIMPLE EXPRESSION FOR THE FOURIER TRANSFORM OF CENTRAL FUNCTIONS

In this section we apply some basic results concerning the structure and representation theory of compact semisimple Lie groups to provide a simple expression for the Fourier transform of central functions defined on such groups. These algebraic preliminaries can be found in Simon's book [14] or alternatively in [7], but we summarize here the main topics. Let G be a compact semisimple Lie group and let $\mathfrak{g}$ be its Lie algebra. In what follows we choose once and for all an explicit maximal torus $\mathbf{T}$ in G while $\mathfrak{h}$ will stand for its Lie algebra. That is, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{g}$. The rank of G will be denoted by r, in particular $\mathbf{T} \simeq \mathbb{T}^r$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with its natural group structure. Also, as it is customary, we consider the complexification $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ –with complex conjugates taken so that $\mathfrak{g}_{\mathbb{R}} = \{Z \in \mathfrak{g}_{\mathbb{C}} : Z = \overline{Z}\} = i\mathfrak{g}$ and similarly $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$– with the complex inner product $\langle \ , \ \rangle$ induced by the Killing form. We also recall that the Weyl group $\mathcal{W}_G$ associated to G can be seen as a set of $r \times r$ unitary matrices W –isometries on $\mathfrak{h}_{\mathbb{R}}$– with integer entries and $\det W = \pm 1$. In particular the set $\mathcal{W}_G^\star = \{W^t : W \in \mathcal{W}_G\}$ becomes a set of isometries on $\mathfrak{h}_{\mathbb{R}}^\star$. The symbol $\mathcal{R}$ will stand for the set of roots while, if we take $H_0 \in \mathfrak{h}_{\mathbb{R}}$ such that $\alpha(H_0) \neq 0$ for any root α, the symbol $\mathcal{R}^+ = \{\alpha \in \mathcal{R} : \alpha(H_0) > 0\}$ denotes the set of positive roots. Finally we shall write Λ_{W} and Λ_{DW} for the weight lattice and the set of dominant weights respectively.

Once we have fixed some notation, let us consider a central function $f : G \to \mathbb{C}$ and a dominant weight $\lambda \in \Lambda_{\mathrm{DW}}$. By the dominant weight theorem there exists a unique $\pi_\lambda \in \widehat{G}$ associated to λ and, since f is central, we can write by Schur's lemma

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \int_G f(g) \overline{\chi_\lambda(g)} d\mu(g) \ 1_{d_\lambda}$$

where d_λ is the degree of π_λ, χ_λ is the character of π_λ and 1_m denotes the $m \times m$ identity matrix. We now recall the definition of the functions A_β appearing in the Weyl character formula. Given $\beta \in \mathfrak{h}_{\mathbb{R}}^\star$, we define the functions $\exp_\beta : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C}$ and $A_\beta : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C}$ by the relations

$$\begin{aligned} \exp_\beta(H) &= e^{2\pi i \langle \beta, H \rangle} \\ A_\beta(H) &= \sum_{W \in \mathcal{W}_G} \det W \ \exp_\beta\big(W(H)\big). \end{aligned}$$

The maximal torus $\mathbf{T}$ is isomorphic via the exponential mapping to the quotient space $\mathfrak{h}_{\mathbb{R}}/\mathrm{L}_{\mathrm{W}}$, where L_{W} is the set of those $H \in \mathfrak{h}_{\mathbb{R}}$ satisfying $\exp(2\pi i H) = \mathbf{1}$. That is, L_{W} is the dual lattice of Λ_{W}. Therefore, the functions $\exp_\beta$ and A_β are

well-defined functions on $\mathbf{T}$ if and only if $\beta \in \Lambda_W$. As it is well known, the integral form

$$\delta = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} \alpha$$

is not necessarily a weight and so the functions $\exp_\delta$ and A_δ could be not well-defined on $\mathbf{T}$. To avoid this difficulty we assume for the moment that G is **simply connected**, this condition on G assures that $\delta \in \Lambda_W$. Hence, applying consecutively the Weyl integration formula and the Weyl character formula, we get

$$
\begin{aligned}
\widehat{f}(\pi_\lambda) &= \frac{1}{d_\lambda |\mathcal{W}_G|} \int_{\mathbf{T}} f(t)\overline{\chi_\lambda(t)} \ |A_\delta(t)|^2 dm(t) \ 1_{d_\lambda} \\
&= \frac{1}{d_\lambda |\mathcal{W}_G|} \int_{\mathbf{T}} f(t) A_\delta(t) \ \overline{A_{\lambda+\delta}(t)} dm(t) \ 1_{d_\lambda}
\end{aligned}
$$

where m denotes the Haar measure on $\mathbf{T}$ normalized so that $m(\mathbf{T}) = 1$. Now, if we write $A_{\lambda+\delta}$ as a linear combination of exponentials, we obtain

$$
\begin{aligned}
\widehat{f}(\pi_\lambda) &= \frac{1}{d_\lambda |\mathcal{W}_G|} \sum_{W \in \mathcal{W}_G} \det W \int_{\mathbf{T}} f(t) A_\delta(t) \ \exp_{-(\lambda+\delta)}(W(t)) dm(t) \ 1_{d_\lambda} \\
&= \frac{1}{d_\lambda} \int_{\mathbf{T}} f(t) A_\delta(t) \ \exp_{-(\lambda+\delta)}(t) dm(t) \ 1_{d_\lambda}
\end{aligned}
$$

since $A_\delta(W(t)) = \det W A_\delta(t)$ and $f(W(t)) = f(t)$. We recall that, taking coordinates with respect to the basis $\{\omega_1, \omega_2, \ldots \omega_r\}$ of fundamental weights, any weight $\lambda \in \Lambda_W$ has integer coordinates. Therefore, we can understand the last expression as the Fourier transform of $f A_\delta$ on the maximal torus $\mathbf{T}$ evaluated at $\lambda + \delta$. Hence we have

$$\tag{1} \widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \ \mathcal{F}_{\mathbf{T}}(f A_\delta)(\lambda + \delta) \ 1_{d_\lambda}$$

for $f : G \to \mathbb{C}$ central and G any compact semisimple simply connected Lie group. When G is not simply connected, a more careful approach is needed. We have $W^t(\delta) \pm \delta \in \Lambda_W$ for all $W \in \mathcal{W}_G$. In particular we note that

$$\exp_{\pm\delta} A_{\lambda+\delta} = \sum_{W \in \mathcal{W}_G} \det W \exp_{W^t(\lambda+\delta)\pm\delta}$$

is a well-defined function on $\mathbf{T}$ for all $\lambda \in \Lambda_{DW}$. This remark allows us to write $\overline{\chi_\lambda} \ |A_\delta|^2 = (\exp_\delta \overline{A_{\lambda+\delta}}) \ (\exp_{-\delta} A_\delta)$ as a well-defined function on $\mathbf{T}$. Henceforth, applying again Schur's lemma, the Weyl integration formula and the Weyl character formula, we get

$$
\begin{aligned}
\widehat{f}(\pi_\lambda) &= \frac{1}{d_\lambda |\mathcal{W}_G|} \sum_{W \in \mathcal{W}_G} \det W \int_{\mathbf{T}} f(t)(\exp_{-\delta} A_\delta)(t) \ \exp_{\delta-W^t(\lambda+\delta)}(t) dm(t) \ 1_{d_\lambda} \\
&= \frac{1}{d_\lambda} \int_{\mathbf{T}} f(t)(\exp_{-\delta} A_\delta)(t) \ \exp_{-\lambda}(t) dm(t) \ 1_{d_\lambda}
\end{aligned}
$$

where the last equality follows from the change of variable $t \mapsto W^t(t)$. That is, we have shown that

$$\tag{2} \widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \ \mathcal{F}_{\mathbf{T}}(f B_\delta)(\lambda) \ 1_{d_\lambda}$$

where $B_\delta = \exp_{-\delta} A_\delta$. This expression is now valid for any compact semisimple Lie group and it coincides with (1) for simply connected ones.

5. A local variant of the Hausdorff-Young inequality on compact semisimple Lie groups

As we mentioned in the introduction, this section is devoted to the proof of a local variant of the Hausdorff-Young inequality on compact semisimple Lie groups. We recall that this result provides the relation $\mathcal{K}(G, q) > 0$ for $1 \leq q \leq 2$, that we needed in section 3.

Theorem 5.1. *Let $1 \leq q \leq 2$ and let G be a compact semisimple Lie group. Then there exists a constant $0 < \mathcal{K}(G, q) \leq 1$ such that, for any open set $\mathcal{U} \subset G$, we have*

$$\sup \left\{ \frac{\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|f\|_{L^q(G)}} : \quad f \quad central, \quad f \in L^q(G), \quad \mathrm{supp}(f) \subset \mathcal{U} \right\} \geq \mathcal{K}(G, q).$$

Since the norms of $\widehat{f}$ and f –on $\mathcal{L}^{q'}(\widehat{G})$ and $L^q(G)$ respectively– do not change under translations of f, we can assume without loss of generality that $\mathcal{U}$ is a neighborhood of $\mathbf{1}$. Before the proof of theorem 5.1 we need some auxiliary results. Let us assume that G is simply connected and let $f : G \to \mathbb{C}$ be a central function. A quick look at relation (1) given above, allows us to write

$$(3) \qquad \widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \ \det W \ \mathcal{F}_{\mathbf{T}}(fA_\delta)(W^t(\lambda + \delta)) \ 1_{d_\lambda}$$

for all $W \in \mathcal{W}_G$. On the other hand, let us denote by P_α the hyperplane of $\mathfrak{h}_{\mathbb{R}}^\star$ orthogonal to α with respect to the complex inner product given by the Killing form. The infinitesimal Cartan-Stiefel diagram is then given by the expression

$$\mathrm{P} = \bigcup_{\alpha \in \mathcal{R}} P_\alpha.$$

Lemma 5.2. *Let G be a compact semisimple simply connected Lie group. Then we have $\{W^t(\lambda + \delta) : \ W \in \mathcal{W}_G, \ \lambda \in \Lambda_{\mathrm{DW}}\} = \Lambda_{\mathrm{W}} \setminus \mathrm{P}$. Moreover, the mapping $(W, \lambda) \in \mathcal{W}_G \times \Lambda_{\mathrm{DW}} \mapsto W^t(\lambda + \delta) \in \Lambda_{\mathrm{W}} \setminus \mathrm{P}$ is injective.*

Proof. Since G is simply connected we have that $\{\lambda + \delta : \lambda \in \Lambda_{\mathrm{DW}}\} = \Lambda_{\mathrm{W}} \cap \mathbf{C}^{\mathrm{int}}$. Here $\mathbf{C}$ stands for the fundamental Weyl chamber and $\mathbf{C}^{\mathrm{int}}$ for its interior. Now, since P and Λ_{W} are invariant under the action of $\mathcal{W}_G^\star$ and for any Weyl chamber C there exists a unique $W \in \mathcal{W}_G$ such that $W^t(\mathbf{C}) = \mathrm{C}$, we obtain the desired equality. Finally, the injectivity follows from the uniqueness mentioned above. $\square$

Proposition 5.3. *Let G be a compact semisimple simply connected Lie group and let $f : G \to \mathbb{C}$ be a central function. Then there exists a constant $\mathcal{A}(G, q)$ depending on G and q, such that*

$$\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})} = \mathcal{A}(G, q) \ \Big[\sum_{\lambda \in \Lambda_{\mathrm{W}} \setminus \mathrm{P}} \frac{|\mathcal{F}_{\mathbf{T}}(fA_\delta)(\lambda)|^{q'}}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda \rangle|^{q'-2}} \Big]^{1/q'}.$$

Proof. Since f is central and G is simply connected, we can apply expression (3) to obtain

$$\begin{aligned} \|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})} &= \Big[\sum_{\lambda \in \Lambda_{\mathrm{DW}}} d_\lambda \|\widehat{f}(\pi_\lambda)\|_{S_{d_\lambda}^{q'}}^{q'} \Big]^{1/q'} \\ &= \Big[\frac{1}{|\mathcal{W}_G|} \sum_{W \in \mathcal{W}_G} \sum_{\lambda \in \Lambda_{\mathrm{DW}}} d_\lambda \Big| \frac{1}{d_\lambda} \mathcal{F}_{\mathbf{T}}(fA_\delta)(W^t(\lambda + \delta)) \Big|^{q'} \|1_{d_\lambda}\|_{S_{d_\lambda}^{q'}}^{q'} \Big]^{1/q'} \end{aligned}$$

Moreover, the Weyl dimension formula for d_λ gives

$$\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})} = \Big[\frac{1}{|\mathcal{W}_G|} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^{q'-2} \sum_{W \in \mathcal{W}_G} \sum_{\lambda \in \Lambda_{\mathrm{DW}}} \frac{|\mathcal{F}_{\mathbf{T}}(fA_\delta)(W^t(\lambda + \delta))|^{q'}}{\prod\limits_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda + \delta \rangle|^{q'-2}}\Big]^{1/q'}.$$

Finally we observe that

$$\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda + \delta \rangle| = \prod_{\alpha \in \mathcal{R}} |\langle W(\alpha), \lambda + \delta \rangle|^{1/2} = \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, W^t(\lambda + \delta) \rangle|$$

since any $W \in \mathcal{W}_G$ is a permutation of the set of roots. Therefore, by lemma 5.2 we have

$$\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})} = \Big[\frac{1}{|\mathcal{W}_G|} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^{q'-2} \sum_{\lambda \in \Lambda_{\mathrm{W}} \backslash \mathrm{P}} \frac{|\mathcal{F}_{\mathbf{T}}(fA_\delta)(\lambda)|^{q'}}{\prod\limits_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda \rangle|^{q'-2}}\Big]^{1/q'}.$$

The proof is completed just by taking $\mathcal{A}(G,q) = \Big[\frac{1}{|\mathcal{W}_G|} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^{q'-2}\Big]^{1/q'}.$ □

We are now ready to give the proof of theorem 5.1 for simply connected groups. Let $\{H_1, H_2, \ldots H_r\}$ be the predual basis of the fundamental weights, any element of L_{W} can be written as a linear combination of $H_1, H_2, \ldots H_r$ with integer coefficients. Then, since $\mathbf{T} \simeq \mathfrak{h}_{\mathbb{R}}/L_{\mathrm{W}}$, we can regard $\mathbf{T}$ as the subset of $\mathfrak{h}_{\mathbb{R}}$

$$\mathfrak{T} = \Big\{ \sum_{k=1}^{r} x_k H_k : \ -1/2 \le x_k < 1/2 \Big\}.$$

On the other hand, let us fix a bounded central function $f_0 : G \to \mathbb{C}$, then f_0 can be understood as a function on $\mathbf{T}$ invariant under the action of $\mathcal{W}_G$. Now, since the Weyl group is generated by a set of reflections in $\mathfrak{h}_{\mathbb{R}}$, f_0 can be regarded as a complex-valued function on $\mathfrak{h}_{\mathbb{R}}$, supported in $\mathfrak{T}$ and symmetric under such reflections. Let us recall that $\{\omega_1, \omega_2, \ldots \omega_r\}$ stands for the basis of fundamental weights. Let $\tau = 1 - 2/q'$, the way we have interpreted the function f_0 allows us to define the function

$$\widehat{I_\tau(f_0 A_\delta)} : \mathfrak{h}_{\mathbb{R}}^\star \longrightarrow \mathbb{C} \qquad \text{as}$$

$$\widehat{I_\tau(f_0 A_\delta)}(\xi) = \frac{1}{\prod\limits_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi \rangle|^\tau} \ \mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)(\xi) \qquad \text{where} \quad \xi = \sum_{k=1}^{r} \xi_k \omega_k.$$

Remark 5.4. The motivation for the notation employed is that in a classical group such as $SU(2)$ the function just defined is nothing but the Fourier transform of the fractional integral operator

$$I_\tau(f)(x) = \frac{1}{\Gamma(\tau)} \int_{-\infty}^{x} f(y)(x-y)^{\tau-1} dy$$

acting on $f_0 A_\delta$. Here lies the main difference with the commutative case –where a Hausdorff-Young inequality of local type has been already investigated, see [1]– since the presence of the degrees d_λ –as a product in proposition 5.3 by the Weyl dimension formula– requires the presence of a factor of $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)$. This fact does not happen in the commutative case since $d_\lambda = 1$ for all $\lambda \in \Lambda_{\mathrm{DW}}$.

Lemma 5.5. *Let G be a compact semisimple simply connected Lie group and let $f : G \to \mathbb{C}$ be a central function. Then we have $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(fA_\delta)(\xi) = 0$ for all $\xi \in \mathrm{P}$.*

Proof. If $\xi \in \mathrm{P}$, there exists a root α such that $\xi \in P_\alpha$. Let S_α be the reflection in P_α, then $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(fA_\delta)(\xi) = \det S_\alpha \ \mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(fA_\delta)(S_\alpha(\xi)) = -\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(fA_\delta)(\xi)$ since, as it is well known, $S_\alpha \in \mathcal{W}_G^\star$. $\square$

The function $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)$ is analytic since $f_0 A_\delta$ has compact support and, by lemma 5.5, it vanishes at

$$\mathrm{P} = \{\xi \in \mathfrak{h}_{\mathbb{R}}^\star : \prod_{\alpha \in \mathcal{R}^+} \langle \alpha, \xi \rangle = 0\}.$$

In particular, since $0 \leq \tau < 1$, $\widehat{I_\tau(f_0 A_\delta)}$ is continuous and takes the value 0 on P. Now we write the norm of this function in terms of a Riemann sum

$$\|\widehat{I_\tau(f_0 A_\delta)}\|_{L^{q'}(\mathfrak{h}_{\mathbb{R}}^\star)} = \lim_{k \to \infty} \Big[\sum_{\lambda \in \Lambda_{\mathrm{W}}} \frac{\mathrm{V}_G}{k^r} \frac{|\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)(k^{-1}\lambda)|^{q'}}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, k^{-1}\lambda \rangle|^{\tau q'}} \Big]^{1/q'},$$

where V_G denotes the volume of a cell of Λ_{W}. Moreover $\phi_k(x) = k^\sigma f_0(kx) A_\delta(kx)$ is supported in $\mathfrak{T}$ and the relation $\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta)(k^{-1}\lambda) = k^{r-\sigma} \mathcal{F}_{\mathbf{T}}(\phi_k)(\lambda)$ is satisfied for all $\lambda \in \Lambda_{\mathrm{W}}$. Taking $\sigma = \tau|\mathcal{R}^+| + r/q$, we obtain

$$\|\widehat{I_\tau(f_0 A_\delta)}\|_{L^{q'}(\mathfrak{h}_{\mathbb{R}}^\star)} = \mathrm{V}_G^{1/q'} \lim_{k \to \infty} \Big[\sum_{\lambda \in \Lambda_{\mathrm{W}} \backslash \mathrm{P}} \frac{|\mathcal{F}_{\mathbf{T}}(\phi_k)(\lambda)|^{q'}}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda \rangle|^{\tau q'}} \Big]^{1/q'},$$

since we know that for $\lambda \in \mathrm{P}$ we get nothing. Finally, let us define $\varphi_k : \mathfrak{h}_{\mathbb{R}} \to \mathbb{C}$ by the relation $\phi_k = \varphi_k A_\delta$. The function φ_k satisfies $\varphi_k(W(x)) = \varphi_k(x)$ for all $W \in \mathcal{W}_G$ and is supported in $k^{-1}\mathfrak{T}$, hence we can understand φ_k as a central function on G. We can also say that, as a consequence of the well known relation

$$(4) \qquad\qquad A_\delta = \exp_{-\delta} \prod_{\alpha \in \mathcal{R}^+} (\exp_\alpha -1),$$

φ_k has no singularities. Therefore proposition 5.3 provides the following relation for some constant $\mathcal{B}(G, q)$ depending on G and q

$$(5) \qquad\qquad \|\widehat{I_\tau(f_0 A_\delta)}\|_{L^{q'}(\mathfrak{h}_{\mathbb{R}}^\star)} = \mathcal{B}(G, q) \ \lim_{k \to \infty} \|\widehat{\varphi_k}\|_{\mathcal{L}^{q'}(\widehat{G})}.$$

On the other hand, since φ_k can be seen as a central function on G, we can estimate the norm of φ_k on $L^q(G)$. By the Weyl integration formula we get

$$
\begin{aligned}
\|\varphi_k\|_{L^q(G)} &= \Big[\frac{1}{|\mathcal{W}_G|} \int_{\mathbf{T}} |\varphi_k A_\delta(t)|^q |A_\delta(t)|^{2-q} dm(t) \Big]^{1/q} \\
&= \Big[\frac{k^{\sigma q}}{|\mathcal{W}_G|} \int_{\mathfrak{T}} |f_0 A_\delta(kx)|^q |A_\delta(x)|^{2-q} dx \Big]^{1/q} \\
&\leq \Big[\frac{(2\pi)^{(2-q)|\mathcal{R}^+|}}{|\mathcal{W}_G|} k^{\sigma q} \int_{\mathfrak{T}} |f_0 A_\delta(kx)|^q \prod_{\alpha \in \mathcal{R}^+} |\alpha(x)|^{2-q} dx \Big]^{1/q},
\end{aligned}
$$

where the last inequality follows from (4). Now, under the change of variable $y = kx$ and taking $\mathcal{C}(G, q) = (2\pi)^{\tau|\mathcal{R}^+|} |\mathcal{W}_G|^{-1/q}$, we obtain

$$\|\varphi_k\|_{L^q(G)} \leq \mathcal{C}(G, q) \ k^{\sigma - \tau|\mathcal{R}^+| - r/q} \Big(\int_{\mathfrak{T}} |f_0 A_\delta(y)|^q \prod_{\alpha \in \mathcal{R}^+} |\alpha(y)|^{\tau q} dy \Big)^{1/q}.$$

Recall that $\text{supp}(f_0 A_\delta) \subset \mathfrak{T}$, therefore the integral over $k\mathfrak{T}$ –the domain of integration after the change of variable– reduces to the same integral over $\mathfrak{T}$. But $\sigma - \tau|\mathcal{R}^+| - r/q = 0$ and the product inside the integral is bounded over $\mathfrak{T}$, say by M_G. Then we can write

$$\tag{6} \|\varphi_k\|_{L^q(G)} \leq \mathcal{C}(G,q) \ \text{M}_G \ \|f_0 A_\delta\|_{L^q(\mathfrak{h}_\mathbb{R})}.$$

In summary, by (5) and (6), we know there exists a constant $\mathcal{D}(G,q)$ depending on G and q such that

$$\mathcal{K}(G,q) = \mathcal{D}(G,q) \ \frac{\|I_\tau \widehat{(f_0 A_\delta)}\|_{L^{q'}(\mathfrak{h}_\mathbb{R}^\star)}}{\|f_0 A_\delta\|_{L^q(\mathfrak{h}_\mathbb{R})}} \leq \liminf_{k\to\infty} \frac{\|\widehat{\varphi_k}\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|\varphi_k\|_{L^q(G)}} \leq 1.$$

Since f_0 is bounded we easily obtain that $f_0 A_\delta \in L^q(\mathfrak{h}_\mathbb{R})$, $\widehat{I_\tau(f_0 A_\delta)} \in L^{q'}(\mathfrak{h}_\mathbb{R}^\star)$ and $\mathcal{K}(G,q) > 0$. Therefore we have found a family $\{\varphi_k : \ k \geq 1\}$ of central functions on G whose supports are eventually in $\mathcal{U}$ and such that their Hausdorff-Young quotient of exponent q is bounded below by a positive constant. This concludes the proof of theorem 5.1 for compact semisimple simply connected Lie groups.

If G is not simply connected, some extra comments have to be made. In any case we shall not give complete proofs of any of them, the details are left to the reader.

(i) Generalization (3) of formula (1) has no meaning here, but we can generalize formula (2) as

$$\widehat{f}(\pi_\lambda) = \frac{1}{d_\lambda} \ \det W \ \mathcal{F}_\mathbf{T}(f B_\delta)(W^t(\lambda + \delta) - \delta) \ 1_{d_\lambda}.$$

This generalization provides a couple of results parallel to lemmas 5.2 and 5.5. Namely,
  – We have $\{W^t(\lambda + \delta) - \delta : \ W \in \mathcal{W}_G, \ \lambda \in \Lambda_{\text{DW}}\} = \Lambda_\text{W} \setminus (\text{P} - \delta)$. The mapping $(W, \lambda) \in \mathcal{W}_G \times \Lambda_{\text{DW}} \mapsto W^t(\lambda + \delta) - \delta \in \Lambda_\text{W} \setminus (\text{P} - \delta)$ is injective.
  – If $f : G \to \mathbb{C}$ is central, then $\mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f B_\delta)(\xi) = 0$ for all $\xi \in \text{P} - \delta$.

(ii) Proposition 5.3 is now replaced by the following identity, valid for central functions $f : G \to \mathbb{C}$

$$\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})} = \mathcal{A}(G,q) \ \Big[\sum_{\lambda \in \Lambda_\text{W} \setminus (\text{P}-\delta)} \frac{|\mathcal{F}_\mathbf{T}(f B_\delta)(\lambda)|^{q'}}{\prod\limits_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda + \delta \rangle|^{q'-2}} \Big]^{1/q'}.$$

(iii) The bases of $\mathfrak{h}_\mathbb{R}^\star$ and $\mathfrak{h}_\mathbb{R}$ respectively which generate Λ_W and L_W with integer coefficients are no longer the basis of fundamental weights and its predual. In fact, the fundamental weights generate the weight lattice of the universal covering group of G, which is a lattice containing Λ_W and strictly bigger than it. Therefore we need to define $\{H_1, H_2, \ldots H_r\}$ and $\{\omega_1, \omega_2, \ldots \omega_r\}$ just as the bases –of $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^\star$ respectively– for which L_W and Λ_W have integer coefficients. Once we have clarified this point, we can define $\mathfrak{T}$ in the same way and regard f_0 as a bounded complex-valued function on $\mathfrak{h}_\mathbb{R}$, supported in $\mathfrak{T}$ and symmetric under the reflections that generate $\mathcal{W}_G$.

(iv) Let us recall that if $\delta \notin \Lambda_\mathcal{W}$, the function A_δ is not well-defined on $\mathbf{T}$. But A_δ is originally defined on $\mathfrak{h}_\mathbb{R}$ and $\delta \notin \Lambda_\mathcal{W}$ is not an obstacle to work with A_δ as a function defined on $\mathfrak{h}_\mathbb{R}$. On the other hand, (ii) leads us to consider –in the

same spirit as in the proof given for simply connected groups– the function

$$\widehat{\widetilde{I_\tau(f_0 B_\delta)}}(\xi) = \frac{1}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi + \delta \rangle|^\tau} \; \mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 B_\delta)(\xi).$$

Now, the remark given about A_δ shows that $\widehat{\widetilde{I_\tau(f_0 B_\delta)}}(\xi) = \widehat{I_\tau(f_0 A_\delta)}(\xi + \delta)$. Hence we can proceed as before expressing the norm of this function in $L^{q'}(\mathfrak{h}_\mathbb{R}^\star)$ as a Riemann sum, but this time we take the lattice $\Lambda_\mathrm{W} + \delta$ instead of Λ_W

$$\|\widehat{\widetilde{I_\tau(f_0 B_\delta)}}\|_{L^{q'}(\mathfrak{h}_\mathbb{R}^\star)} = \lim_{k \to \infty} \Big[\sum_{\lambda \in \Lambda_\mathrm{W} + \delta} \frac{\mathrm{V}_G}{k^r} \frac{|\mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta)(k^{-1}\lambda)|^{q'}}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, k^{-1}\lambda \rangle|^{\tau q'}} \Big]^{1/q'}.$$

(v) It is not difficult to check that $\mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta)(k^{-1}\lambda) = k^{r - \sigma} \mathcal{F}_\mathbf{T}(\varphi_k B_\delta)(\lambda - \delta)$, where φ_k is defined as we did above. Hence we get

$$
\begin{aligned}
\|\widehat{\widetilde{I_\tau(f_0 B_\delta)}}\|_{L^{q'}(\mathfrak{h}_\mathbb{R}^\star)} &= \mathrm{V}_G^{1/q'} \lim_{k \to \infty} \Big[\sum_{\lambda \in \Lambda_\mathrm{W} \backslash (\mathrm{P} - \delta)} \frac{|\mathcal{F}_\mathbf{T}(\varphi_k B_\delta)(\lambda)|^{q'}}{\prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \lambda + \delta \rangle|^{\tau q'}} \Big]^{1/q'} \\
&= \mathcal{B}(G, q) \lim_{k \to \infty} \|\widehat{\varphi_k}\|_{\mathcal{L}^{q'}(\widehat{G})}.
\end{aligned}
$$

Finally, to estimate the norm of φ_k on $L^q(G)$, we follow the same arguments. This completes the proof of theorem 5.1 and, consequently, the proof of theorem 3.1.

Remark 5.6. Let $\{\mathcal{U}_n : n \geq 1\}$ be a basis of neighborhoods of $\mathbf{1}$, and let

$$\mathcal{K}(G, q) = \inf_{n \geq 1} \sup \left\{ \frac{\|\widehat{f}\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|f\|_{L^q(G)}} : f \; \text{central}, \; f \in L^q(G), \; \mathrm{supp}(f) \subset \mathcal{U}_n \right\}.$$

This constant does not depend on the chosen basis and theorem 5.1 states that $0 < \mathcal{K}(G, q) \leq 1$ for any $1 \leq q \leq 2$ and any compact semisimple Lie group. However, it would be interesting to find the exact value of that constant. Sharp constants for the Hausdorff-Young inequality were investigated in [2], [3] or [13]. In the local case, if $\mathcal{B}_q = \sqrt{q^{1/q}/q'^{1/q'}}$ stands for the Babenko-Beckner constant, it is already known that $\mathcal{K}(\mathbb{T}, q) = \mathcal{B}_q$. Andersson proved it for q' an even integer in [1] and Sjölin completed the proof, see [15]. Also it is obvious that $\mathcal{K}(G, 1) = \mathcal{K}(G, 2) = 1$ for any compact group G. In the general case, a detailed look at the proof of theorem 5.1 gives that the constant $\mathcal{K}(G, q)$ is the supremum of

$$|\mathcal{W}_G|^\tau \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^\tau \; \mathrm{V}_G^{-1/q'} \; \lim_{k \to \infty} \frac{\left(\int_{\mathfrak{h}_\mathbb{R}^\star} |\mathcal{F}_{\mathfrak{h}_\mathbb{R}}(f_0 A_\delta)(\xi)|^{q'} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi \rangle|^{2 - q'} d\xi \right)^{1/q'}}{\left(\int_{\mathfrak{h}_\mathbb{R}} |f_0 A_\delta(x)|^q \; |k^{|\mathcal{R}^+|} A_\delta(x/k)|^{2 - q} dx \right)^{1/q}}$$

for $1 < q \leq 2$, where the supremum runs over the family of functions $f_0 : \mathfrak{h}_\mathbb{R} \to \mathbb{C}$, supported in $\mathfrak{T}$ and symmetric under the reflections generating the Weyl group of G. If $\mathcal{K}_{f_0}(G, q)$ denotes the expression given above, then one easily gets that

$\mathcal{K}_{f_0}(G, q)$ equals

$$\frac{|\mathcal{W}_G|^\tau}{(2\pi)^{\tau|\mathcal{R}^+|}\mathrm{V}_G^{1/q'}} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \delta \rangle|^\tau \frac{\left(\int_{\mathfrak{h}_{\mathbb{R}}^\star} |\mathcal{F}_{\mathfrak{h}_{\mathbb{R}}}(f_0 A_\delta(\xi))|^{q'} \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, \xi \rangle|^{2-q'} d\xi \right)^{1/q'}}{\left(\int_{\mathfrak{h}_{\mathbb{R}}} |f_0 A_\delta(x)|^q \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2-q} dx \right)^{1/q}}.$$

Moreover, taking $q = 2$ and by Plancherel theorem on compact groups, it follows that $\mathrm{V}_G = 1$. The boundedness of this expression can be regarded as a weighted Hausdorff-Young inequality of Pitt type, see [4] for more on this topic.

As we pointed out in the introduction, the growth of $\mathcal{C}_q^1(l^{p'}(n), G)$ remains open for $1 \le p < q \le 2$. We end this paper with some remarks about this problem.

Remark 5.7. In theorem 3.1 we found an extremal function $\Phi_n = (\varphi_1, \varphi_2, \ldots \varphi_n)$, such that

$$\mathcal{C}_q^1(l^p(n), G) \ge \frac{\|\widehat{\Phi}_n\|_{\mathcal{L}_{l^{p}(n)}^{q'}(\widehat{G})}}{\|\Phi_n\|_{L_{l^{p}(n)}^{q}(G)}} \ge \mathcal{K}(G, q) \; n^{1/p - 1/q}.$$

Our functions $\varphi_1, \varphi_2, \ldots \varphi_n$ satisfied two crucial properties, namely

(P1) The norm of $\widehat{\varphi}_k(\pi)$ on $S_{d_\pi}^{q'}$ does not depend on k for any $\pi \in \widehat{G}$.

(P2) $\varphi_1, \varphi_2, \ldots, \varphi_n$ have pairwise disjoint supports on G.

The idea was to compare the norms of $\widehat{\Phi}_n$ and Φ_n with $n^{1/p}$ and $n^{1/q}$ respectively. To this end, properties (P1) and (P2) were the conditions to be required since they provided suitable simplifications for the original expressions of such norms. Now, if we replace $l^p(n)$ by $l^{p'}(n)$ in the relation above, we want to compare the norms of $\widehat{\Phi}_n$ and Φ_n with $n^{1/q'}$ and $n^{1/p'}$ respectively. Notice that $1/p - 1/q = 1/q' - 1/p'$. For that, we require these other properties on $\varphi_1, \varphi_2, \ldots \varphi_n$

(P3) The absolute value $|\varphi_k(g)|$ does not depend on k for any $g \in G$.

(P4) $\widehat{\varphi}_1, \widehat{\varphi}_2, \ldots, \widehat{\varphi}_n$ have pairwise disjoint supports on $\widehat{G}$.

In the introduction we recalled that the growth of $\mathcal{C}_q^1(l^p(n), G)$ and $\mathcal{C}_q^1(l^{p'}(n), G)$ can be understood as dual problems with respect to the Fourier transform operator. Now, these properties justify this point. Assuming properties (P3) and (P4), we get $\mathcal{C}_q^1(l^{p'}(n), G) \ge \mathcal{K}'(G, q, n) \; n^{1/q' - 1/p'}$, where $\mathcal{K}'(G, q, n)$ is given by

$$\mathcal{K}'(G, q, n) = \left(\frac{1}{n} \sum_{k=1}^n \left[\frac{\|\widehat{\varphi}_k\|_{\mathcal{L}^{q'}(\widehat{G})}}{\|\varphi_k\|_{L^q(G)}} \right]^{q'} \right)^{1/q'}.$$

Hence, if we define $\mathcal{K}'(G, q) = \inf_{n \ge 1} \mathcal{K}'(G, q, n)$, it remains to see that $\mathcal{K}'(G, q) > 0$. We do not know if this inequality holds for any compact semisimple Lie group and any $1 \le q \le 2$.

Remark 5.8. We do not know if properties (P3) and (P4) are compatible. However, given $f_0 \in L^2(G)$ continuous and any sequence of positive numbers $\{\varepsilon_n : \; n \ge 1\}$ decreasing to 0, it is not difficult to see that there exists a system $\Phi = \{\varphi_n : \; n \ge 1\}$ of trigonometric polynomials on G satisfying

1. The functions $\widehat{\varphi}_1, \widehat{\varphi}_2, \ldots$ have pairwise disjoint supports on $\widehat{G}$.
2. The estimate $|\varphi_n| \le |f_0| + \varepsilon_n$ holds in G.
3. The estimate $|\varphi_n| \ge |f_0| - \varepsilon_n$ holds outside Ω_n, where $\mu(\Omega_n) \to 0$ as $n \to \infty$.

Remark 5.9. As it is well-known, $\mathcal{C}_q^1(l^{p'}(n), G) = n^{1/p-1/q}$ for any compact abelian group G. This equality follows by taking $\varphi_1, \varphi_2, \ldots \varphi_n$ to be a collection of n pairwise distinct characters. This motivates us to see what happens when we consider the irreducible characters of a compact semisimple Lie group. Let χ_λ be the character of the irreducible representation π_λ, let us consider the function $\Phi_n(g) = (d_{\lambda_1}^\tau \chi_{\lambda_1}(g), d_{\lambda_2}^\tau \chi_{\lambda_2}(g), \ldots d_{\lambda_n}^\tau \chi_{\lambda_n}(g))$, where $\lambda_1, \lambda_2, \ldots \lambda_n$ are pairwise distinct dominant weights and $\tau = 1 - 2/q'$. Then we have

$$\|\widehat{\Phi}_n\|_{\mathcal{L}_{l^{p'}(n)}^{q'}(\widehat{G})} = \Big(\sum_{k=1}^n d_{\lambda_k} \|d_{\lambda_k}^\tau \widehat{\chi}_{\lambda_k}(\pi_k)\|_{S_{d_{\lambda_k}}^{q'}}^{q'} \Big)^{1/q'} = n^{1/q'}.$$

On the other hand, applying consecutively the Weyl integration formula and the Weyl character formula, we get

$$\|\Phi_n\|_{L_{l^{p'}(n)}^q(G)} = \Big(\frac{1}{|\mathcal{W}_G|} \int_{\mathbf{T}} \Big(\sum_{k=1}^n |d_{\lambda_k} A_\delta(t)|^{\tau p'} |A_{\lambda_k+\delta}(t)|^{p'} \Big)^{q/p'} dm(t) \Big)^{1/q}.$$

However these relations do not provide optimal growth. For instance, in the simplest case $G = SU(2)$ it can be checked that there exists a constant $\mathcal{K}_{p,q}$ depending on p and q such that

$$\frac{1}{n^{1/p'}} \Big(\int_{SU(2)} \|\Phi_n(g)\|_{l^{p'}(n)}^q d\mu(g) \Big)^{1/q} \geq \mathcal{K}_{p,q} \ n^\tau.$$

Remark 5.10. If we try to find out why our attempts to get optimal growth have failed, we need to revisit the proof of theorem 3.1. The point is that we required the functions $\varphi_1, \varphi_2, \ldots \varphi_n$, not only to satisfy properties (P1) and (P2), but also to be translations of a common function. This was essential in section 3 and here the obstacle lies in the fact that we can not take translations since the dual object has not a group structure. This is the main difference with the abelian case where, since the dual object is a group, multiplication by a character in G becomes a translation in the other side of the Fourier transform operator.

Remark 5.11. The quantized Rademacher system associated to a probability space $(\Omega, \mathcal{M}, \mu)$, an index set Σ, and a family $\{d_\sigma : \sigma \in \Sigma\}$ of positive integers is defined by a collection $\mathcal{R} = \{\rho^\sigma : \Omega \to O(d_\sigma)\}_{\sigma \in \Sigma}$ of independent random orthogonal matrices, uniformly distributed on the orthogonal group $O(d_\sigma)$. In [9] we define the notions of $\mathcal{R}$-type, $\mathcal{R}$-cotype and strong $\mathcal{R}$-cotype of an operator space E. Moreover, we show that

$$\begin{array}{lll}
\text{Fourier type } p & \Rightarrow & \text{strong } \mathcal{R}\text{-cotype } p' \\
\text{Fourier cotype } p' & \Rightarrow & \mathcal{R}\text{-type } p.
\end{array}$$

This implications allow us to work with the quantized Rademacher system where other techniques are available to study the growth of $\mathcal{C}_q^1(l^{p'}(n), G)$.

Remark 5.12. Of course, the growth of $\mathcal{C}_q^1(l^{p'}(n), G)$ is trivially optimal when we work with compact groups with infinitely many inequivalent irreducible representations of the same degree d_0. The unitary groups $U(n)$ are the simplest non-commutative examples of this degenerate case. Also, it is not difficult to check that $\mathcal{C}_2^1(l^{p'}(n), G) = n^{1/2-1/p'}$ by the Plancherel theorem for compact groups.

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